Vanishing theorems for Shimura varieties

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1 Brief overview of Shimura varieties
1 Brief overview of Shimura varieties
2 Conjectures about vanishing of cohomology
1. Brief overview of Shimura varieties
2. Conjectures about vanishing of cohomology
3. Perfectoid geometry of Shimura varieties
1. Brief overview of Shimura varieties
2. Conjectures about vanishing of cohomology
3. Perfectoid geometry of Shimura varieties
4. Results for completed cohomology
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2. Conjectures about vanishing of cohomology
3. Perfectoid geometry of Shimura varieties
4. Results for completed cohomology
5. Results at finite level
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For \(K \subset G(\mathbb{A}_f)\) a neat compact open subgroup, let \(X_K/E\) be the corresponding Shimura variety, which satisfies

\[
X_K(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K
\]
- $G = \text{GL}_2$, 

$X_K = \mathbb{H}^\pm$, the $X_K/\mathbb{Q}$ are modular curves:

$$X_K(\mathbb{C}) = \text{GL}_2(\mathbb{Q})/\mathbb{H}^\pm \times \text{GL}_2(\mathbb{A}_f)/K = \bigtimes_{i=1}^n \Gamma_i \mathbb{H},$$

where each $\Gamma_i \subseteq \text{SL}_2(\mathbb{Z})$ is a congruence subgroup.
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• Shimura varieties of Hodge type: admit a closed embedding

  \[(G, X) \hookrightarrow (\tilde{G}, \tilde{X}),\]

  where $(\tilde{G}, \tilde{X})$ is a Siegel datum. Example: $G$ unitary similitude group.
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• Beyond abelian type. Examples: those associated to Dynkin diagrams of types $E_6$, $E_7$.
The spherical Hecke algebra $\mathbb{T}$ acts on $H^*_c(X_K(\mathbb{C}), \mathbb{C})$. 
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• The systems of Hecke eigenvalues that occur can be described in terms of *automorphic representations* of $G$. 
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If $\mathfrak{m} \subset \mathbb{T}$ is a non-Eisenstein maximal ideal, we expect

$$H^i_c(X_K(\mathbb{C}), \mathbb{C})_{\mathfrak{m}} \neq 0 \text{ only if } i = d := \dim_E X_K.$$
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  This is predicted by Arthur’s conjectures, and essentially follows from work of Borel–Wallach, Franke.
If \( m \subset \mathbb{T} \) is a non-Eisenstein maximal ideal, we expect

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This is conjectured by Calegari–Geraghty, Emerton. Why?
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What is it good for?

- Taylor–Wiles patching.
- For modular curves, used to establish the compatibility with $p$-adic local Langlands ($\ell = p$).
- Useful in studying Bloch–Kato conjecture.
Let $N_0 \subset G(\mathbb{Q}_p)$ be a compact subgroup. Define

$$\tilde{H}^i_{(c)}(K^p N_0, \mathbb{F}_p) := \lim_{\longrightarrow} H^i_{(c)}(X_{K^p K_p}(\mathbb{C}), \mathbb{F}_p).$$
Let $\mathcal{N}_0 \subset G(\mathbb{Q}_p)$ be a compact subgroup. Define

$$\tilde{H}^i_{(c)}(K^p \mathcal{N}_0, \mathbb{F}_p) := \lim_{\mathcal{N}_0 \subseteq K_p} H^i_{(c)}(X_{K^p K_p}(\mathbb{C}), \mathbb{F}_p).$$

- For $\mathcal{N}_0 = \{1\}$, we obtain Emerton’s completed cohomology.
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- For $G_{\mathbb{Q}_p}$ split, $N$ unipotent radical of $B \subset G_{\mathbb{Q}_p}$, $N_0 = N(\mathbb{Z}_p)$, this leads to Hida’s ordinary completed cohomology.
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We expect vanishing of $\tilde{H}_c^i$ whenever $i > d$. 
Let $N_0 \subset G(\mathbb{Q}_p)$ be a compact subgroup. Define

$$\tilde{H}^i_{(c)}(K^p N_0, \mathbb{F}_p) := \lim_{\text{as } N_0 \subseteq K_p} H^i_{(c)}(X_{K^p K_p}(\mathbb{C}), \mathbb{F}_p).$$

- For $N_0 = \{1\}$, we obtain Emerton’s completed cohomology.
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We expect vanishing of $\tilde{H}^i_c$ whenever $i > d$.

For $N_0 = \{1\}$, we also expect vanishing of $\tilde{H}^i$ whenever $i > d$. Calegari–Emerton conjecture, motivated by heuristics from $p$-adic Langlands programme.
Choose a rational prime $p$, and a prime $p | p$ of $E$. Set

$$X_K^* := (X_K^* \times_E E_p)^{\text{ad}}.$$
Choose a rational prime $p$, and a prime $p \mid p$ of $E$. Set

$$\mathcal{X}_K^* := (\mathcal{X}_K^* \times_E E_p)^{\text{ad}}.$$ 

Let $\mu$ be the Hodge cocharacter of the Shimura datum. This determines a parabolic subgroup $P_\mu \subset G_E$. Set

$$\mathcal{F}_\ell := ((G_E/P_\mu) \times_E E_p)^{\text{ad}}.$$
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**Theorem 1 (Scholze, C-Scholze)**

There exists a perfectoid space $\mathcal{X}_{K_p}^* = \varprojlim_{K_p} \mathcal{X}_{K_p K_p}^*$ and a morphism of adic spaces

$$\pi_{\text{HT}} : \mathcal{X}_{K_p}^* \to \mathcal{F}_\ell$$
\( \pi_{HT} \) measures the relative position of the Hodge–Tate filtration. For the modular curve, we have:

\[
\chi \in \mathcal{X}_{Kp}(C, \mathcal{O}_C) \leftrightarrow (E, \alpha : T_pE \cong \mathbb{Z}_p^2) \mapsto \text{Lie}E(1) \subset T_pE \otimes_{\mathbb{Z}_p} C \overset{\alpha}{\cong} C^2.
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\end{align*}
\]

\( \pi_{\text{HT}} : \mathcal{X}^*_{K^p} \rightarrow \mathcal{F} \ell \) is \( \mathbb{T} \)- and \( G(\mathbb{Q}_p) \)-equivariant.
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\( \pi_{\text{HT}} : \mathcal{X}_{K^p}^* \rightarrow \mathcal{F} \ell \) is \( \mathbb{T} \)- and \( G(\mathbb{Q}_p) \)-equivariant.

\( \pi_{\text{HT}} \) is “affinoid”: there exists an open cover of \( \mathcal{F} \ell \) by affinoid \( U_i \) such that each \( \pi_{\text{HT}}^{-1}(U_i) \) is affinoid perfectoid.
If $(G, X)$ is a Shimura datum of abelian type, then

$$\tilde{H}^i_{(c)}(K^p, \mathbb{F}_p) = 0 \text{ whenever } i > d.$$ 

This is a theorem of Scholze and Hansen–Johannson. The case of $\tilde{H}^i_c(K^p, \mathbb{F}_p)$ is based on purely geometric techniques.
If $(G, X)$ is a Shimura datum of abelian type, then

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**Theorem 2 (C–Gulotta–Johansson + Hsu–Mocz–Reinecke–Shih)**

Assume that $(G, X)$ is a Shimura datum of Hodge type, and $G_{\mathbb{Q}_p}$ is split. Let $N_0 := N(\mathbb{Z}_p)$. Then

$$\tilde{H}^i_c(K^p N_0, \mathbb{F}_p) = 0 \text{ whenever } i > d.$$
- **$p$-adic Hodge theory:**

\[
R\Gamma_{et,c} \left( \mathcal{X}_{K^p N_0}, \mathbb{F}_p \right) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \cong R\Gamma_{et} \left( \mathcal{X}^{*}_{K^p N_0}, \mathcal{I}^+/p \right),
\]

where $\mathcal{I}^+ \subseteq \mathcal{O}^+$ is the ideal of sections that vanish at the boundary.
- \( p \)-adic Hodge theory:

\[
R\Gamma_{\text{et},c} \left( \mathcal{X}_{K \mathcal{P} N_0}, \overline{\mathbb{F}}_p \right) \otimes_{\mathbb{F}_p} \mathcal{O}_C / p \cong R\Gamma_{\text{et}} \left( \mathcal{X}_{K \mathcal{P} N_0}^*, \mathcal{I}^+ / p \right),
\]

where \( \mathcal{I}^+ \subseteq \mathcal{O}^+ \) is the ideal of sections that vanish at the boundary.

- The Bruhat stratification into \( B(\mathbb{Q}_p) \)-orbits

\[
\mathcal{F} \ell = \bigsqcup_{w \in W / W_{P_\mu}} \mathcal{F} \ell^w,
\]

which descends to \( \mathcal{F} \ell / N_0 \).
- $p$-adic Hodge theory:

$$R\Gamma_{et,c}(\mathcal{X}_{K^p N_0}, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathcal{O}_C/p \cong R\Gamma_{et}(\mathcal{X}_{K^p N_0}^*, \mathcal{I}^+/p),$$

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- The Bruhat stratification into $B(\mathbb{Q}_p)$-orbits

$$\mathcal{F}\ell = \bigsqcup_{w \in W/W_{P\mu}} \mathcal{F}\ell^w,$$

which descends to $\mathcal{F}\ell/N_0$.

- By quantifying when different subsets of $(\mathcal{X}_K^*)_K$ become perfectoid, we show that the cohomological amplitude of $R\pi_{HT}/N_0, \star \mathcal{I}^{+a}/p$ restricted to $\mathcal{F}\ell^w/N_0$ lies in $[0, d - \dim \mathcal{F}\ell^w]$. 
For the modular curve, the geometry of the reduction mod $p$:

\[ \overline{X}^* \cong \overline{X}_{\Gamma_0(p)} \]

\[ \text{Frob}_p \]

\[ \overline{X}^*,\text{ord} = \overline{X}^*_{\Gamma_0(p)} \sqcup \overline{X}^*_{\Gamma_0(p)} \]
For the modular curve, the geometry of the reduction mod $p$:

$\overline{X}^\ast \cong \overline{X}_{\Gamma_0(p)}$  

$\xrightarrow{\text{Frob}_p} \overline{X}$  

$\overline{X}_{\Gamma_0(p)}^\ast, \text{ord} = \overline{X}_{\Gamma_0(p)}^\ast, \text{anti} \sqcup \overline{X}_{\Gamma_0(p)}^\ast, \text{can}$

matches the Bruhat stratification:

$\mathbb{P}^{1, \text{ad}} = \mathbb{A}^{1, \text{ad}} \sqcup \{ \infty \}$.  

In particular, $X_{K_{\rho N_0}}^\ast, \text{anti}$ is perfectoid (Ludwig).
Let $F = F^+ \cdot E$ be a CM field, with $F^+ \neq \mathbb{Q}$. 
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Let $\mathfrak{m} \subset \mathbb{T}$ be in the support of $H^i_{(c)}(X_K, \overline{F}_\ell)$. Assume that $\overline{\rho}_\mathfrak{m}$ is \textit{generic}: auxiliary condition at $p \neq \ell$, guarantees all lifts to characteristic 0 are \textit{principal series} at $p$. 

\begin{itemize}
  \item \textbf{Theorem 3 (C-Scholze)}
    \begin{enumerate}
      \item If $X_K$ is compact, then $H^i_{(c)}(X_K, \overline{F}_\ell) \mathfrak{m} = 0$ unless $i = d$.
      \item If $G$ is quasi-split and length $(\rho_\mathfrak{m}) \leq 2$, then $H^i_{(c)}(X_K, \overline{F}_\ell) \mathfrak{m} = 0$ unless $i \leq d$, $H^i(X_K, \overline{F}_\ell) \mathfrak{m} = 0$ unless $i \geq d$.
    \end{enumerate}
\end{itemize}
Let $F = F^+ \cdot E$ be a CM field, with $F^+ \neq \mathbb{Q}$.

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**Theorem 3 (C-Scholze)**

1. If $X_K$ is compact, then $H^i_{(c)}(X_K, \overline{F}_\ell)_m = 0$ unless $i = d$.

2. If $G$ is quasi-split and $\text{length}(\overline{\rho}_m) \leq 2$, then

   $H^i_c(X_K, \overline{F}_\ell)_m = 0$ unless $i \leq d$,

   $H^i(X_K, \overline{F}_\ell)_m = 0$ unless $i \geq d$. 
The Newton stratification

$$\mathcal{F}_\ell = \bigsqcup_{b \in B(G, \mu)} \mathcal{F}_\ell^b$$

and the identification of the fibers over $\mathcal{F}_\ell^b$ with perfectoid Igusa varieties $Ig^b$ (Mantovan product formula).
The Newton stratification

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The complexes \((R\pi_{HT*}\mathcal{F}_\ell)_m\) behave like perverse sheaves.
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- The complexes \( (R\pi_{HT*} \mathbb{F}_\ell)_m \) behave like perverse sheaves.
- Computation of \( R\Gamma(\text{Ig}^b, \mathbb{Q}_\ell)_m \) using trace formula (Shin).
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- The complexes $(R^\pi_{HT*} \mathbb{F}_\ell)_m$ behave like perverse sheaves.
- Computation of $R\Gamma(Ig^b, \mathbb{Q}_\ell)_m$ using trace formula (Shin).

**Remark**

Boyer proves a stronger result for Shimura varieties of *Harris–Taylor type*, going beyond the generic case. There is also forthcoming work of Koshikawa.
For the modular curve, we have $B(G, \mu) = \{\text{ord}, \text{ss}\}$:

$$
\begin{align*}
\mathcal{X}^*_{Kp} & = \mathcal{X}^*_{Kp} \sqcup \mathcal{X}^\text{ss}_{Kp} \\
\mathbb{P}1,\text{ad} & = \mathbb{P}1,\text{ad}(\mathbb{Q}_p) \sqcup \Omega
\end{align*}
$$

- Let $D/\mathbb{Q}$ be the quaternion algebra ramified at $p, \infty$. The fibers $\mathcal{I}_\text{ss}$ of $\pi_{\text{HT}}$ over $\Omega$ can be identified with Shimura sets for $D^\times$ (Howe).
For the modular curve, we have $B(G, \mu) = \{\text{ord}, \text{ss}\}$:

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\begin{align*}
\mathcal{X}_{K^p}^* & = & \mathcal{X}_{K^p}^{*,\text{ord}} & \sqsupset & & \mathcal{X}_{K^p}^{\text{ss}} \\
\mathbb{P}^1, \ad & \downarrow & \mathbb{P}^1, \ad(Q_p) & \sqsupset & & \Omega
\end{align*}
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- Let $D/\mathbb{Q}$ be the quaternion algebra ramified at $p, \infty$. The fibers $\text{Ig}^{\text{ss}}$ of $\pi_{HT}$ over $\Omega$ can be identified with Shimura sets for $D^\times$ (Howe).
- If $\pi$ is an automorphic representation of $\text{GL}_2(\mathbb{A})$ such that $\pi_f^p$ contributes to $R\Gamma(\text{Ig}^{\text{ss}}, \mathbb{Q}_\ell)$, then $\pi_p$ cannot be a principal series representation.
Let $G = \text{Res}_{F/Q} \text{GL}_2$, $F$ totally real, so that the $X_K$ are Hilbert modular varieties.
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### Theorem 4 (C-Tamiozzo, in progress)

Let \( \ell \geq 3 \) and \( m \subset T \) be in the support of \( H^*_c(X_K, \mathbb{F}_\ell) \) such that \( \text{Im}(\bar{\rho}_m) \supset \text{SL}_2(\mathbb{F}_\ell) \). Then

\[
H^i_c(X_K, \mathbb{F}_\ell)_m = 0 \quad \text{unless} \quad i = d.
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- This strengthens results of Dimitrov in the Fontaine–Laffaille case.
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unless $i = d$.

- This strengthens results of Dimitrov in the Fontaine–Laffaille case.

- Key idea: work with an auxilliary prime $p$ that splits completely in $F$. Replace the direct computation of Igusa cohomology with the geometric Jacquet–Langlands functoriality established by Tian–Xiao.