Serre's conjecture and two notions of minimal weight

Hanneke Wiersema

May 31, 2020

King's College London

Galois representations

We write $G_{\mathbb{Q}}$ for the absolute Galois group of $\mathbb{Q}$. One way to study this group is to study its representations. We will look at mod p Galois representations, e.g:

Example

The mod p cyclotomic character $\omega : G_{\mathbb{Q}} \to \overline{\mathbb{F}}_p^{\times}$ defined by

$$\sigma(\zeta_p) = \zeta_p^{\omega(\sigma)}$$

is a 1-dimensional mod p Galois representation.

For this talk we will look at

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p).$$

Serre's conjecture [Khare-Wintenberger]

Let $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be irreducible, continuous and odd.

Theorem (Serre's conjecture)
There exists a modular form f such that $\rho_f \cong \rho$. Moreover, one can explicitly describe the minimal weight $k(\rho)$ and level $N(\rho)$ that one can take for f.

This $k(\rho)$ is completely determined by ρ_p, the restriction of ρ to $G_{\mathbb{Q}_p}$.

Example
Suppose ρ_p is reducible and

$$\rho|_{I_p} \sim \begin{pmatrix} \omega^a & 0 \\ 0 & 1 \end{pmatrix}$$

with $2 \leq a \leq p-3$ and ω the mod p cyclotomic character, then $k(\rho) = a + 1$.

Reformulation to algebraic modularity

We write $\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p^2$ for the $(k-2)$-th symmetric power of the standard representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on $\overline{\mathbb{F}}_p^2$

Proposition (Ash-Stevens)
Let $k \geq 2$. Then ρ is modular of level N and weight k if and only if ρ appears in $H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p{}^2)$. Moreover, ρ is modular of level N and weight k if and only if ρ appears in $H^1(\Gamma_1(N), V)$, with V a Jordan–Hölder constituent of $\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p{}^2$.

We can explicitly describe such constituents!

Example
The representation $\mathrm{Sym}^p\,\overline{\mathbb{F}}_p{}^2$ has two constituents: $\mathrm{Sym}^1\,\overline{\mathbb{F}}_p{}^2$ and $\det \otimes \mathrm{Sym}^{p-2}\,\overline{\mathbb{F}}_p{}^2$.

Serre weights

Definition (Serre weights)
The V are irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$,

$$V_{t,s} = \det^s \otimes \mathrm{Sym}^{t-1} \overline{\mathbb{F}}_p^2, \quad 0 \leq s < p-1, 1 \leq t \leq p$$

We call these *Serre weights*.

Buzzard, Diamond and Jarvis define a set of Serre weights $W(\rho)$:

Theorem (The BDJ conjecture in the classical case)
If $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ *is modular, then*

$$W(\rho) = \{V | \rho \text{ is modular of weight } V\}.$$

The weight set $W(\rho)$ - examples

Remark
The recipe for $W(\rho)$ depends purely on the local representation ρ_p.

Example
Let

$$\rho|_{I_p} \sim \begin{pmatrix} \omega^a & 0 \\ 0 & 1 \end{pmatrix}$$

as before, then

$$W(\rho) = \{V_{a,0}, V_{p-1-a,a}\}.$$

Question: How to compare this with $k(\rho)$?

The minimal algebraic weight

Definition (Minimal weight)

$$k_{\min}(V_{t,s}) = \min_k \{k \geq 2 \mid V_{t,s} \in \mathrm{JH}(\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p^2)\}$$

e.g. for $V_{t,0} = \mathrm{Sym}^{t-1}\,\overline{\mathbb{F}}_p^{\,2}$, we see $k_{\min}(V_{t,0}) = t+1$.

Proposition

$$k_{\min}(V_{t,s}) = \begin{cases} s(p+1)+t+1 & s+t<p \\ (s+1)(p+1)+tp-p^2 & s+t \geq p \end{cases}$$

Definition: We set $k_{\min}(W(\rho)) = \min_k\{k_{\min}(V_{t,s}) \mid V_{t,s} \in W(\rho)\}$.

Minimal algebraic weight

Example
Again let

$$\rho|_{I_p} \sim \begin{pmatrix} \omega^a & 0 \\ 0 & 1 \end{pmatrix}$$

as before, recall $W(\rho) = \{V_{a,0}, V_{p-1-a,a}\}$.

We find $k_{\min}(V_{a,0}) = a + 1$ and $k_{\min}(V_{p-1-a,a}) = ap + p$, so

$$k_{\min}(W(\rho)) = a + 1 = k(\rho)$$

Theorem (Equality of two weight invariants I)

$$k(\rho) = k_{\min}(W(\rho))$$

Generalisation to the totally real field case

Generalisations

- $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$
- $\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, K totally real number field

- weight of a mod p modular form
- weight of a mod p Hilbert modular form

These are the objects for *geometric modularity*, we again study algebraic modularity in which the weights are representations of $\mathrm{GL}_2(k_{\mathfrak{p}})$.

Serre weights in the totally real field case

Let p be an odd prime and let

$$\rho : G_K \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be a continuous, totally odd, irreducible representation with K a totally real field in which p is unramified and for simplicity assume p is inert. Let $k_{\mathfrak{p}}$ be the residue field at $\mathfrak{p}$, $\mathfrak{p}$ lying above p. Let $\Sigma = \{\tau : k_{\mathfrak{p}} \to \overline{\mathbb{F}}_p\}$.

Definition (More general Serre weights)

$$V_{\vec{b},\vec{a}} = \bigotimes_{\tau \in \Sigma} \left(\det{}^{a_\tau} \otimes_{k_{\mathfrak{p}}} \mathrm{Sym}^{b_\tau - 1} k_{\mathfrak{p}}^2\right) \otimes_{k_{\mathfrak{p}},\tau} \overline{\mathbb{F}}_p.$$

with $\Sigma = \{\tau : k_{\mathfrak{p}} \to \overline{\mathbb{F}}_p\}$ *and* $b_\tau \leq p$ *for all* $\tau \in \Sigma$*.*

More general BDJ conjecture [Gee et al.]

Theorem (The BDJ conjecture)
If $\rho : G_K \to GL_2(\overline{\mathbb{F}}_p)$ is modular, then

$$W(\rho) = \{V | \rho \text{ is modular of weight } V\}.$$

Question
How do we define analogues of $k_{\min}(V_{t,s})$ and $k_{\min}(W(\rho))$?

We first introduce a partial ordering.

Remark
For simplicity we will restrict ourselves to the quadratic case - but all definitions to come have higher degree analogues.

Hasse cone and the partial ordering

We set

$$\Xi_{\mathrm{Ha}}^{\mathbb{Z}} = \left\{ x \begin{pmatrix} -1 \\ p \end{pmatrix} + y \begin{pmatrix} p \\ -1 \end{pmatrix} \in \mathbb{Z}^2 \mid x, y \geq 0 \right\},$$

where the vectors are weights of partial Hasse invariants.

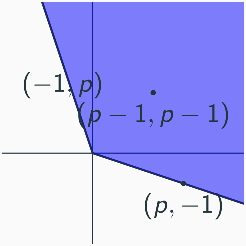

Figure 1: The Hasse cone for K quadratic

Definition: We say $\vec{k} \leq_{\mathrm{Ha}} \vec{k}'$ when $\vec{k}' - \vec{k} \in \Xi_{\mathrm{Ha}}^{\mathbb{Z}}$.

Minimal weight cone

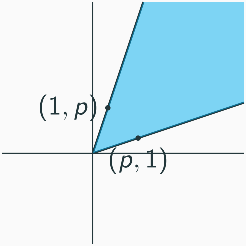

Figure 2: The minimal cone in the quadratic case

We have

$$\Xi^{\mathbb{Q}}_{\text{min}} = \left\{ (x, y) \in \mathbb{Q}^2 \mid px \geq y, py \geq x \right\}.$$

Minimal algebraic weight

We define

$$k_{\mathrm{min}}(V_{\vec{b},\vec{a}}) = \min_{\geq \mathrm{Ha}} \left\{ \vec{k} \in \mathbb{Z}^2_{\geq 2} \cap \Xi^{\mathbb{Q}}_{\mathrm{min}} \mid V_{\vec{b},\vec{a}} \in \mathrm{JH}\left(\bigotimes_{\tau \in \Sigma} \mathrm{Sym}^{\vec{k}_\tau - 2}\, k_{\mathfrak{p}}^2 \otimes_\tau \overline{\mathbb{F}}_p \right) \right\},$$

e.g. for $V_{(b_0,b_1),(0,0)} = \mathrm{Sym}^{b_0-1}\, k_{\mathfrak{p}}^2 \otimes \mathrm{Sym}^{b_1-1}\, k_{\mathfrak{p}}^2$, we see $k_{\mathrm{min}}(V_{(b_0,b_1),(0,0)}) = (b_0+1, b_1+1)$.

Conjecture

Let p be odd and $k_{\mathrm{min}}(V_{\vec{b},\vec{a}})$ be as above, then we have

$$k_{\mathrm{min}}(V_{\vec{b},\vec{a}}) = \sum_{\tau \in \Sigma} (a_\tau(e_\tau + p e_{\mathrm{Fr}^{-1}\circ\tau}) + (b_\tau+1)e_\tau) - \sum_{\substack{\tau \in \Sigma \mid \\ a_\tau + b_\tau \geq p}} (p - b_\tau)(p e_{\mathrm{Fr}^{-1}\circ\tau} - e_\tau).$$

We analogously define

$$k_{\mathrm{min}}(W(\rho)) = \min_{\geq \mathrm{Ha}} \{ k_{\mathrm{min}}(V_{\vec{b},\vec{a}}) \mid V_{\vec{b},\vec{a}} \in W(\rho) \}$$

Minimal algebraic weight vs minimal geometric weight

Question: Is it possible to define $k(\rho)$ such that also in the general case we find: $k(\rho) = k_{\text{min}}(W(\rho))$?

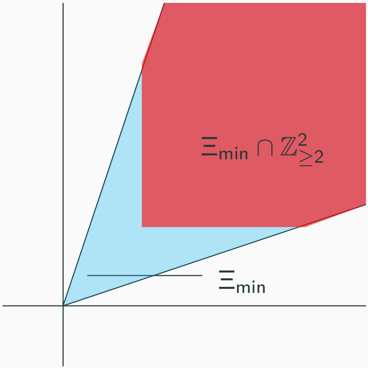

Figure 3: Different weight spaces in the quadratic case

Questions?