DISCONNECTED JULIA SETS

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INTRODUCTION

The connectivity properties of the Julia set for a polynomial have an intimate relationship with the dynamical properties of the finite critical points. For example, if all critical points iterate to infinity, then the Julia set J is totally disconnected, and the polynomial p restricted to J is topologically conjugate to the one-sided shift on d = deg(p) symbols. On the other hand, if the orbits of all of the finite critical points are bounded, then J is connected. In this paper, we discuss other possibilities, and in particular, we indicate how to construct symbolic codings for the components of the Julia set for a large class of cubic polynomials. These cubics will have one critical point which iterates to infinity and another whose orbit remains bounded. Using these two orbits, we define a kneading sequence with two symbols, and given certain kneading sequences, we show how to reconstruct the dynamics of these cubics using the Douady-Hubbard [8] theory of polynomial-like maps.

In Section 1, we establish our notation and summarize the known results about symbolic codings and the Julia set with a

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particular emphasis on the quadratic case. Then, in Section 2, we describe our approach for cubics and its relationship to the Branner-Hubbard decomposition of the space of cubics. At the end of that section, we state a few important unresolved questions regarding the dynamics of cubics. For results that are implicitly used in this paper, the reader should consult the exposition [1], and the reader should also see the paper by Branner in this volume for a more elaborate discussion of the parameter space of cubics. In fact, we have made a serious effort to keep our notation consistent with that presentation.

1. NOTATION AND BACKGROUND MATERIAL

We consider polynomials $p(z)$ as functions of the Riemann sphere $\bar{C} = C \cup \{\infty\}$ and the associated discrete dynamical systems they generate. If $\deg(p) \geq 2$, the Fatou-Julia theory applies, and therefore, we have a disjoint, completely invariant decomposition

$$\bar{C} = J \cup N$$

where the Julia set J is the closure of the repelling periodic points and the domain N is the domain of normality for the family $\{p^n\}$ of iterates of the polynomial. The point at infinity plays a unique and important role. It is a superattracting fixed point, and it belongs to a completely invariant component of N. That is, this component is invariant under both the map p and its inverse p^{-1}. In fact, this component also has another characterization in terms of its dynamical properties as the stable set of ∞:

$$W^s(\infty) = \{z \in \bar{C} \mid p^n(z) \to \infty \text{ as } n \to \infty\}.$$

Following Douady and Hubbard, we focus on the "filled-in" Julia set K of the polynomial defined by the equation

$$K = \bar{C} - W^s(\infty).$$

In this paper, we are mostly concerned with polynomials for which K is disconnected, and we describe the dynamics of the components of K using symbolic codings. Figures 1 and 2 illustrate two filled-in Julia sets for two different types of polynomials. The black regions are the filled-in Julia set and the shading of the stable manifold of infinity roughly corresponds to levels of the "rate of escape" map that is defined next.

We make frequent use of the "rate of escape to infinity" map $h\colon C \to R^+ \cup \{0\}$ defined by

$$h(z) = \lim_{k\to\infty} \frac{1}{d^k} \log_+ |p^k(z)| \qquad \text{where}$$

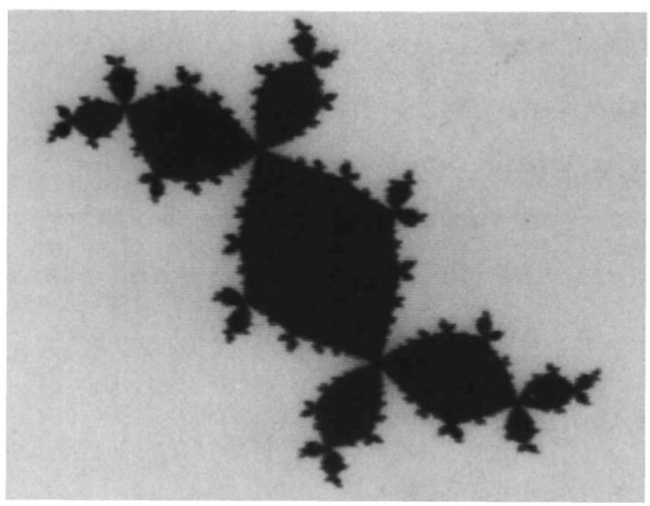

Fig. 1. The black region is the filled-in Julia set of the quadratic polynomial $z \mapsto z^2 + v$ where $v \approx -0.12256117 + 0.74486177i$. The value of v is chosen so that the origin (which is a critical point) is periodic of period three.

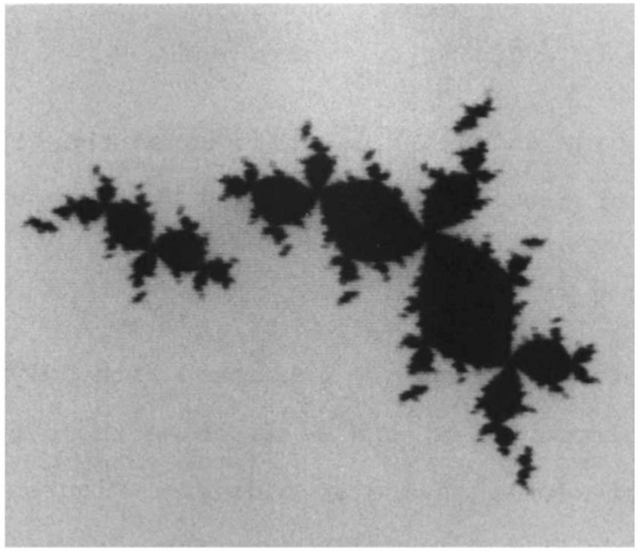

Fig. 2. The black region is the filled-in Julia set of the cubic polynomial $z \mapsto z^3 - 3a^2z + b$ where $b = 0.8$ and $a \approx -0.5769525 + 0.175i$. This kind of Julia set is the main object of interest in this paper. Unlike the Julia set illustrated in Figure 1, one finite critical point escapes to infinity and, therefore, the filled-in Julia set consists of infinitely many components. We characterize these components in Section 2.

$$\log_+(x) = \begin{cases} \log(x) & x \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

In fact, h is continuous on C and harmonic on $W^s(\infty)$. Note $K = h^{-1}(0)$. Near infinity, we can give an alternate description using the conjugacy of p to the map $z \mapsto z^d$. Recall that there exists a unique analytic homeomorphism $\psi: U_1 \to U_2$ where U_1 and U_2 are open subsets containing infinity such that $\psi(\infty) = \infty$, $\psi(p(z)) = [\psi(z)]^d$, and $D\psi_\infty = Id$. Using ψ, we obtain another formula for h(z) as

$$h(z) = \log|\psi(z)| \text{ when } z \in U_1.$$

From this formula, it is easy to see that $h(p(z)) = d \cdot h(z)$.

Before discussing the unusual symbol spaces we use in the second half of this paper, it is useful to elaborate on the two extreme cases mentioned in the introduction. First of all, we define the one-sided shift $\sigma|\Sigma_d$ on d symbols as the topological space

$$\Sigma_d = \prod_{k=0}^{\infty} \{1,2,\ldots,d\}$$

(where $\{1,\ldots,d\}$ is given the discrete topology and Σ_d is given the associated product topology), and the shift map $\sigma: \Sigma_d \to \Sigma_d$ is the d-to-1 endomorphism defined by

$$[\sigma(\{s_i\})]_i = s_{i+1} .$$

Note that this map has d fixed points, many periodic points of each period, points which are eventually periodic but not periodic, and aperiodic points with dense orbits. Later, we will find it easier to use symbols which are more mnemonic than the numbers from 1 to d, but the symbol set will always be equipped with the discrete topology.

Using $\sigma|\Sigma_d$, we can give a modern statement of the classical result concerning the two extremes mentioned in the introduction. Let C be the set of finite critical points of the polynomial p(z).

Theorem 1. If $C \subset W^s(\infty)$, then $p|J$ is topologically conjugate to the map $\sigma|\Sigma_d$. On the other hand, if $C \subset K$, then J is connected.

In the quadratic case, every polynomial is analytically conjugate to one of the form

$$q_v(z) = z^2 + v.$$

In this form, $C = \{0\}$ for all q_v, and the first part of Theorem

1 applies if and only if

$$0 \mapsto v \mapsto v^2 + v \mapsto (v^2 + v)^2 + v \mapsto \ldots \to \infty.$$

When this happens, $h(0) > 0$, and using h, we can define the conjugacy $\phi: J \to \Sigma_2$ as follows. The level curve $L = h^{-1}(h(0))$ is a pinched circle which bounds two finite disks D_1 and D_2. Then

$$[\psi(z)]_k = \begin{cases} 1 \iff q^k(z) \in D_1 \\ 2 \iff q^k(z) \in D_2 . \end{cases}$$

This dichotomy (Theorem 1 applied to quadratics) motivates Douady and Hubbard's extensive analysis of the Mandelbrot set M. The set M is defined by

$$M = \{v \in C \mid J_{q_v} \text{ is connected}\},$$

and as Figure 3 indicates, it is remarkably complicated with an extremely interesting fractal structure.

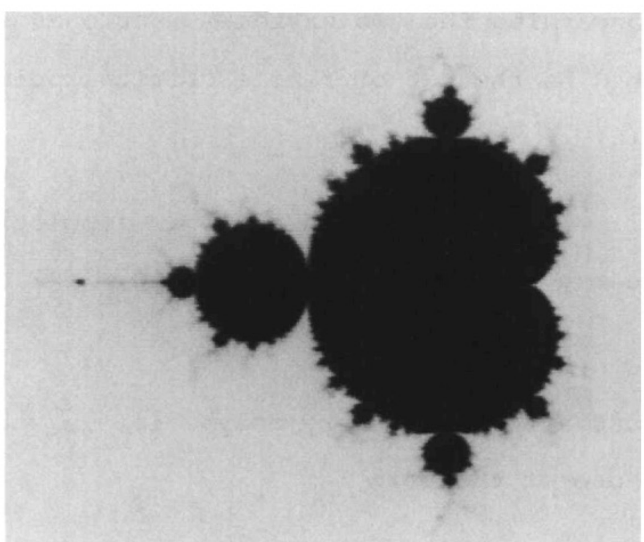

Fig. 3. The Mandelbrot set. See Mandelbrot [16, 17] for many more illustrations which indicate its complicated, yet regular structure. See also [9] and [10] for a detailed discussion of the dynamics of quadratics.

As we saw above, $\sigma|\Sigma_2$ can be used to study the dynamics of $q_v|J$ when $v \notin M$. An analogous question can be asked about the case where $v \in M$. This question was studied by Guckenheimer [12] and Jacobsen [14, 15] with the additional hypothesis that $p|J$ is expanding. Since their work is similar to what we describe in the next section, we give a brief statement of their results.

Theorem 2. (Jacobsen and Guckenheimer) If every critical point of a polynomial is attracted to some periodic sink (the point at infinity is allowed), then there exists a quotient with finite fibers of Σ_d which is topologically conjugate to $p|J$.

Consider two examples in the quadratic case. The first is the fundamental quadratic $z \mapsto z^2$. Recall that J is the unit circle. The quotient of Σ_2 in this case is generated by the identification $011\bar{1} \sim 100\bar{0}$. Extend this identification to all of Σ_2 in the minimal way such that there exists a well-defined quotient map $\sigma\colon (\Sigma/\sim) \to (\Sigma/\sim)$. In other words, $11\bar{1} \sim 00\bar{0}$, and $a_0a_1\ldots a_n011\bar{1} \sim a_0a_1\ldots a_n100\bar{0}$. The second example is the quadratic q_v whose Julia set is Douady's rabbit (see Figure 1 and recall that $\partial K = J$). In that case, we need one identification in addition to those of the first example to generate the quotient. It is $001\overline{001} \sim 010\overline{010} \sim 100\overline{100}$. One then adds the minimal set of identifications necessary to produce a well-defined quotient of the shift map.

2. SYMBOLIC CODINGS FOR CUBICS

In this section, we focus on the dynamics of cubics, and in particular on the intermediate case not covered in Theorem 1. Our approach is similar to that of Guckenheimer and Jacobsen in that we employ symbol sequences to study these maps, but it differs in that we associate a symbol for each component of K rather than for each point in K. The dynamics of the individual components is then determined using the Douady-Hubbard theory of polynomial-like maps. Although the coding in our approach is less precise than the Guckenheimer/Jacobsen coding, it has the advantage of applying to more general situations -- namely we do not need to make the assumption that $p|J$ is expanding.

For the remainder of the paper, we assume that p is a cubic and that the set of finite critical points consists of two distinct points c_1 and c_2 where $c_1 \in W^s(\infty)$ and $c_2 \in K$. First of all, consider the energy function and the structure of its level sets. As in the quadratic case, the level curve $L = h^{-1}(h(c_1))$ is a pinched curve (pinched at c_1) which bounds two finite disks -- denoted A and B. In fact, we fix our notation so that $c_2 \in B$, and therefore, $\deg(p|A) = 1$ and $\deg(p|B) = 2$. Given this decomposition, we define the A-B kneading sequence for p from the orbit of c_2 in K. This sequence $\{k_i\}$ is a sequence of the letters A and B by

$$k_i = \begin{cases} A & \text{if } p^i(c_2) \in A \\ B & \text{if } p^i(c_2) \in B \end{cases}$$

for $i = 0,1,2,\ldots$. In this section, we discuss a symbolic coding of the components of K for three distinct cases of $\{k_i\}$,

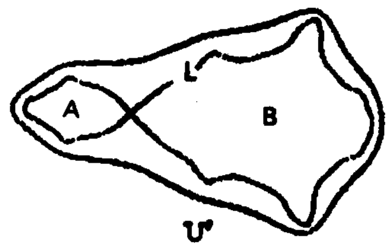

Fig. 4. The disk U' (for the cubic $z^3 - 1.47z + 0.8$) defined by the rate of escape function h and its value for $p(c_1)$. That is, $U' = h^{-1}\{[0,h(p(c_1))]\}$. The level curve L and the two disks A and B that it bounds are all subsets of U'.

and then we state some open questions regarding the remaining cases.

A key ingredient is the Douady-Hubbard theory of polynomial-like maps [9]. Their idea is that often, in the dynamics of high degree polynomials and even in transcendental functions, one can find regions on which the dynamics is really determined by a low degree polynomial.

<u>Definition</u>. Let U and U' be two simply-connected domains in C such that U is a relatively compact subset of U'. If $f: U \to U'$ is a proper, holomorphic map of degree d, then we say that f is polynomial-like of degree d on U. Associated to every polynomial-like map $f: U \to U'$, there exists a "filled-in" Julia set K_f defined by

$$K_f = \{z \mid f^n(z) \in U \text{ for } n = 0,1,2,\ldots\}.$$

The basic theorem of Douady and Hubbard ([7] and [9]) which gives this notion its strength is the following result.

<u>Theorem 3</u>. (Douady and Hubbard) If $f: U \to U'$ is polynomial-like of degree d, then there exists a polynomial q

of degree d such that $q|K_q$ is quasi-conformally conjugate to $f|K_f$.

This is precisely what is happening in the case at hand. We have two polynomial-like maps $f_1 = p|A$ and $f_2 = p|B$ such that $\deg(f_k) = k$. If the A-B kneading sequence of c_2 is BBB..., then $f_2|K_{f_2}$ has the dynamics of some quadratic q_v where $v \in M$, and if the kneading sequence is anything else, then $f_2|K_{f_2}$ is topologically conjugate to the one-sided shift on two symbols.

Before we state and prove our results, we should pause to relate this situation to the Branner-Hubbard decomposition of the parameter space of cubics described elsewhere in this volume. They discuss the dynamics of cubics in terms of the family

$$p_{a,b}(z) = z^3 - 3a^2z + b$$

where $(a,b) \in C^2$. The critical points are therefore $\pm a$. Let's suppose that $c_1 = +a$, and $c_2 = -a$. Then, according to their results, if we fix both the rate of escape of +a to infinity (i.e., the value of h(+a)) and the angle of escape to infinity, then we are left with a two-dimensional subspace of cubics which is trefoil cloverleaf T. Moreover, the structure of parameter space inside of each leaf does not change as we move from leaf to leaf. Many cubics in T will also have $-a \in W^s(\infty)$, and consequently, Theorem 1 applies. However, it is interesting to relate the A-B kneading sequence to the structure inside T. Inside of each leaf of T, there exists an entire Mandelbrot set of cubics whose A-B kneading sequence is BBB... (see Figure 5) and another Mandelbrot set whose A-B kneading sequence is BABABA... (see Figures 5 and 6). In addition,

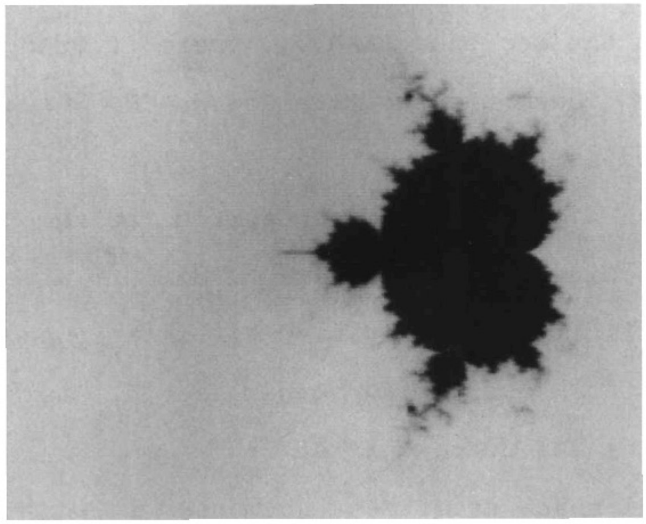

Fig. 5. A slice of the cubic parameter space with b = 0.8. The black regions indicate the areas where one critical point escapes to infinity while the orbit of the other is bounded. The largest Mandelbrot set corresponds to the kneading sequence BBB... .

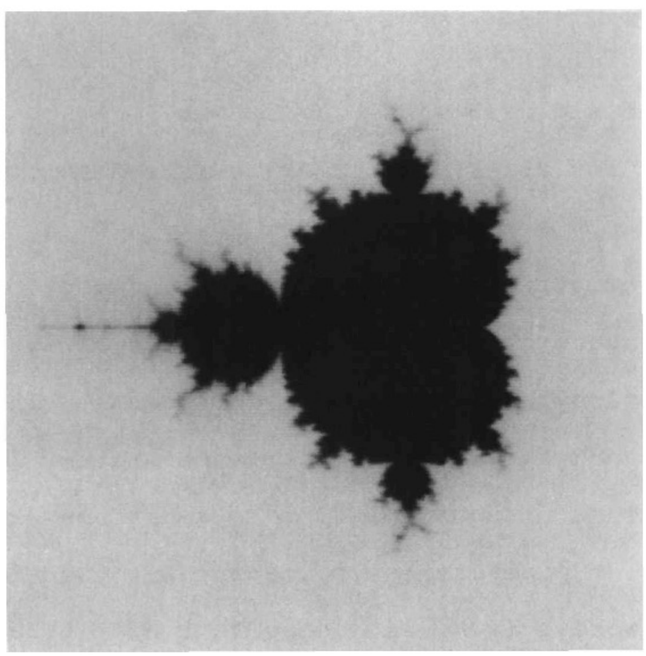

Fig. 6. An enlargement of a small area along the horizontal axis of Figure 5. The length of this side is about 3/1000 the length of the side in Figure 5. The largest Mandelbrot set corresponds to the kneading sequence BABABA... .

associated to the sequence BAAA..., there is exactly one cubic in each leaf. These are the three cases for which we construct a symbolic coding.

As one can immediately conclude from Theorem 3, the case where the kneading sequence is BBB... differs widely from the other possibilities. In this case K_{f_2} is a connected, filled-in Julia set for a quadratic. Otherwise, K_{f_2} is a Cantor set. We consider the BBB... case first.

Our symbol space describes the manner in which components of K_p map, and we give a complete topological description of $p|K_p$ if we know the dynamics of $f_2|K_2$. We use the symbol space Σ' which is a σ-invariant subset of the one-sided shift Σ_4 on the four symbols $\{1,2,3,B\}$. A sequence $s \in \Sigma'$ if and only if

(1) $s_k = B \Rightarrow s_{k+1} = B$,

(2) $s_n = B$ and $s_{n-1} \neq B \Rightarrow s_{n-1} = 1$, and

(3) if $s_k \neq B$ for all k, then there exists a subsequence s_{n_i} such that $s_{n_i} = 1$ for all i.

The theorem states that there is a conjugacy between $\sigma|\Sigma'$ and the space of components of K_p. As always, we use the Hausdorff topology on the space of components. That is, two components K_1 and K_2 are within ε of one another if every point in K_1 is within ε of some point of K_2 and vice-versa. To state the theorem, let $\tilde{p}$ denote the component-wise version of the map p. In other words, if K_1 is a component of K_p, then $\tilde{p}(A)$ is the component p(A).

<u>Theorem 4</u>. Suppose p is a cubic polynomial whose Julia set is disconnected but not totally disconnected. Using the notation of this section, suppose the A-B kneading sequence

associated to c_2 is BBB... . Then there exists a homeomorphism $\phi: \Sigma' \to$ {components of K_p} such that $\phi \circ \sigma = \tilde{p} \circ \phi$. Moreover, if $s \in \Sigma'$ is an element of Σ_3, then $\phi(s)$ is a point.

Remark. In all of the proofs of this section, we use invariant sets of rays to infinity which are intimately connected with the conjugacy at infinity of $p(z)$ to the map $z \mapsto z^3$. Basically, they are found in the following manner. Near infinity, we have a polar coordinate system associated to the cubic via the conjugacy ψ (as discussed in Section 1). A ray of angle α is the inverse image under ψ of an angular ray of angle $e^{2\pi i\alpha}$ from infinity in the standard coordinate system. The only difficulty involving these rays comes when we try to extend them so that they limit on the Julia set. In this case, we must consider the effects of critical points contained in $W^s(\infty)$ and expansion properties of the map $p|J$. We do not belabor these points here. Whenever we need to choose certain rays, we will be careful to choose those which behave as we claim. If the reader is interested in more detail, he should consult the paper [2] where this topic is treated in the generality needed here.

Proof. We choose an invariant ray ℓ_1 from infinity to a fixed point in K_2. To see that such a ray exists suppose that α_1 and α_2 are the two angles corresponding to the rays from infinity that limit on the critical point c_1. Then, since the angles 0 and 1/2 are fixed by $z \mapsto z^3$ and since α_1 and α_2 differ by 1/3, either 0 or 1/2 must enter B. Moreover, if the angle of the ray through $p(c_1)$ equals either α_1 or α_2, then it also equals either 0 or 1/2, and the remaining invariant ray will enter B and will be disjoint from the forward orbit of c_1. Consequently, there is always at least one invariant ray from infinity which limits on a fixed point of K_2.

We define the conjugacy using ℓ_1. Actually, it is best to first define the inverse map of ϕ. Using f_2^{-1}, we find another ray ℓ_2 from infinity such that $f_2(\ell_2) = \ell_1$. Then the set $B - (\ell_1 \cup \ell_2)$ can be written as $U_2' \cup U_3' \cup K_2$ as illustrated in Figure 7. The map f_2 wraps both U_2' and U_3' entirely around the slit annulus $U' - (K_2 \cup \ell_1)$. If K is a component of K_p, we define

$$[\phi^{-1}(K)]_i = \begin{cases} 1 \text{ if } p^i(K) \subset A \\ B \text{ if } p^i(K) = K_2 \\ 2 \text{ if } p^i(K) \subset U_2' \\ 3 \text{ if } p^i(K) \subset U_3'. \end{cases}$$

The definition of ϕ is slightly more involved. Let $V = U' - (K_2 \cup \ell_1)$. Then V is a simply connected domain which does not contain any critical values of the map p. Therefore we can define three inverse maps of p, namely

$$I_1: U' \to A, \quad I_2: V \to U_2', \quad \text{and} \quad I_3: V \to U_3' .$$

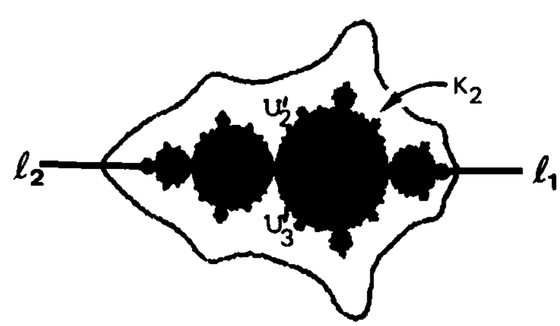

Fig. 7. The decomposition of B into $U_2' \cup K_2 \cup U_3'$ for the cubic $z^3 - 1.47z + 0.8$.

Since there are essentially two different kinds of sequences in Σ', we define ϕ in two steps. First, suppose $s_n = B$ and $s_{n-1} \neq B$ for $s \in \Sigma'$. Then

$$\phi(s) = I_{s_1} \circ I_{s_2} \circ \dots \circ I_{s_{n-1}}(K_2)$$

Note that $s_{n-1} = 1$, and consequently, K_2 is a subset of the domain of definition of $I_{s_{n-1}}$. Secondly, if $s_i \neq B$ for all i, let s_{n_j} be the subsequence of indices such that $s_{n_j} = 1$ for all j. Then let

$$M_j = I_{s_1} \circ \dots \circ I_{s_{(n_j-1)}}(\bar{A}).$$

and define

$$\phi(s) = \bigcap_{j=1}^{\infty} M_j.$$

In this case, the restrictions of the maps I_1, I_2 and I_3 to $\bar{U}_1$ are all strong contractions in the hyperbolic metric on V. Therefore, $\phi(s)$ must be a point. □

The second case is that of a periodic kneading sequence of period 2. In other words, the sequence is BABABA... . Although we have not worked out all the details in the general case of a periodic kneading sequence, we expect that the final result for a periodic sequence will have a similar statement to the specific case we now consider. Since the kneading sequence is periodic of period two, we consider three level sets of h. In addition to $L = h^{-1}(h(c_1))$, we consider $p(L)$ and $p^{-1}(L)$ (see Figure 8).

Note that $p^{-1}(L)$ consists of two components and bounds five finite disks. The orbit of c_2 will alternate between two

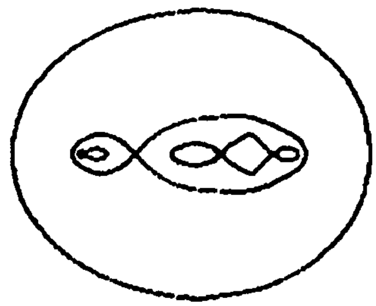

Fig. 8. Three level sets of h and the corresponding domains that they enclose for the cubic $z^3 - 3.637z + 0.8$.

of them which we label A' and B' where $A' \subset A$ and $B' \subset B$. The map $p^2|B'$ is polynomial-like of degree two, and it has a connected filled-in Julia set which we denote K_B. The map $p|A'$ maps a region K_A homeomorphically on K_B. Since the kneading sequence is BABABA..., the orbit of c_2 is contained in the union $K_A \cup K_B$. Our symbolic coding in this case is based on the following geometric decomposition of U'. Take a ray ℓ_1 from infinity whose angle is periodic of period two under $z \to z^3$ on S^1 and which limits onto a periodic point of period two in K_A. Then the inverse $(p|B)^{-1}$ applied to ℓ_1 yields two other rays from infinity ℓ_2 and ℓ_3. With these rays, we get a decomposition of U' (see Figure 9) which is similar to the decomposition we used in Theorem 4 and which gives our symbolic coding. If $V = U' - (K_A \cup \ell_1)$, then we have three inverse maps I_1, I_2, and I_3 defined on U', V, and V respectively. The construction of the coding proceeds in a similar manner to the proof of Theorem 4 with a slightly different symbol space Σ' which is now a subspace of $\Pi\{1,2,3,A,B\}$. However, before we give a precise description, we introduce a bit of terminology which simplifies the definition. Given a sequence $s \in \Pi\{1,2,3\}$, its associated A-B sequence is gotten by replacing 1 by A and

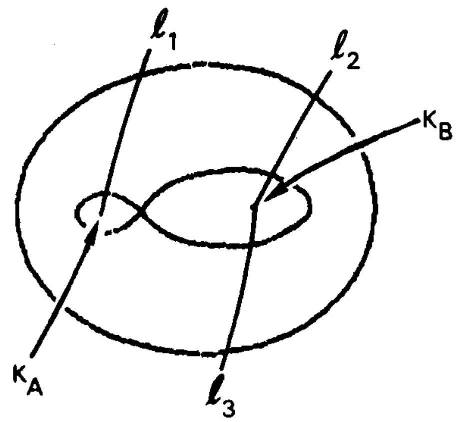

Fig. 9. The sets K_A and K_B and the rays from infinity which define the partition of U' for the cubic $z^3 - 3.637z + 0.8$.

2 and 3 by B. A sequence $s \in \Sigma'$ for this kneading sequence if and only if

(1) $s_k = B \Rightarrow s_{k+1} = A$,

(2) $s_k = A \Rightarrow s_{k+1} = B$,

(e) B and 1 are the only symbols that can precede A,

(4) A, 1, and 2 are the only symbols that can precede B, and

(5) if $s \in \Sigma_3$, then its associated A-B sequence cannot end with the sequence $BA\overline{BA}$.

<u>Theorem 5</u>. Suppose the A-B kneading sequence is BABABA... and Σ' is the σ-invariant subspace of Σ_5 defined above. Then there exists a conjugacy σ: $\Sigma' \to$ {components of K_p}. Those components which are images of sequences not containing the symbols A or B are points.

The final case which we consider is the kneading sequence BAAA... . We call this sequence preperiodic because, although the sequence is not periodic, it is eventually periodic (under the shift map applied to kneading sequences). This case is quite different from the other two considered because the filled-in Julia set K is totally disconnected. However, K (which equals J) is not topologically conjugate to $\sigma|\Sigma_3$ because it contains a fixed point with only two distinct preimages rather than three. Although we construct a symbolic coding for this case, its topological properties are quite different from those used in the proof of Theorem 4. We should also note that this case was first studied by Brolin [6, Theorem 13.8] to provide a counterexample to a conjecture of Fatou regarding critical points which are contained in the Julia set. Brolin proved that J was totally disconnected. R. Devaney actually pointed out how the techniques described in this section can be used to study this case. As always, we start with the level set $L = h^{-1}(h(c_1))$ which is a pinched circle. The fact that this kneading sequence is BAAA... implies that $p(c_2)$ is the repelling fixed point α which is the only member of $K_{p|A}$. First, note that the component of K containing α is the singleton $\{\alpha\}$ because that component is contained in the nested intersection $\cap_{k=0}^{\infty}(f_1)^{-1}(A)$. The boundaries of those sets are not in K, and they contract to a point. Now we choose a ray from infinity ℓ which limits on α.

Let V be U' - ℓ (see Figure 10). On V, there exist three

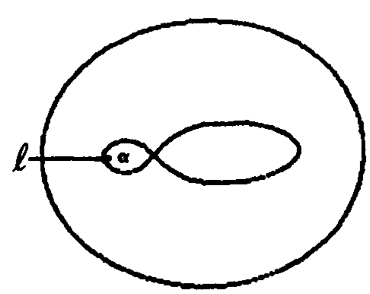

Fig. 10. The decomposition of U' induced by the ray ℓ for the cubic $z^3 - 4.03474z + 0.8$.

inverse functions I_1, I_2, and I_3 of the map p which map V into disjoint subsets of V. We can now characterize every component (and every point in K) just as we have done above. Let Σ' be the subset of Σ_3 consisting of all sequences which contain either infinitely many 2's or 3's. Then there exists a conjugacy ϕ from Σ' to the set of components of K which do not eventually map onto α. It is defined by

$$\phi(\{s_i\}) = \bigcap_{k=1}^{\infty} I_{s_0} \circ I_{s_1} \circ \ldots \circ I_{s_k}(V),$$

and one can prove that each such component is actually a point because these holomorphic functions are strict contractions on B in the Poincaré metric on V. The only remaining components of K which are left to analyze are the ones which are eventually mapped to α. Therefore, they are $\{\alpha\}$, $\{c_2\}$, and all the singletons corresponding to the inverse images of c_2. Therefore, we have the following theorem.

Theorem 6. If the A-B kneading sequence for the cubic p(z) is BAAA..., then the map $p|J$ is topologically conjugate to the quotient $(\Sigma_3/\sim)$ where $\sim$ is the smallest equivalence relation which is generated by the identification 2111... $\sim$ 3111... and which yields a space on which the quotient $\tilde{\sigma}:(\Sigma_3/\sim) \to (\Sigma_3/\sim)$ is well-defined.

3. SUMMARY AND OPEN PROBLEMS

In this paper, we have introduced a kneading sequence with two symbols as a device to help in our study of the dynamics of cubics with one critical point iterating to infinity and with another critical point whose orbit is bounded. We were then able to explicitly construct conjugacies of the component space of the filled-in Julia sets to manageable symbol spaces for three different types of kneading sequences -- BBB...., BABABA..., and BAAA... . We expect that these constructions indicate how symbolic codings should be constructed when the kneading sequence is periodic or preperiodic. We conclude with three unresolved questions which are fundamental to this study.

Problems. (1) Can the A-B kneading sequence be aperiodic? If so, what is the associated dynamics?

(2) If the kneading sequence is preperiodic, is the Julia set a Cantor set?

(3) For a given kneading sequence, calculate the number of its Mandelbrot sets in each leaf of the trefoil clover.

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