An approach to characterizing the local Langlands conjecture over p-adic fields

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- ${}^LG$ will denote the Weil form of the L-group (i.e. ${}^LG = \widehat{G} \rtimes W_F$).

The local Langlands conjecture

A Langlands correspondence

A *Langlands correspondence* for G is a finite-to-one association

$$\mathrm{LL} : \left\{ \begin{matrix} \text{Admissible representations} \\ \text{of } G(F) \end{matrix} \right\} \to \left\{ \begin{matrix} L\text{-parameters} \\ \text{of } G \end{matrix} \right\}$$

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L-packets

An *L-packet* for LL is a set (necessarily finite) of the form $\Pi_G(\psi) := \mathrm{LL}^{-1}(\psi)$ for some L-parameter ψ of G.

The local Langlands conjecture (cont.)

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- $G = \mathrm{GSp}_4$ (Gan–Takeda)

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The GL_n case

In the case of $G = \mathrm{GL}_n$ one can take the standard desiderata that LL is compatible with tensor products, local class field theory, L-functions, and ϵ-factors. These properties do uniquely characterize the correspondence.

Scholze's characterization of LLC for GL_n

Theorem (Scholze, 2013)

For every $\tau \in W_F^+$ and $h \in C_c^\infty(\mathrm{GL}_n(\mathcal{O}), \mathbb{Q})$ there exists a function $f_{\tau,h} \in C_c^\infty(\mathrm{GL}_n(F), \mathbb{Q})$ such that for any admissible representation π of $\mathrm{GL}_n(F)$ the equality

$$\mathrm{tr}(f_{\tau,h} \mid \pi) = \mathrm{tr}(\tau \mid \mathrm{LL}(\pi)(\chi))\,\mathrm{tr}(h \mid \pi)$$

holds, and this uniquely characterizes LL.

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- They show up as terms in the trace formula existing within the framework of the Langlands–Kottwitz(–Scholze) method.
- They have generalized versions beyond GL_n for more general PEL type situations (Scholze) and in most abelian type situations (forthcoming work of the author).

The Scholze–Shin conjecture

Conjecture (Scholze-Shin)

Let G be an unramified group over $\mathbb{Q}_p$ with $\mathbb{Z}_p$-model $\mathcal{G}$ and let μ be a dominant cocharacter of $G_{\overline{\mathbb{Q}_p}}$ with reflex field E. Let $\tau \in W_{\mathbb{Q}_p}^+$ and let $h \in C_c^\infty(\mathcal{G}(\mathbb{Z}_p), \mathbb{Q})$. Then, for every supercuspidal L-parameter ψ

$$S\Theta_\psi(f_{\tau,h}) = \mathrm{tr}\left(\tau \mid (r_{-\mu} \circ \psi)\,|_{W_E}\, |\cdot|_E^{-\langle \rho, \mu \rangle}\right) S\Theta_\psi(h).$$

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Notation

- $S\Theta_\psi(f) := \sum_{\pi \in \Pi_G(\psi)} r_\pi \, \mathrm{tr}(f \mid \pi)$—the *stable character for* ψ.
- $r_{-\mu}$ is the representation of ${}^L G$ whose restriction to $\widehat{G}$ has highest weight $\mu^\vee$.

The Scholze–Shin conjecture (cont.)

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Second natural question

Does the Scholze–Shin equations uniquely characterize LL for groups other than GL_n?

The Scholze–Shin conjecture (cont.)

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- Some PEL cases (Scholze and Scholze–Shin).
- It's known to hold (appropriately interpreted) in some cases of the form $G = D^\times$ (Shen).
- It's known to hold in the case of unitary groups (Bertoloni Meli–Y.)

Setup for main result

Supercuspidal L-parameters

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Slogan for supercuspidal L-parameters

Supercuspidal L-parameters should be those whose L-packet consists entirely of supercuspidals.

Setup for main result (cont.)

Elliptic hyperendoscopic group

An *extended elliptic hyperendoscopic datum* is a sequence of tuples of data $(H_1, s_1, {}^L\eta_1), \ldots, (H_k, s_k, {}^L\eta_k)$ such that $(H_1, s_1, {}^L\eta_1)$ is an extended elliptic endoscopic datum of G, and for $i > 1$, the tuple $(H_i, s_i, {}^L\eta_i)$ is an extended elliptic endoscopic datum of H_{i-1}.

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They are a set of groups for which an L-parameter could factorize through and which is sufficiently large to study the packet structure of a parameter.

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Example for hyperendoscopic groups

If $G = U(n)$ then the elliptic hyperendoscopic groups for G are those of the form $U(a_1) \times \cdots \times U(a_m)$ with $a_1 + \cdots + a_m = n$.

Setup for main result (cont.)

Supercuspidal local Langlands correspondence

A *supercuspidal local Langlands correspondence* for a group G (assumed quasi-split for simplicity) is an association

$$\Pi_H : \left\{ \begin{array}{c} \text{Equivalence classes of} \\ \text{Supercuspidal } L\text{-parameters} \\ \text{for } H \end{array} \right\} \to \left\{ \begin{array}{c} \text{Finite subsets} \\ \text{of } \mathrm{Irr}^{\mathrm{sc}}(H(F)) \end{array} \right\}$$

for every elliptic hyperendoscopic group H of G satisfying the following conditions:

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- **(ECI)** The endoscopic character identities hold.

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Scholze–Shin equations

We say that Π_H satisfies the *Scholze–Shin equations* relative to $\{\varphi^{\mu}_{\tau,h}\}$ if the following equation holds for all ψ:

$$S\Theta_\psi(\varphi_{\tau,h}) = \mathrm{tr}\left(\tau \mid (r_{-\mu} \circ \psi) \mid_{W_E} \mid \cdot \mid_E^{-\langle \rho,\mu\rangle}\right) S\Theta_\psi(h).$$

The main result

Theorem (Bertoloni Meli–Y.)

Suppose that G is a 'good' group and that Π^1 and Π^2 are two supercuspidal Langlands correspondences for G which satisfy the Scholze–Shin equations for the same Scholze–Shin datum $\{\varphi^{\mu}_{\tau,h}\}$. Then, $\Pi^1 = \Pi^2$ and the bijections in **(Bij)** *are the same.*

The notion of 'good'

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Examples/Non-examples

Some examples of ‘good groups’:

Examples	Non-examples
GL_n	SL_n
U_n	SO_{2n}
SO_{2n+1}	Sp_{2n}
PGL_n	E_8
G_2	

Broad idea of proof

- If $\Pi^1_H(\psi) = \{\pi\} = \Pi^2_H(\psi')$ then taking h such that $\Theta_\pi(h) \neq 0$ we see that

$$\mathrm{tr}(\tau \mid r_{-\mu} \circ \psi) = \frac{\Theta_\psi(\varphi^\mu_{\tau,h})}{\Theta_\pi(h)} = \mathrm{tr}(\tau \mid r_{-\mu} \circ \psi')$$

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implies that $\psi \sim \psi'$.
- The desiderata for a supercuspidal Langlands correspondence non-trivially satisfy *atomic stability*—the fact that if S is a set of representations for which some linear combination is stable, then S is a union of L-packets—this in turn implies that if $\Pi^1_H(\psi) = \{\pi\}$ then $\{\pi\} = \Pi^2_H(\psi')$ for some ψ'.

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- If $\Pi^1_H(\psi) = \{\pi\}$ then $\Pi^1_H(\psi) = \Pi^2_H(\psi)$.
- Every ψ can be written as $\eta \circ \psi^{H'}$ for $\psi^{H'}$ a parameter of some hyperendoscopic group H' of H (and thus of G) such that $\Pi^1_{H'}(\psi^{H'})$ is a singleton.

An application

Theorem (Bertoloni Meli–Y.)

Let $E/\mathbb{Q}_p$ be an unramified extension and F the quadratic subextension of E. Let G be the quasi-split unitary group $U_{E/F}(n)^*$ associated to E/F. Then, the local Langlands correspondence for G (as in the work of Mok) satisfies the Scholze–Shin conjecture and is uniquely characterized by this condition.

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- Opens up question of whether there is a useful version of our result in the function field setting.

Thanks for listening!