Name: ____________________________ **Time Started:** ___________**Time Stopped:**___________

MATH 513 Final Exam

Instructions: You have five hours to complete this exam. You should take the exam on your own with aid from your text book, class notes, and homework assignments. **You are not allowed to use a calculator, a computer, or any other books or notes.** The exam consists of EIGHT questions. Show all of your work and justify the steps in your proofs if you want to receive full credit. The exam is due in my office by **5pm on Tuesday, April 22nd**.

You will need to use your own paper. Before you start, write your name and starting time on the top. When you finish, write down what time you stopped the exam, staple your pages together, place the exam in the envelope, and slide the envelope under my office door.

Problem	Possible	Score
1	16	
2	20	
3	10	
4	10	
5	12	
6	12	
7	10	
8	15	
TOTAL	105	

1. (16 pts) Clearly label the following statements as being either true or false.

   (a) If T is a linear transformation over an n-dimensional vector space and the characteristic polynomial splits into n distinct linear factors, then T is diagonable.

   (b) If the determinant of $T \in L(V,V)$ is zero where $\dim(V) < \infty$, then T is not diagonable.

   (c) Matrices with the same characteristic polynomials are similar.

   (d) Every vector space has a unique zero vector.

   (e) If A and B are similar matricies, then they have the same determinant.

   (f) Let V be a vector space with an inner product. If $(x,y) = 0$ for all $x \in V$, then $y = 0$.

   (g) Every finite dimensional vector space is isomorphic to its dual.

   (h) Let V and W be arbitrary dual vector spaces with respect to a nondegenerate bilinear form B and $V_1 \subseteq V$ a subspace. Then $V_1^{\perp}$ is also a subspace of V.

2. (20 pts) Let $A \in M_{n\times n}(\mathbb{R})$.

   (a) Suppose that $\alpha + i\beta \in \mathbb{C}$ is an eigenvalue for A with eigenvector $v + iw \in \mathbb{C}_n$ where $\alpha, \beta \in \mathbb{R}$ and $v, w \in \mathbb{R}_n$. Prove that $\alpha - i\beta$ is also an eigenvalue for A and has $v - iw$ as an eigenvector.

   (b) Suppose A is a symmetric matrix, i.e. $A = A^t$. Prove that A has only real eigenvalues (no complex eigenvalues).
   Hint: Using the notation of (a), compute $(v + iw)^t A(v - iw)$ *in two different ways noting that* $(A(v + iw))^t = (v + iw)^t A^t$.

   (c) We say that a matrix $A \in M_{n\times n}(\mathbb{R})$ is orthogonally diagonalizable if $A = S^t DS$ where $SS^t = S^t S = I$ and D is diagonal, $S, D \in M_{n\times n}(\mathbb{R})$.
   Prove that if A is orthogonally diagonalizable, then A is symmetric.

   (d) Suppose A is symmetric. Is A always similar to an upper triangular matrix? Justify your answer.

   (e) Is the following matrix diagonable? Justify your answer.

$$B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix},$$

3. (10 pts) Let T be a linear transformation $T : \mathbb{R}_3 \to \mathbb{R}_3$ whose matrix with respect to the standard basis on $\mathbb{R}_3$ is

$$A = \begin{pmatrix} 3 & -1 & -1 \\ 1 & 1 & 0 \\ 0 & 4 & 5 \end{pmatrix}.$$

Find an ordered basis for $\mathbb{R}_3$ so that the matrix for T with respect to this basis is upper triangluar. (You do not have to find the matrix for T with respect to this basis)

4. (10 pts) Which of the following matricies are similar? (Justify your answer)

$$A = \begin{pmatrix} 10 & 4 \\ -9 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \quad C = \begin{pmatrix} 8 & 1 \\ -16 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 6 & 3 \\ -2 & 1 \end{pmatrix}.$$

5. (12 pts) What are the six possible Jordan Canonical forms for $T \in L(V, V)$, up to rearrangement of blocks on the diagonal, if the characteristic polynomial of T is

$$h(x) = (x-1)(x-2)^2(x-3)^3 \quad ?$$

6. (12 pts) Let V an W be finite dimensional vector spaces over a field F. Let $T \in L(V, W)$. Define $T^t : W^* \to V^*$ by $T^t f = fT$ for $f \in W^*$ (i.e. $(T^t f)(v) = f(T(v))$ for $v \in V$).

   (a) Prove that T^t is a linear transformation from W^* to V^*.

   (b) Prove that if $U \subset V$ is a proper subspace, $U \neq V$, then there exists a nonzero linear function $f \in V^*$ such that $f(u) = 0$ for all $u \in U$.

   (c) Prove that T is onto if and only if T^t is one-to-one.

7. (10 pts) A matrix $A \in M_{n \times n}(F)$ is skew-symmetric if $A^t = -A$.

   (a) Prove that the set of all $n \times n$ skew-symmetric matricies is a subspace of $M_{n \times n}(F)$.

   (b) Find a basis for this subspace.

   (c) What is the dimension of this subspace?

8. (15 pts) Each row in the following figure summarizes data collected from the coefficient matrix $A \in M_{m\times n}(\mathbb{R})$ of a system of m linear equations in n unknowns

$$x_1c_1 + \cdots + x_nc_n = b.$$

The Solution Always Exists column refers to whether the system has at least one solution for all vectors $b \in \mathbb{R}_m$. The unique solution column refers to whether or not there is a unique solution whenever a solution exists. The nullity of A is the dimension of the solution space of the homogeneous system

$$x_1c_1 + \cdots + x_nc_n = 0.$$

Fill in the missing data from the table. If the data is inconsistent, write impossible.

Size of A	Solution Always Exists	Unique Solution	rank(A)	nullity(A)
3×4			3	
4×3	yes			
5×5				2
4×4		yes		
3×2			3	