Review Problems for the Test on March 31, 1999
Partial Solutions
MEAM 502 Differential Equation Methods in Mechanics

1. What is a general expression of the second order partial differential equations defined on a domain in $\mathbf{R}^n$?

$$\sum_{i,j=1}^{n} a_{ij}(\mathbf{x},u)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(\mathbf{x},u)\frac{\partial u}{\partial x_i} + c(x,u)u = f\left(\mathbf{x},u,\frac{\partial u}{\partial x_i}\right)$$

2. Suppose that two coordinates (x,y) are obtained by a mapping from another coordinate system (ξ,η). Transform the differential equation in the system (x,y): $-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$ to the one in the coordinate system (ξ,η). Furthermore, if the mapping $\begin{cases} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{cases}$ and the first derivatives of a function g are calculated in the coordinate system (ξ,η), find the way to compute the first derivatives of g in the coordinate system (x,y).

Noting that

$$\begin{Bmatrix} \frac{\partial g}{\partial \xi} \\ \frac{\partial g}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \frac{\partial g}{\partial x} \\ \frac{\partial g}{\partial y} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1} \begin{Bmatrix} \frac{\partial g}{\partial \xi} \\ \frac{\partial g}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \frac{\partial g}{\partial \xi} \\ \frac{\partial g}{\partial \eta} \end{Bmatrix}$$

we have

$$-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=-\begin{Bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{Bmatrix}^T\begin{Bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{Bmatrix}u=-\left(\begin{bmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x}\\ \frac{\partial\xi}{\partial y} & \frac{\partial\eta}{\partial y}\end{bmatrix}\begin{Bmatrix}\frac{\partial}{\partial\xi}\\ \frac{\partial}{\partial\eta}\end{Bmatrix}\right)^T\left(\begin{bmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x}\\ \frac{\partial\xi}{\partial y} & \frac{\partial\eta}{\partial y}\end{bmatrix}\begin{Bmatrix}\frac{\partial u}{\partial\xi}\\ \frac{\partial u}{\partial\eta}\end{Bmatrix}\right)$$

$$=-\left(\begin{Bmatrix}\left\{\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2\right\}\frac{\partial}{\partial\xi}+\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right)\frac{\partial}{\partial\eta}\\ \left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right)\frac{\partial}{\partial\xi}+\left\{\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right\}\frac{\partial}{\partial\eta}\end{Bmatrix}^T\right)\begin{Bmatrix}\frac{\partial u}{\partial\xi}\\ \frac{\partial u}{\partial\eta}\end{Bmatrix}-\left(\begin{Bmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{Bmatrix}^T\begin{bmatrix}\frac{\partial\xi}{\partial x} & \frac{\partial\eta}{\partial x}\\ \frac{\partial\xi}{\partial y} & \frac{\partial\eta}{\partial y}\end{bmatrix}\right)\begin{Bmatrix}\frac{\partial u}{\partial\xi}\\ \frac{\partial u}{\partial\eta}\end{Bmatrix}$$

$$=-\left\{\left(\frac{\partial\xi}{\partial x}\right)^2+\left(\frac{\partial\xi}{\partial y}\right)^2\right\}\frac{\partial^2 u}{\partial\xi^2}-2\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x}+\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right)\frac{\partial^2 u}{\partial\xi\partial\eta}-\left\{\left(\frac{\partial\eta}{\partial x}\right)^2+\left(\frac{\partial\eta}{\partial y}\right)^2\right\}\frac{\partial^2 u}{\partial\eta^2}$$

$$-\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\partial^2\xi}{\partial y^2}\right)\frac{\partial u}{\partial\xi}-\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right)\frac{\partial u}{\partial\eta}$$

3. What is the steepest descent method to solve a system of linear equations $\mathbf{Ax}=\mathbf{b}$? Here we shall assume symmetry of the coefficient matrix $\mathbf{A}$.

It is an iteration method to solve a given system of linear equations $\mathbf{Ax}=\mathbf{b}$ such that for a given initial guess $\mathbf{x}_0$, a solution is obtained by

$$\mathbf{x}_{k+1}=\mathbf{x}_k-\alpha_k\left(\mathbf{Ax}_k-\mathbf{b}\right) \quad , \quad k=0,1,2,......$$

where α_k are determined to solve the following one-dimensional minimization problem:

$$f(\alpha_k)=\min_{\alpha_k}\frac{1}{2}\mathbf{x}_{k+1}{}^T\mathbf{Ax}_{k+1}-\mathbf{x}_{k+1}{}^T\mathbf{b} \quad \Leftrightarrow \quad \frac{\partial}{\partial\alpha_k}\left(\frac{1}{2}\mathbf{x}_{k+1}{}^T\mathbf{Ax}_{k+1}-\mathbf{x}_{k+1}{}^T\mathbf{b}\right)=0$$

$$\Leftrightarrow \quad \alpha_k=\frac{\mathbf{x}_k{}^T\mathbf{A}^T\left(\mathbf{Ax}_k-\mathbf{b}\right)}{\mathbf{x}_k{}^T\mathbf{A}^T\mathbf{Ax}_k}$$

4. What is the conjugate gradient method to solve a system of linear equations $\mathbf{Ax}=\mathbf{b}$? Here we shall assume symmetry of the coefficient matrix $\mathbf{A}$.

It is an iteration method to solve a given system of linear equations $\mathbf{Ax}=\mathbf{b}$ such that for a given initial guess $\mathbf{x}_0$ & $\mathbf{x}_1$, a solution is obtained by

$$\mathbf{x}_{k+1}=\mathbf{x}_k-\beta_k\left(\mathbf{x}_k-\mathbf{x}_{k-1}\right)-\alpha_k\left(\mathbf{Ax}_k-\mathbf{b}\right) \quad , \quad k=1,2,......$$

where α_k & β_k are determined to solve the following two-dimensional minimization problem:

$$f(\alpha_k, \beta_k) = \min_{\alpha_k} \frac{1}{2}\mathbf{x}_{k+1}{}^T \mathbf{A}\mathbf{x}_{k+1} - \mathbf{x}_{k+1}{}^T \mathbf{b}$$

$$\Leftrightarrow \begin{cases} \dfrac{\partial}{\partial \alpha_k}\left(\dfrac{1}{2}\mathbf{x}_{k+1}{}^T \mathbf{A}\mathbf{x}_{k+1} - \mathbf{x}_{k+1}{}^T \mathbf{b}\right) = 0 \\ \dfrac{\partial}{\partial \beta_k}\left(\dfrac{1}{2}\mathbf{x}_{k+1}{}^T \mathbf{A}\mathbf{x}_{k+1} - \mathbf{x}_{k+1}{}^T \mathbf{b}\right) = 0 \end{cases}$$

Solving the two-dimensional minimization problem, we have

$$\begin{cases} \mathbf{r}_k{}^T(\mathbf{r}_k - \mathbf{A}\mathbf{d}_k\beta_k - \mathbf{A}\mathbf{r}_k\alpha_k) = 0 \quad , \quad \mathbf{r}_k = \mathbf{A}\mathbf{x}_k - \mathbf{b} \\ \mathbf{d}_k{}^T(\mathbf{r}_k - \mathbf{A}\mathbf{d}_k\beta_k - \mathbf{A}\mathbf{r}_k\alpha_k) = 0 \quad , \quad \mathbf{d}_k = \mathbf{x}_k - \mathbf{x}_{k-1} \end{cases}$$

$$\Leftrightarrow$$

$$\begin{Bmatrix} \alpha_k \\ \beta_k \end{Bmatrix} = \begin{bmatrix} \mathbf{r}_k{}^T \mathbf{A}\mathbf{r}_k & \mathbf{r}_k{}^T \mathbf{A}\mathbf{d}_k \\ \mathbf{d}_k{}^T \mathbf{A}\mathbf{r}_k & \mathbf{d}_k{}^T \mathbf{A}\mathbf{d}_k \end{bmatrix}^{-1} \begin{Bmatrix} \mathbf{r}_k{}^T \mathbf{r}_k \\ \mathbf{d}_k{}^T \mathbf{r}_k \end{Bmatrix}$$

5. What is the Newton's method to solve a system of nonlinear equations $\mathbf{f}(\mathbf{u}) = \mathbf{0}$?

$$\mathbf{u}_{k+1} = \mathbf{u}_k - [\nabla_{\mathbf{u}}\mathbf{f}(\mathbf{u}_k)]^{-1}\mathbf{f}(\mathbf{u}_k) \quad , \quad k = 0,1,2,..... \quad \text{for a given } \mathbf{u}_0$$

6. State the fixed point theorem.

If a function f satisfies the condition, for $0 < \alpha < 1$,

$$\| f(x) - f(y) \| \leq \alpha \| x - y \| \quad , \quad \forall x, y$$

where $\|.\|$ is a norm defined in a normed linear space V, then there is a unique fixed point x in V:

$$x = f(x)$$

and it can be obtained by the iteration method

$$x_{k+1} = f(x_k) \quad , \quad k = 1,2,....., \quad \text{for a given} \quad x_1 .$$

7. What is a norm?

A norm $\|.\|$ is a function from a linear space V into $[0,+\infty)$ such that

1) $\|x\| \geq 0 \quad , \quad \forall x \in V \qquad and \qquad \|x\| = 0$ if and only if $x = 0$
2) $\|\alpha x\| = |\alpha|\|x\| \quad , \quad \forall \alpha \in (-\infty,+\infty),\ x \in V$
3) $\|x+y\| \leq \|x\| + \|y\| \quad , \quad \forall x, y \in V$

8. What is a scalar product (or inner product)?

A scalar product $(.,.)$ is a function from a product space $V \times V$ of a linear space V into **R** such that

1) $(x,x) \geq 0 \quad , \quad \forall x \in V \qquad \& \qquad (x,x) = 0$ if and only if $x = 0$
2) $(x,y) = \overline{(y,x)} \quad , \quad \forall x, y \in V$
3) $(\alpha x + \beta y, z) = \alpha(x,z) + \beta(y,z) \quad , \quad \forall x, y, z \in V\ , \alpha, \beta \in \mathbf{C}$
4) $(x, \alpha y + \beta z) = \overline{\alpha}(x,y) + \overline{\beta}(x,z) \quad , \quad \forall x, y, z \in V\ , \alpha, \beta \in \mathbf{C}$

9. What is a normed linear space?

A linear space with a norm defined.

10. What is the strong convergence of a sequence $\{f_n\}$ in a normed linear space?

If a sequence $\{f_n\}$ is strongly convergent to an element f if $\lim_{n \to +\infty} \|f_n - f\| = 0$.

11. What is the weak convergence of a sequence $\{f_n\}$ in a scalar product (or inner product) space?

If a sequence $\{f_n\}$ is weakly convergent to an element f if $\lim_{n \to +\infty} (g, f_n - f) = 0$, $\forall g$, where $(.,.)$ is a scalar product.

12. Define the best approximation of an arbitrary element $f \in V$, where V is a Hilbert space with an inner product (f,g), $f, g \in V$.

The best approximation $f_K = P_K f$ of an arbitrary element of a Hilbert space V onto a closed subspace K of V, is defined by

$$f_K \in K \quad : \quad \|f_K - f\| \le \|v - f\| \quad , \quad \forall v \in K$$

and it is characterized by the solution of

$$\mathrm{Re}(v, f_K - f) = 0 \quad , \quad \forall v \in K .$$

If K is a closed convex set of V, then the best approximation $f_K = P_K f$ is characterized by

$$\mathrm{Re}(v - f_K, f_K - f) \ge 0 \quad , \quad \forall v \in K$$

13. What is a convex set in a linear space V?

A set K is said to be convex if $(1-\alpha)x + \alpha y \in K$, $\forall x, y \in K$, $\alpha \in [0,1]$.

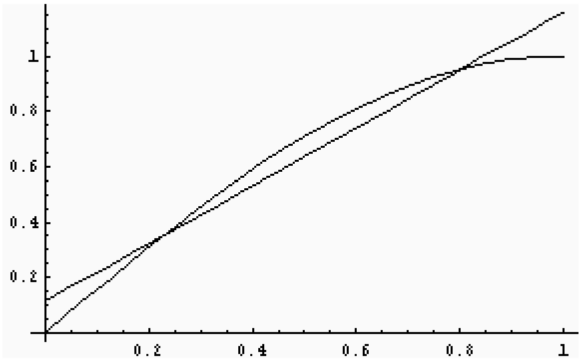
14. Find the best approximation of a continuous function $f(x) = \sin\left(\frac{\pi}{2}x\right)$ defined on an interval $(0,1)$ as an element of $L^2(0,1)$, onto the closed linear subspace $K = \{v \in L^2(0,1) \mid v(x) \text{ is spanned by } \{1 \quad x\}, \text{ that is, } v(x) = c + dx \text{ for some } c \text{ and } d \}$.

Using the characterization of the best approximation, we have

$$\left(1, c + dx - \sin\left(\frac{\pi}{2}x\right)\right) = 0$$
$$\left(x, c + dx - \sin\left(\frac{\pi}{2}x\right)\right) = 0$$

where the best approximation of f is assumed by $f_K = c + dx$, and $(f, g) = \int_0^1 fg\,dx$ is a scalar product of a Hilbert space $L^2(0,1)$. Thus,

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{Bmatrix} c \\ d \end{Bmatrix} = \begin{Bmatrix} \frac{2}{\pi} \\ \frac{4}{\pi^2} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} c \\ d \end{Bmatrix} = \frac{1}{\pi^2} \begin{Bmatrix} 8\pi - 24 \\ 12\pi - 48 \end{Bmatrix}$$



15. What is the Lagrange interpolation?

Choosing a n+1 number of discrete points $x_1, x_2, \ldots\ldots, x_{n+1}$, a given function $f(x)$ is approximated by a n degree polynomial by

$$f(x) \approx f_h(x) = \sum_{i=1}^{n+1} f(x_i) L_i(x) \quad , \quad L_i(x) = \prod_{\substack{j=1 \\ i \neq i}}^{n+1} \frac{x - x_j}{x_i - x_j}$$

16. If the function $f(x) = \sin\left(\frac{\pi}{2} x\right)$ is interpolated by a linear polynomial by using the nodal points $x = 0$ and 1, how can we estimate the interpolation error?

17. State briefly the element free Galerkin method?

18. What are the Haar scaling and mother wavelet functions?

19. State difference between Wavelet and Fourier transformations.

20. Define the wavelet functions.