Consider an oscillatory coefficient shown in below

\[ a(x) \]

and the functional

\[ F(u) = \frac{1}{2} \int_0^l \left[ a(x) \left( \frac{du}{dx} \right)^2 + \sin(2\pi x)u^2 \right] dx - \int_0^l (1 + x^2)u dx \]

for the minimization problem

\[ \min_u F(u) \]

among the admissible functions such that \( u(0) = u(1) = 0 \).

(1) Find the first variation \( \delta F \) of the functional \( F \) and its Euler’s equation by setting \( \delta F = 0 \) for every \( \delta u = 0 \).

(2) Find the homogenized coefficient \( a^H \) and homogenized functional \( F^H(u^H) \), and then solve the homogenized problem to find \( u^H \) that minimizes the homogenized functional \( F^H(u^H) \). Furthermore, estimate the maximum difference of \( u^H - u \) and \( \frac{d}{dx}(u - u^H) \) based on the homogenization asymptotic expansion.
(3) Assume the discontinuously oscillated coefficient \(a(x)\) by \(\tilde{a}(x)\) in terms of trigonometric functions (by applying the Fourier series to approximate a function), find the minimizer \(\tilde{u}\) of the approximated functional

\[
\tilde{F}(u) = \frac{1}{2} \int_0^l \left[ \tilde{a}(x) \left( \frac{du}{dx} \right)^2 + \sin(2\pi x)u^2 \right] dx - \int_0^l (1 + x^2) u dx.
\]

Note that \(\tilde{a}(x)\) contains only few terms of sine and cosine functions.

(4) Find the differences of \(\tilde{u} - u^H\) and \(\frac{d}{dx} (\tilde{u} - u^H)\). Plot these so that we can look at the differences.

(5) How can we find the limit of the differences \(u - u^H\) and \(\frac{d}{dx} (u - u^H)\) by passing to the limit of approximation of \(a(x)\) ?

(6) (optional: if you know FEM or FDM) Solve the original problem by FEM or FDM, and compare these results with the one obtained by the homogenization method.