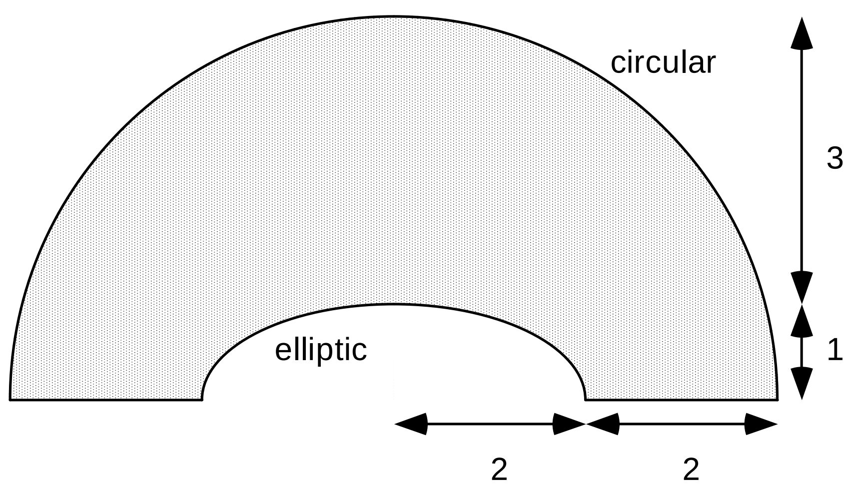
Homework #1 : 1999 Winter

MEAM 502 Differential Equation Methods in Mechanics

Solve the following boundary value problem

$$-\Delta u = 2 \quad \text{in} \quad \Omega$$
$$u = 0 \quad \text{on} \quad \Gamma$$

by the finite difference method by applying a coordinate transformation method, where Ω is a domain in the two-dimensional space, shown in the figure, and Γ is its boundary. To do this, you may use MATLAB or MATHEMATICA.



Hint *Examine the coordinate transformation*

$$a = (1-\eta) + 2(1+\eta)$$
$$b = \frac{1}{2}(1-\eta) + 2(1+\eta)$$
$$\theta = \frac{\pi}{2}(1-\xi)$$

from the square domain $-1 \le \xi, \eta \le +1$*, and also*

$$x = a\cos\theta = (3+\eta)\cos\left(\frac{\pi}{2}(1-\xi)\right)$$
$$y = b\sin\theta = \frac{1}{2}(5+3\eta)\sin\left(\frac{\pi}{2}(1-\xi)\right).$$

Answer

Coordinate transformation

$$x = (3+\xi)\cos\left(\frac{\pi}{2}(1-\eta)\right)$$
$$y = \frac{1}{2}(5+3\xi)\sin\left(\frac{\pi}{2}(1-\eta)\right)$$

Noting that

$$\begin{Bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \xi} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{Bmatrix}$$

we have

$$\frac{\partial \xi}{\partial x} = J_{11}^{-1}\frac{\partial \xi}{\partial \xi} + J_{12}^{-1}\frac{\partial \xi}{\partial \eta} = J_{11}^{-1} \quad , \quad \frac{\partial \eta}{\partial x} = J_{11}^{-1}\frac{\partial \eta}{\partial \xi} + J_{12}^{-1}\frac{\partial \eta}{\partial \eta} = J_{12}^{-1}$$
$$\frac{\partial \xi}{\partial y} = J_{21}^{-1}\frac{\partial \xi}{\partial \xi} + J_{22}^{-1}\frac{\partial \xi}{\partial \eta} = J_{21}^{-1} \quad , \quad \frac{\partial \eta}{\partial y} = J_{21}^{-1}\frac{\partial \eta}{\partial \xi} + J_{22}^{-1}\frac{\partial \eta}{\partial \eta} = J_{22}^{-1}$$

Thus, the original Poisson equation will be transformed into

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = -\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= -\left(\frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \xi}{\partial x}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta}\right) - \left(\frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}\right)\left(\frac{\partial \xi}{\partial y}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial u}{\partial \eta}\right)$$

$$= -\left(J_{11}^{-1}\frac{\partial}{\partial \xi} + J_{12}^{-1}\frac{\partial}{\partial \eta}\right)\left(J_{11}^{-1}\frac{\partial u}{\partial \xi} + J_{12}^{-1}\frac{\partial u}{\partial \eta}\right) - \left(J_{21}^{-1}\frac{\partial}{\partial \xi} + J_{22}^{-1}\frac{\partial}{\partial \eta}\right)\left(J_{21}^{-1}\frac{\partial u}{\partial \xi} + J_{22}^{-1}\frac{\partial u}{\partial \eta}\right)$$

$$= -\left(J_{11}^{-1}\frac{\partial}{\partial \xi} + J_{12}^{-1}\frac{\partial}{\partial \eta}\right)\left(J_{11}^{-1}\frac{\partial u}{\partial \xi} + J_{12}^{-1}\frac{\partial u}{\partial \eta}\right) - \left(J_{21}^{-1}\frac{\partial}{\partial \xi} + J_{22}^{-1}\frac{\partial}{\partial \eta}\right)\left(J_{21}^{-1}\frac{\partial u}{\partial \xi} + J_{22}^{-1}\frac{\partial u}{\partial \eta}\right)$$

$$= -\left\{\left(J_{11}^{-1}\right)^2 + \left(J_{21}^{-1}\right)^2\right\}\frac{\partial^2 u}{\partial \xi^2}$$

$$-\left(J_{11}^{-1}J_{12}^{-1} + J_{12}^{-1}J_{11}^{-1} + J_{21}^{-1}J_{22}^{-1} + J_{22}^{-1}J_{21}^{-1}\right)\frac{\partial^2 u}{\partial \xi \partial \eta}$$

$$-\left\{\left(J_{12}^{-1}\right)^2 + \left(J_{22}^{-1}\right)^2\right\}\frac{\partial^2 u}{\partial \eta^2}$$

$$-\left(J_{11}^{-1}\frac{\partial J_{11}^{-1}}{\partial \xi} + J_{12}^{-1}\frac{\partial J_{11}^{-1}}{\partial \eta} + J_{21}^{-1}\frac{\partial J_{21}^{-1}}{\partial \xi} + J_{22}^{-1}\frac{\partial J_{21}^{-1}}{\partial \eta}\right)\frac{\partial u}{\partial \xi}$$

$$-\left(J_{11}^{-1}\frac{\partial J_{12}^{-1}}{\partial \xi} + J_{12}^{-1}\frac{\partial J_{12}^{-1}}{\partial \eta} + J_{21}^{-1}\frac{\partial J_{22}^{-1}}{\partial \xi} + J_{22}^{-1}\frac{\partial J_{22}^{-1}}{\partial \eta}\right)\frac{\partial u}{\partial \eta}$$

$$= -a_{11}\frac{\partial^2 u}{\partial \xi^2} - a_{12}\frac{\partial^2 u}{\partial \xi \partial \eta} - a_{22}\frac{\partial^2 u}{\partial \eta^2} - b_1\frac{\partial u}{\partial \xi} - b_2\frac{\partial u}{\partial \eta} = 2$$

where

$$a_{11} = \left(J_{11}^{-1}\right)^2 + \left(J_{21}^{-1}\right)^2$$

$$a_{12} = J_{11}^{-1}J_{12}^{-1} + J_{12}^{-1}J_{11}^{-1} + J_{21}^{-1}J_{22}^{-1} + J_{22}^{-1}J_{21}^{-1}$$

$$a_{22} = \left(J_{12}^{-1}\right)^2 + \left(J_{22}^{-1}\right)^2$$

$$b_1 = J_{11}^{-1}\frac{\partial J_{11}^{-1}}{\partial \xi} + J_{12}^{-1}\frac{\partial J_{11}^{-1}}{\partial \eta} + J_{21}^{-1}\frac{\partial J_{21}^{-1}}{\partial \xi} + J_{22}^{-1}\frac{\partial J_{21}^{-1}}{\partial \eta}$$

$$b_2 = J_{11}^{-1}\frac{\partial J_{12}^{-1}}{\partial \xi} + J_{12}^{-1}\frac{\partial J_{12}^{-1}}{\partial \eta} + J_{21}^{-1}\frac{\partial J_{22}^{-1}}{\partial \xi} + J_{22}^{-1}\frac{\partial J_{22}^{-1}}{\partial \eta}$$

We shall evaluate these by using MATHEMATICA:

```
x=(3+s)*Cos[Pi*(1-t)/2];
y=(5+3*s)*Sin[Pi*(1-t)/2]/2;
JM={{D[x,s],D[y,s]},{D[x,t],D[y,t]}};
MatrixForm[JM]
JMI=Simplify[Inverse[JM]];
MatrixForm[JMI]
a11=JMI[[1,1]]^2+JMI[[2,1]]^2;
a11=Simplify[a11];
a12=2*(JMI[[1,1]]*JMI[[1,2]]+JMI[[2,1]]*JMI[[2,2]]);
a12=Simplify[a12];
a22=JMI[[1,2]]^2+JMI[[2,2]]^2;
a22=Simplify[a22];
b1=JMI[[1,1]]*D[JMI[[1,1]],s]+JMI[[1,2]]*D[JMI[[1,1]],t]+
    JMI[[2,1]]*D[JMI[[2,1]],s]+JMI[[2,2]]*D[JMI[[2,1]],t];
b1=Simplify[b1];
b2=JMI[[1,1]]*D[JMI[[1,2]],s]+JMI[[1,2]]*D[JMI[[1,2]],t]+
    JMI[[2,1]]*D[JMI[[2,2]],s]+JMI[[2,2]]*D[JMI[[2,2]],t];
b2=Simplify[b2];
Print["a11=",FortranForm[a11]]
Print["a12=",FortranForm[a12]]
Print["a22=",FortranForm[a22]]
Print["b1=",FortranForm[b1]]
Print["b2=",FortranForm[b2]]
```

$$\begin{pmatrix} \text{Cos}\left[\frac{1}{2}\pi(1-t)\right] & \frac{3}{2}\,\text{Sin}\left[\frac{1}{2}\pi(1-t)\right] \\ \frac{1}{2}\pi(3+s)\,\text{Sin}\left[\frac{1}{2}\pi(1-t)\right] & -\frac{1}{4}\pi(5+3s)\,\text{Cos}\left[\frac{1}{2}\pi(1-t)\right] \end{pmatrix}$$

$$\begin{pmatrix} \frac{(5+3\,s)\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]}{7+3\,s+2\,\mathrm{Cos}[\pi t]} & \frac{6\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]}{7\,\pi+3\,\pi\,s+2\,\pi\,\mathrm{Cos}[\pi t]} \\ \frac{2\,(3+s)\,\mathrm{Cos}\left[\frac{\pi t}{2}\right]}{7+3\,s+2\,\mathrm{Cos}[\pi t]} & -\frac{4\,\mathrm{Sin}\left[\frac{\pi t}{2}\right]}{\pi\,(7+3\,s+2\,\mathrm{Cos}[\pi t])} \end{pmatrix}$$

```
"a11="¥[InvisibleSpace](61 + 54*s + 13*s**2 + (11 - 6*s - 5*s**2)*Cos(Pi*t))/(
      2.*(7 + 3*s + 2*Cos(Pi*t))**2)
"a12="¥[InvisibleSpace](2*(3 + 5*s)*Sin(Pi*t))/(Pi*(7 + 3*s + 2*Cos(Pi*t))**2)
"a22="¥[InvisibleSpace](2*(13 + 5*Cos(Pi*t)))/(
      Pi**2*(7 + 3*s + 2*Cos(Pi*t))**2)
"b1="¥[InvisibleSpace](
      201 + 192*s + 39*s**2 + 5*(15 + 14*s + 3*s**2)*Cos(Pi*t) -
        2*(3 + 5*s)*Cos(2*Pi*t))/(2.*(7 + 3*s + 2*Cos(Pi*t))**3)
"b2="¥[InvisibleSpace](2*(4 - 15*s + 5*Cos(Pi*t))*Sin(Pi*t))/(
      Pi*(7 + 3*s + 2*Cos(Pi*t))**3)
```

Now, we shall consider the finite difference approximation of the second order partial differential equation

$$-a_{11}\frac{\partial^2 u}{\partial \xi^2}-a_{12}\frac{\partial^2 u}{\partial \xi \partial \eta}-a_{22}\frac{\partial^2 u}{\partial \eta^2}-b_1\frac{\partial u}{\partial \xi}-b_2\frac{\partial u}{\partial \eta}=2$$

Applying the central difference schemes, we have

$$-a_{11,i,j}\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta\xi^2}-a_{12,i,j}\frac{u_{i+1,j+1}-u_{i+1,j-1}-u_{i-1,j+1}+u_{i-1,j-1}}{4\Delta\xi\Delta\eta}$$
$$-a_{22,i,j}\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta\eta^2}-b_{1,i,j}\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta\xi}-b_{2,i,j}\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta\eta}=2$$

From this, we obtain

$$u_{i,j} = \frac{1}{2\left(\frac{a_{11,i,j}}{\Delta\xi^2} + \frac{a_{22,i,j}}{\Delta\eta^2}\right)} \left\{ \left(\frac{a_{11,i,j}}{\Delta\xi^2} + \frac{b_{1,i,j}}{2\Delta\xi}\right) u_{i+1,j} + \left(\frac{a_{11,i,j}}{\Delta\xi^2} - \frac{b_{1,i,j}}{2\Delta\xi}\right) u_{i-1,j} \right.$$

$$+ \left(\frac{a_{22,i,j}}{\Delta\eta^2} + \frac{b_{2,i,j}}{2\Delta\eta}\right) u_{i,j+1} + \left(\frac{a_{22,i,j}}{\Delta\eta^2} - \frac{b_{2,i,j}}{2\Delta\eta}\right) u_{i,j-1}$$

$$\left. + \frac{a_{12,i,j}}{4\Delta\xi\Delta\eta} \left(u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}\right) + 2 \right\}$$

and then, for any $0 < \omega$, we have

$$u_{i,j} = (1-\omega)u_{i,j} + \frac{\omega}{2\left(\frac{a_{11,i,j}}{\Delta\xi^2} + \frac{a_{22,i,j}}{\Delta\eta^2}\right)} \left\{ \left(\frac{a_{11,i,j}}{\Delta\xi^2} + \frac{b_{1,i,j}}{2\Delta\xi}\right) u_{i+1,j} + \left(\frac{a_{11,i,j}}{\Delta\xi^2} - \frac{b_{1,i,j}}{2\Delta\xi}\right) u_{i-1,j} \right.$$

$$+ \left(\frac{a_{22,i,j}}{\Delta\eta^2} + \frac{b_{2,i,j}}{2\Delta\eta}\right) u_{i,j+1} + \left(\frac{a_{22,i,j}}{\Delta\eta^2} - \frac{b_{2,i,j}}{2\Delta\eta}\right) u_{i,j-1}$$

$$\left. + \frac{a_{12,i,j}}{4\Delta\xi\Delta\eta} \left(u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}\right) + 2 \right\}$$

This suggests the following successive over-relaxation iteration scheme:

$$u_{i,j}{}^{(k+1)} = (1-\omega)u_{i,j}{}^{(k)} + \frac{\omega}{2\left(\frac{a_{11,i,j}}{\Delta\xi^2} + \frac{a_{22,i,j}}{\Delta\eta^2}\right)} \left\{ \left(\frac{a_{11,i,j}}{\Delta\xi^2} + \frac{b_{1,i,j}}{2\Delta\xi}\right) u_{i+1,j}{}^{(k)} + \left(\frac{a_{11,i,j}}{\Delta\xi^2} - \frac{b_{1,i,j}}{2\Delta\xi}\right) u_{i-1,j}{}^{(k+1)} \right.$$

$$+ \left(\frac{a_{22,i,j}}{\Delta\eta^2} + \frac{b_{2,i,j}}{2\Delta\eta}\right) u_{i,j+1}{}^{(k)} + \left(\frac{a_{22,i,j}}{\Delta\eta^2} - \frac{b_{2,i,j}}{2\Delta\eta}\right) u_{i,j-1}{}^{(k+1)}$$

$$\left. + \frac{a_{12,i,j}}{4\Delta\xi\Delta\eta} \left(u_{i+1,j+1}{}^{(k)} - u_{i+1,j-1}{}^{(k)} - u_{i-1,j+1}{}^{(k+1)} + u_{i-1,j-1}{}^{(k+1)}\right) + 2 \right\}$$

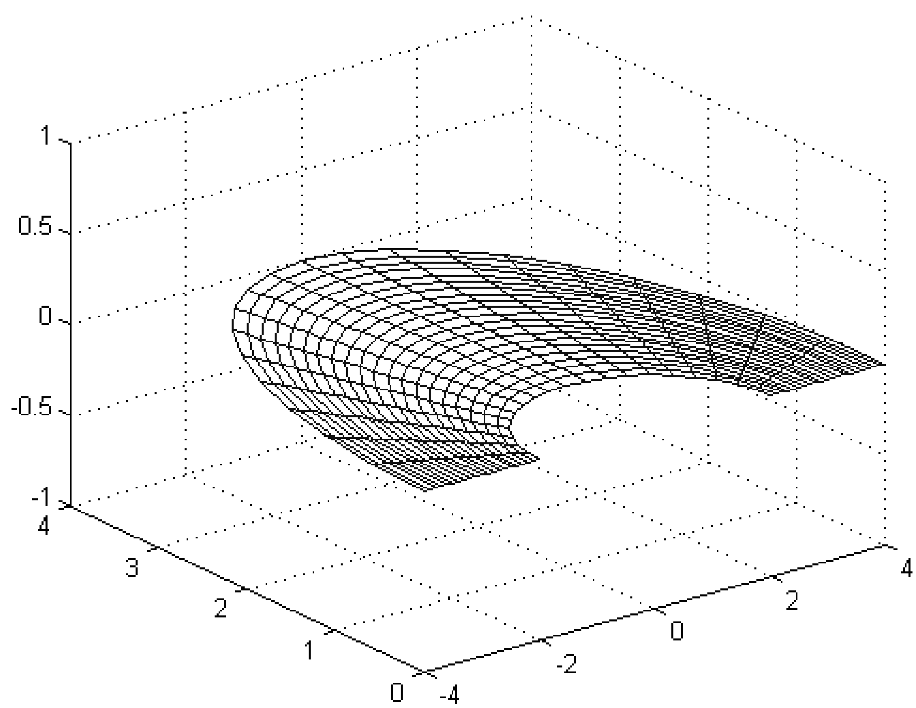
for $k = 0,1,2,.....$

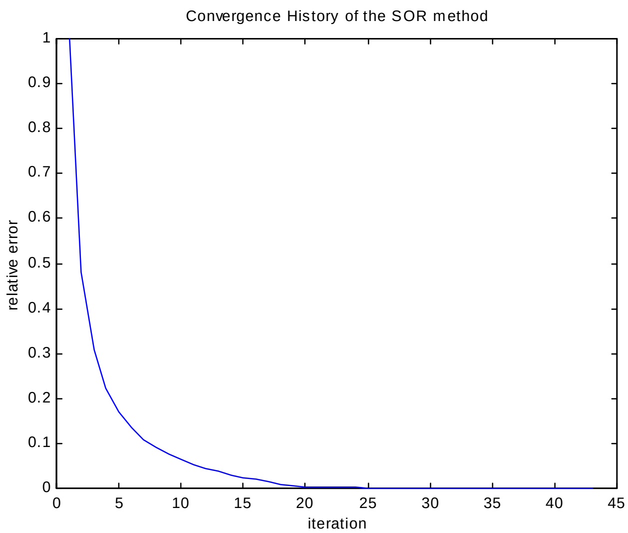
Using MATLAB, we shall compute the distribution of the function u:

```
% Homework #1
% MEAM 502
%
______________________________________________________________________
__________________
% set up a grid in the computational domain
%
ds=0.1;
dt=0.1;
xl=-1:ds:1;
yl=-1:dt:1;
n=size(xl,2);
for i=1:n
   for j=1:n
      x(i,j)=(3+yl(j))*cos(pi*(1-xl(i))/2);
      y(i,j)=(5+3*yl(j))*sin(pi*(1-xl(i))/2)/2;
      R(i,j)=0;
   end
end
mesh(x,y,R)
pause
%
% for the successive over-relaxation method
%
omega=1.75;
u=zeros(n);
iteration=0;
maxiteration=100;
error=1;
tolerance=10^(-5);
errorh=[];
```

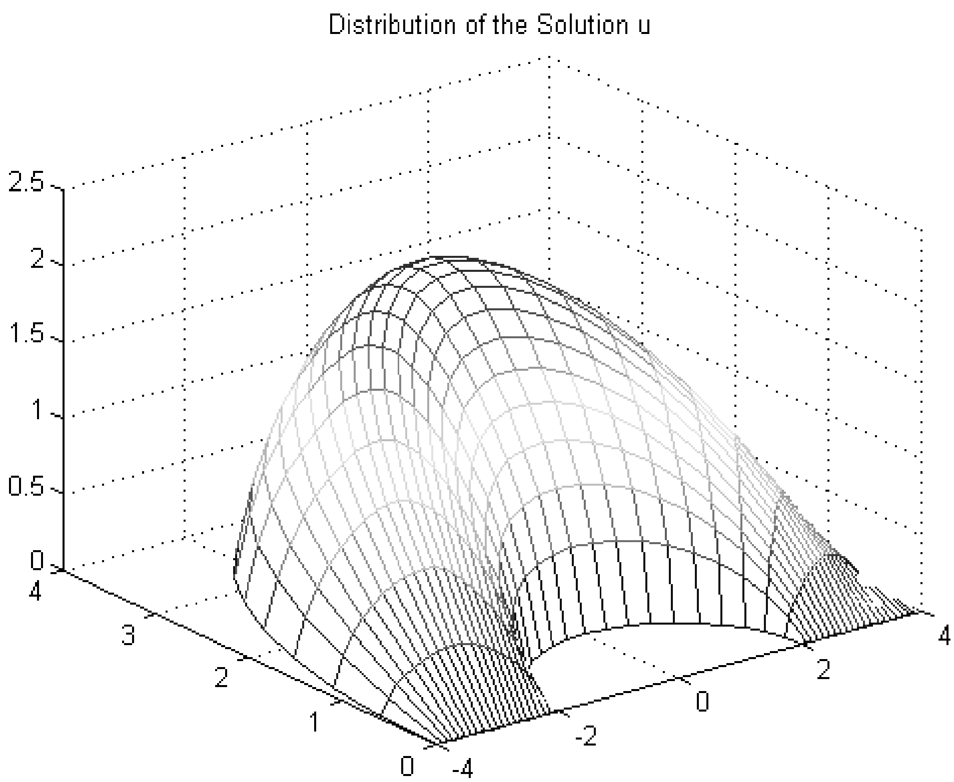
```
%
while error>tolerance
%
errorn=0;
errord=0;
for i=2:n-1
for j=2:n-1
s=xl(i);
t=yl(j);
%
a11=(61+54*s+13*s^2+(11-6*s-
5*s^2)*cos(pi*t))/(2.*(7+3*s+2*cos(pi*t))^2);
a12=(2*(3+5*s)*sin(pi*t))/(pi*(7+3*s+2*cos(pi*t))^2);
a22=(2*(13+5*cos(pi*t)))/(pi^2*(7+3*s+2*cos(pi*t))^2);
b1=(201+192*s+39*s^2+5*(15+14*s+3*s^2)*cos(pi*t)-
2*(3+5*s)*cos(2*pi*t))/(2.*(7+3*s+2*cos(pi*t))^3);
b2=(2*(4-15*s+5*cos(pi*t))*sin(pi*t))/(pi*(7+3*s+2*cos(pi*t))^3);
%
a1=0.5*(a11/ds^2+0.5*b1/ds)/(a11/ds^2+a22/dt^2);
a2=0.5*(a11/ds^2-0.5*b1/ds)/(a11/ds^2+a22/dt^2);
a3=0.5*(a22/dt^2+0.5*b2/dt)/(a11/ds^2+a22/dt^2);
a4=0.5*(a22/dt^2-0.5*b2/dt)/(a11/ds^2+a22/dt^2);
a5=0.125*(a12/(ds*dt))/(a11/ds^2+a22/dt^2);
a6=0.5*2/(a11/ds^2+a22/dt^2);
res=a1*u(i+1,j)+a2*u(i-1,j)+a3*u(i,j+1)+a4*u(i,j-1)+a5*(u(i+1,j+1)-
u(i+1,j-1)-u(i-1,j+1)+u(i-1,j-1))+a6;
uij=(1-omega)*u(i,j)+omega*res;
errorn=errorn+(uij-u(i,j))^2;
errord=errord+uij^2;
u(i,j)=uij;
end
end
iteration=iteration+1;
```

```
error=sqrt(errorn)/sqrt(errord);
errorh(iteration)=error;
if iteration>maxiteration, break, end
%
end
%
plot(errorh)
title('Convergence History of the SOR method')
xlabel('iteration')
ylabel('relative error')
pause
%
mesh(x,y,u)
title('Distribution of the Solution u')
```





Convergence History of the SOR method
relative error
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
5
10
15
20
25
30
35
40
45



Distribution of the Solution u
2.5
2
1.5
1
0.5
0
4
3
2
1
0
-4
-2
0
2
4