Mod p points on Shimura varieties of parahoric level

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Structure of the talk

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Introduction to the Langlands-Rapoport conjecture and a quick survey of previous work

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Idea of the proof

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Understanding these integral models has interesting applications, e.g. construction of Galois representations (Deligne, Langlands), Ribet's proof of the ϵ-conjecture.

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For example $G = \mathrm{GL}_2, X = \mathbb{H}^{\pm}$ and $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ or $K_p = \Gamma_0(p)$, then $E = \mathbb{Q}$ and the integral models from the previous slide are 'good'.

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Here $X_p(\phi)$ is an affine Deligne-Lusztig variety of level K_p.

Previous Work

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Theorem (Kisin, 2008 and 2013)

Let (G, X) *be a Shimura datum of abelian type, let* $p > 2$ *and suppose that* $G_{\mathbb{Q}_p}$ *is unramified and that* K_p *is hyperspecial. Then the Langlands-Rapoport conjecture holds for* (G, X, p)*.*

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Theorem (Zhou, 2017)

Let (G, X) be a Shimura datum of Hodge type, let $p > 2$ and suppose that $G_{\mathbb{Q}_p}$ is residually split, then isogeny classes have Rapoport-Zink uniformisation for arbitrary parahorics K_p.

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Theorem 1 (-)

Suppose that G has no factors of type A and that $\mathbf{Sh}_K(G, X)$ is proper. Then the Langlands-Rapoport conjecture holds for the Kisin-Pappas integral models of $\mathbf{Sh}_K(G, X)$.

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Remarks

The assumption that $G_{\mathbb{Q}_p}$ is unramified can be removed for most (G, X).

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For Hodge type Shimura varieties, the integral models do not have a moduli interpretation, which makes it difficult to make the above strategy work. We can still associate a p-divisible group with extra structures X to an $\overline{\mathbb{F}}_p$-point, but it is no longer clear that the fiber only depends on this X.

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The LR conjecture holds for the Shimura variety in the top left corner if and only if the diagram is Cartesian.

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This last result is new even for $S_{K',\overline{\mathbb{F}}_p}(G,X)$!

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