1. (a) True. Note the word *distinct*.

   (b) False. Example $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is diagonal already.

   (c) False. Consider A and B in question 4.

   (d) True.

   (e) True. $A = S^{-1}BS \Rightarrow D(A) = D(S^{-1})D(B)D(S) = D(S)^{-1}D(B)D(S) = D(B)$.

   (f) True. $(y, y) = 0 \iff y = 0$.

   (g) True. They have the same dimension and are defined over the same field.

   (h) False. $V_1^{\perp}$ is a subspace of W.

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2. (a) By assumption $\alpha + i\beta \in \mathbb{C}$ is an eigenvalue for A with eigenvector $v + iw \in \mathbb{C}_n$, i.e.

$$Av + iAw = A(v + iw) = (\alpha + i\beta)(v + iw) = (\alpha v - \beta w) + i(\alpha w + \beta v).$$

   Thus, $Av = (\alpha v - \beta w)$ and $Aw = (\alpha w + \beta v)$. So:

$$A(v - iw) = Av - iAw = (\alpha v - \beta w) - i(\alpha w + \beta v) = (\alpha - i\beta)(v - iw).$$

   (b) Suppose $\alpha + i\beta \in \mathbb{C}$ is an eigenvalue for A with eigenvector $v + iw \in \mathbb{C}_n$. Then from part (a), we know that $\alpha - i\beta$ is also an eigenvalue for A with eigenvector $v - iw$.

$$(v + iw)^t A(v - iw) = (v + iw)^t(\alpha - i\beta)(v - iw) = (\alpha - i\beta)(v + iw)^t(v - iw).$$

$$\begin{aligned}(v + iw)^t A(v - iw) &= (A^t(v + iw))^t(v - iw) \\ &= (A(v + iw))^t(v - iw) \qquad \text{(because } A \text{ is symmetric)} \\ &= ((\alpha + i\beta)(v + iw))^t(v - iw) \\ &= (\alpha + i\beta)(v + iw)^t(v - iw).\end{aligned}$$

   Thus, $(\alpha - i\beta)(v + iw)^t(v - iw) = (\alpha + i\beta)(v + iw)^t(v - iw) \Rightarrow \beta = 0$.

   (c) Suppose $A = S^tDS$. Then $A^t = (S^tDS)^t = S^tD^tS^{t^t} = S^tDS = A$.

   (d) If A is symmetric, then all its eigenvalues are real so its characteristic polynomial splits. Thus it is similar to an upper triangular matrix.

   (e) The characteristic polynomial of B is $x^3 - 5x^2 + 7x - 3 = (x-3)(x-1)^2$. $E_1 := S([0, 1, 0], [-1, 0, 1])$ and $E_3 = S([1, 0, 1])$. Since the dimension of the eigenspaces equal the multiplicities of the eigenvalues, the matrix is similar to the diagonal matrix

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

3. The characteristic polynomial of T is $h(x) = x^3 - 9x^2 + 24x - 16 = (x-1)(x-4)^2$. Calculating the eigenspaces we see that $E_1 = S([0,-1,1])$ and $E_4 = S([3,1,-4])$. Since $\dim(E_4) \neq 2$, we need to calculate the generalized eigenspace for the eigenvalue 4, $K_4 := S([3,1,-4],[1,1,4])$. So we have found an ordered basis for $\mathbb{R}_3$ which makes the matrix for T with respect to this basis upper triangular, namely

$$\{[0,-1,1],[3,1,-4],[1,1,4]\}.$$

In this case the matrix for T is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & -2 \\ 0 & 0 & 4 \end{pmatrix}.$$

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4. Matricies are similar if and only if they have the same Rational Normal Forms.

$$RCF(A) = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \quad RCF(B) = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \quad RCF(C) = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, \quad RCF(D) = \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}.$$

Thus, only A and C are similar.

5.

$$\begin{pmatrix} 1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&0&0&0\\0&0&0&3&0&0\\0&0&0&0&3&0\\0&0&0&0&0&3 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\0&2&1&0&0&0\\0&0&2&0&0&0\\0&0&0&3&0&0\\0&0&0&0&3&0\\0&0&0&0&0&3 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\0&2&1&0&0&0\\0&0&2&0&0&0\\0&0&0&3&1&0\\0&0&0&0&3&0\\0&0&0&0&0&3 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0&0\\0&2&1&0&0&0\\0&0&2&0&0&0\\0&0&0&3&1&0\\0&0&0&0&3&1\\0&0&0&0&0&3 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&0&0&0\\0&0&0&3&1&0\\0&0&0&0&3&0\\0&0&0&0&0&3 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0\\0&2&0&0&0&0\\0&0&2&0&0&0\\0&0&0&3&1&0\\0&0&0&0&3&1\\0&0&0&0&0&3 \end{pmatrix}.$$

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6. (a) $T^t(\alpha f + \beta g) = (\alpha f + \beta g)(T^t) = \alpha f T^t + \beta g T^t = \alpha T^t f + \beta T^t g$.

   (b) Since U is a proper subspace of V, there exists a basis $\{v_1, \ldots, v_k\}$ for U that extends to a basis $\{v_1, \ldots, v_n\}$ for V where $k < n$. If we consider the dual basis $\{f_1, \ldots, f_n\}$ for V^* where $f_i(v_j) = 0$ if $i \neq j$ then the basis vector f_n for V^* will vanish on U but is not the zero function since $f_n(v_n) = 1$.

   (c) ($\Rightarrow$) Suppose $T^t f = 0$ for some linear function $f \in W^*$. Then $(T^t f)(v) = f(T(v)) = 0$ for all $v \in V$. Since $T(V) = W$ by assumption, this means that $f(w) = 0$ for all $w \in W$. Hence f is the zero function and T^t is one-to-one.

   ($\Leftarrow$) Suppose $T(V)$ is a proper subset of W. Then by (b), there exists a nonzero function $f \in W^*$ such that $f(w) = 0$ for all $w \in T(V)$, i.e. $f(T(v)) = 0$ for all $v \in V$. But $f(T(v)) = (T^t f)(v)$ implies that $(T^t f)(v) = 0$ for all $v \in V$. By assumption T^t is one-to-one, so f must be the zero function. This contradicts the fact from (b) that f is nonzero, so $T(V)$ cannot be a proper subset of W. Hence $T(V) = W$ and T is onto.

7. (a) Suppose A and B are skew symmetric and $\alpha \in F$. $(A+B)^t = A^t + B^t = -A + -B = -(A+B)$. $(\alpha A)^t = \alpha A^t = -\alpha A$. So the subset is closed under addition and scalar multiplication thus is a subspace.

   (b) A basis for the subspace is the set of matricies $\{A_{i,j} | 1 \leq i < j \leq n\}$ where all of the entries of $A_{i,j}$ are 0 except for the $(i,j)^{th}$ entry which is 1 and the $(j,i)^{th}$ entry which is -1.

   (c) dimension$= \frac{n(n-1)}{2}$.

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8. The last line is impossible because the rank of a 3×2 matrix can be at most 2. The second line is impossible because the rank of the matrix can be at most 3 so the columns cannot span $\mathbb{R}_4$.

Size of A	Solution Always Exists	Unique Solution	rank(A)	nullity(A)
3×4	yes	no	3	1
4×3	yes	incon	incon	incon
5×5	no	no	3	2
4×4	yes	yes	4	0
3×2	incon	incon	3	incon