Homework #4
MEAM 501 Analytical Methods in Mechanical Engineering

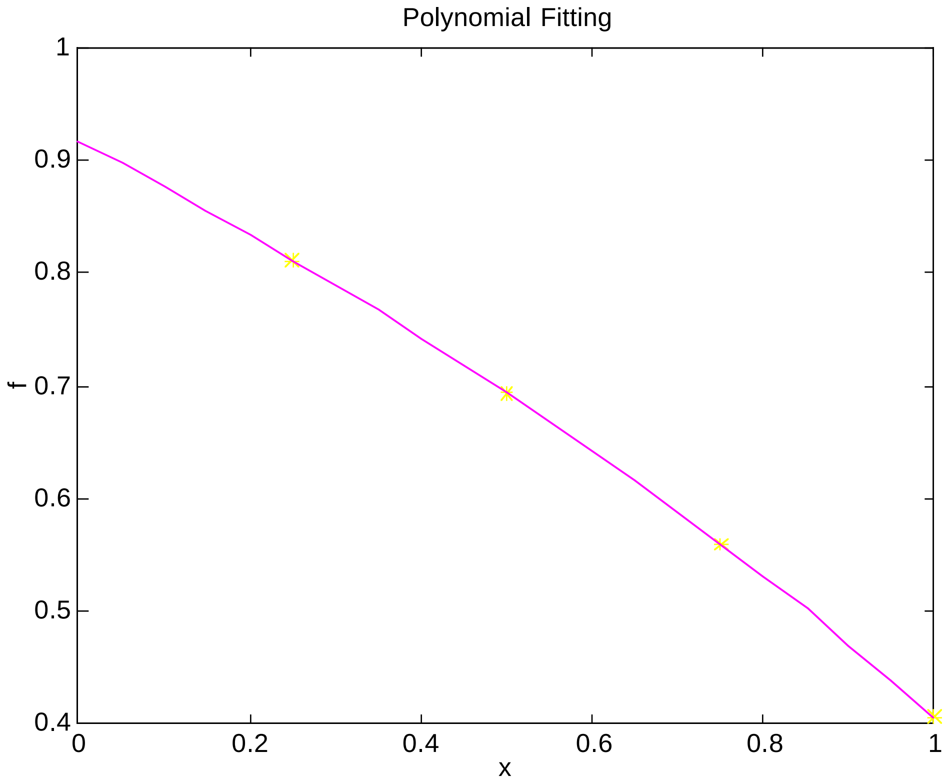
1997 Fall

1. (Problem 4.4) (a) Find the polynomial fitted to data points 2,3,4, and 5 of the following data :

k	xk	f(xk)
1	0	0.9162
2	0.25	0.8109
3	0.5	0.6931
4	0.75	0.5596
5	1.0	0.4055

(b) Transform the polynomial to clustered form. (c) Transform the polynomial to factorized form.

```
x=[0.25,0.5,0.75,1];
y=[0.8109,0.6931,0.5596,0.4055];
a=polyfit(x,y,3)
roots(a)
xi=0:0.05:1;
yi=polyval(a,xi);
plot(x,y,'*',xi,yi)
xlabel('x')
ylabel('f')
title('Polynomial Fitting')

a =

   -0.0523   -0.0472   -0.4129    0.9179


ans =

  -1.2102 + 3.1794i
  -1.2102 - 3.1794i
   1.5174
```



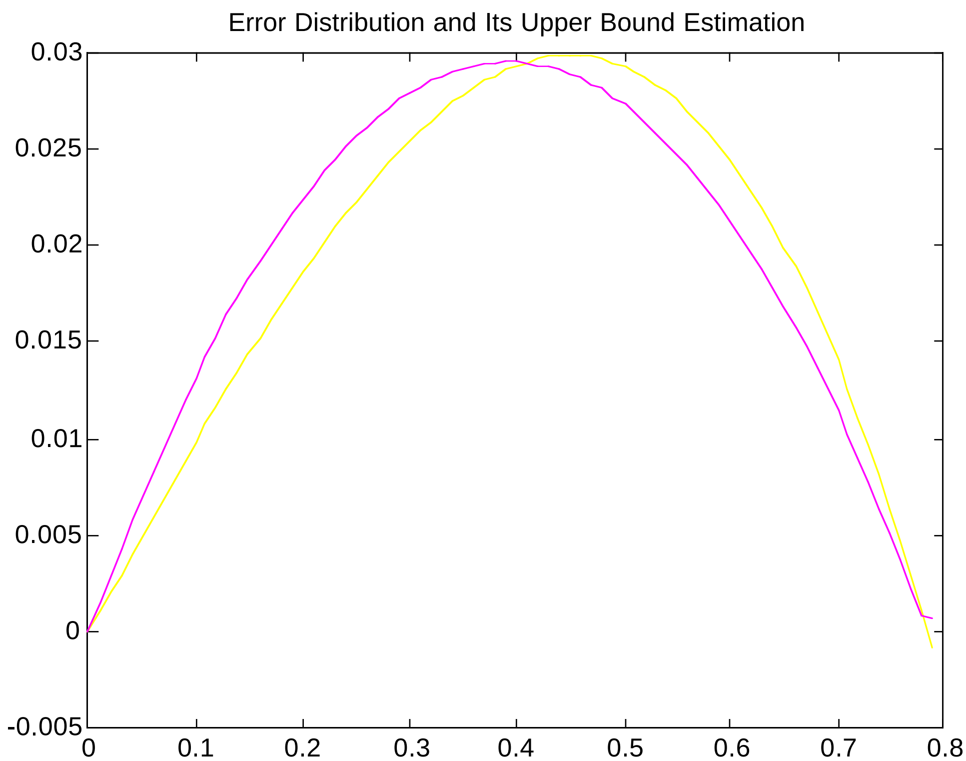
$$f_k(x) = -0.0523x^3 - 0.0472x^2 - 0.4129x + 0.9179$$
$$= ((-0.0523x - 0.0472)x - 0.4129)x + 0.9179$$
$$= -(x + 1.2102 - 3.1794i)(x + 1.2102 + 3.1794i)(x - 1.5174)$$

2. (Problem 4.10 & 4.11) Write a linear interpolation formula that approximate $\sin(x)$ in the interval of $0 \le x \le \pi/4$ using the values at $x = 0$ and $x = \pi/4$. Find the maximum error of the interpolation and at what x it occurs by plotting the error.

Knowing that max |f”| ~ 0.3827 in $0 \le x \le \pi/4$, predict the maximum possible error of the linear interpolation determined in above using Eq.(4.2.3).

$$g(x) = \frac{\frac{\pi}{4} - x}{\frac{\pi}{4} - 0}\sin(0) + \frac{x - 0}{\frac{\pi}{4} - 0}\sin\left(\frac{\pi}{4}\right) = \frac{4x}{\pi}\frac{1}{\sqrt{2}} = \frac{2\sqrt{2}}{\pi}x$$

$$e(x) = \sin(x) - g(x)$$



$$\left|e(x)\right| \le \frac{1}{2}\left|x\left(x-\frac{\pi}{4}\right)\right| \max_{\xi\in\left(0,\frac{\pi}{4}\right)}\left|f''(\xi)\right| \approx \frac{0.3827}{2}\left|x\left(x-\frac{\pi}{4}\right)\right| \quad \Rightarrow \quad \max_{x}\left|e(x)\right| \le 0.0295$$

```
xi=0:0.01:pi/4+0.01;
ei=sin(xi)-2*sqrt(2)*xi/pi;
eu=0.3827*abs(xi.*(xi-pi/4))/2;
plot(xi,ei,xi,eu)
title('Error Distribution and Its Upper Bound Estimation')
emax=0.3827*abs(pi/8*(pi/8-pi/4))/2
```

3. (Problem 4-16) (a) Write the Lagrange interpolation that passes through the following data points :

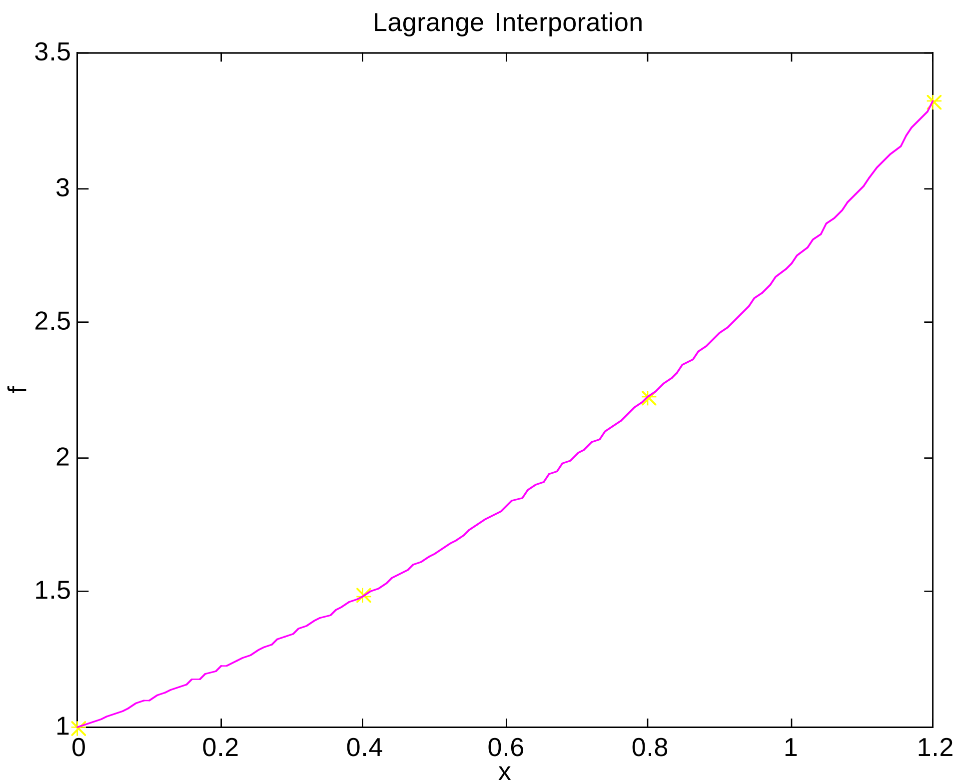
x	0	0.4	0.8	1.2
f	1.0	1.491	2.225	3.320

(b) Knowing f''''(0.6) = 1.822, estimate the error at x=0.2, 0.6, and 1.0 by Eq.(4.5.4) with $\xi_i = x_m$.

(c) Given the fact that the data table has been obtained from f(x) = exp(x), evaluate error of the interpolation formula at x = 0.2, 0.6, and 1.0 by e(x) = f(x) - g(x) = exp(x) - g(x).

```
% HW#4_4 ( Problem 4.16 )
x=[0,0.4,0.8,1.2];
f=[1.,1.491,2.225,3.320];
xi=0:0.01:1.2;
```

```
yi=Lagran_(x,f,xi);
plot(x,f,'*',xi,yi)
title('Lagrange Interporation')
xlabel('x')
ylabel('f')
a=polyfit(x,f,3)
roots(a)
% error estimation
f4=1.822;
xj=[0.2,0.6,1.0];
yj=Lagran_(x,f,xj)
ee=abs(exp(xj)-yj)
ej=ones(size(xj))*f4;
for i=1:size(x')
        ej=ej.*(xj-x(i))/i;
end
ej=abs(ej)


function fi=Lagran_(x,f,xi)
fi=zeros(size(xi));
np1=length(f);
for i=1:np1
        z=ones(size(xi));
        for j=1:np1
                if i~=j, z=z.*(xi-x(j))/(x(i)-x(j)); end
        end
        fi=fi+z*f(i);
end
return


a =

    0.3073    0.3906    1.0221    1.0000


ans =

  -0.1099 + 1.7558i
  -0.1099 - 1.7558i
  -1.0514


yj =

    1.2225    1.8203    2.7200


ee =

    0.0011    0.0019    0.0017


ej =

    0.0018    0.0011    0.0018
```



Lagrange Interpolation of f by using the data, is

$$f_i(x) = 0.3073x^3 + 0.3906x^2 + 1.0221x + 1$$

Assuming that

$$f''''(0.6) = 1.822 \quad \text{as well as} \quad f''''(\xi) \approx f''''(0.6) = 1.822 \text{ for } \xi \in (0,1.2)$$

the interpolation error can be estimated by

$$|e(x)| \le \left| \frac{1}{(n+1)!} \prod_{i=1}^{n+1} (x - x_i) \right| \max_{\xi \in (a,b)} \left| f^{(n+1)}(\xi) \right|$$

that is

$$|e(x)| \le \left| \frac{1}{(n+1)!} \prod_{i=1}^{n+1} (x - x_i) \right| \max_{\xi \in (a,b)} \left| f^{(n+1)}(\xi) \right| \approx 1.822 \left| \frac{1}{(n+1)!} \prod_{i=1}^{n+1} (x - x_i) \right|$$

x	fj (nodal value)	ej (estimated)	ee (exact error)
0.2	1.2225	0.018	0.011
0.6	1.8203	0.011	0.019
1.0	2.7200	0.018	0.017

4 (Problem 4.22) Approximate

$$y = \frac{1+x}{1+2x+3x^2} \quad , \quad x \in (0,5)$$

by the Lagrange interpolation of order 4, and evaluate the error by e(x) = y - g(x). Work according to the following steps : (a) determine the points, (b) write the Lagrange interpolation, (c) calculate the error for each increment of 0.2 in x, and (d) plot the error distribution.

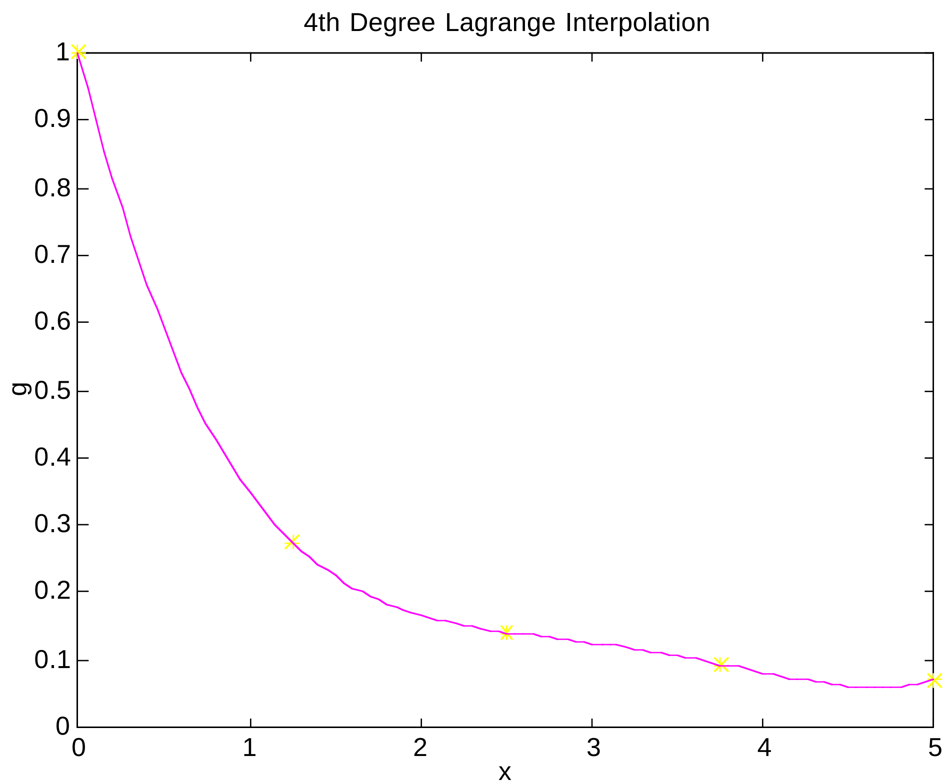
(a) For order 4 Lagrange interpolation, we must introduce 5 points. To this end, let us choose five points with equal distance :

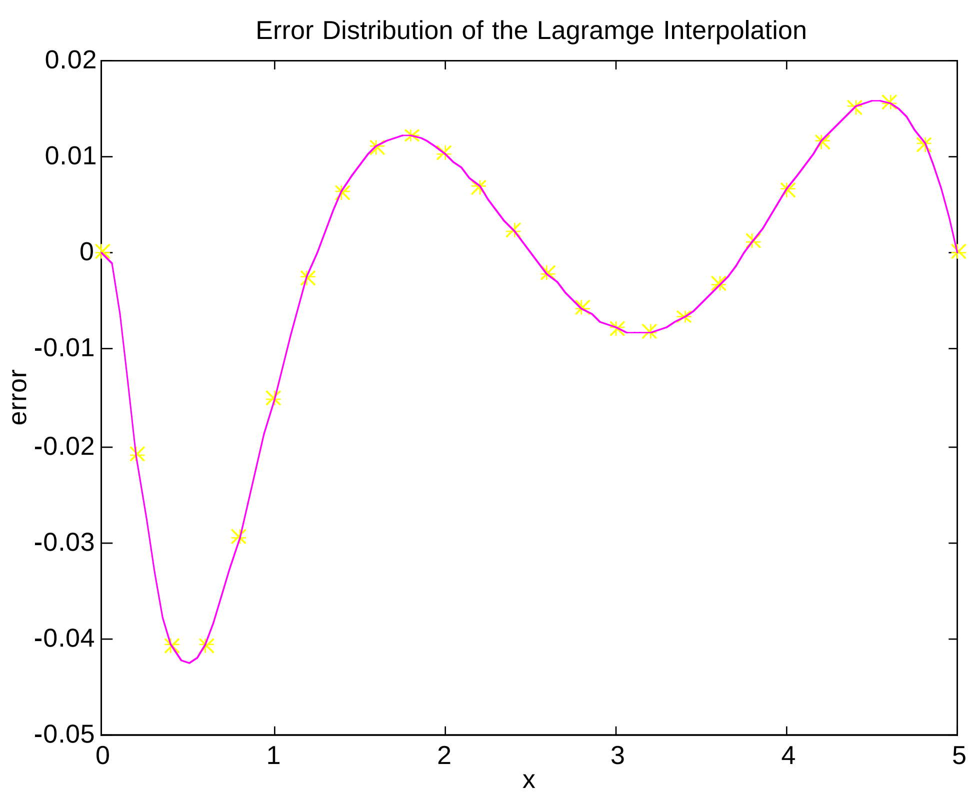
x = 0, 5/4, 10/4, 15/4, and 5.

(b), (c), and (d)

```
% Homework #3_4 : Problem 4.22
x=0:5/4:5
f=(1+x)./(1+2*x+3*x.^2)
xi=0:0.05:5;
fi=Lagran_(x,f,xi);
plot(x,f,'*',xi,fi)
title('4th Degree Lagrange Interpolation')
xlabel('x')
ylabel('g')
pause
xe=0:0.2:5;
fe=Lagran_(x,f,xe);
ee=(1+xe)./(1+2*xe+3*xe.^2)-fe;
ei=(1+xi)./(1+2*xi+3*xi.^2)-fi;
plot(xe,ee,'*',xi,ei)
title('Error Distribution of the Lagramge Interpolation')
xlabel('x')
ylabel('error')
```

```
x =

         0    1.2500    2.5000    3.7500    5.0000


f =

    1.0000    0.2748    0.1414    0.0937    0.0698
```





5. The Legendre polynomials satisfy the orthogonality relation :

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0 & \text{if} \quad m \neq n \\ \dfrac{2}{2n+1} & \text{if} \quad m = n \end{cases}$$

Werify the foregoing relation by computing

$$a_{mn} = \int_{-1}^{1} P_m(x)P_n(x)dx$$

for m, n = 1, ..., 5.

Noting that

$$p_{n+m}(x) = p_n(x)p_m(x) = \sum_{i=1}^{n+1} c_i x^{n-i+1} \sum_{j=1}^{m+1} d_j x^{m-j+1} = \sum_{i=1}^{n+1}\sum_{j=1}^{m+1} c_i d_j x^{n-i+1+m-j+1}$$

we can develop a MATLAB script program to take the inner product of the Legendre polynomials :

```
% Homework #4_5 : Problem 4.27
% compute the coefficients of the Legendre polynomials
cef=[];
for i=1:5
	cef(i,1:i+1)=Legen_pw(i);
end
cef
% inner product of the Legendre polynomials
A=zeros(5);
for m=1:5
	for n=m:5
		p=zeros(1,m+n+1);
		for i=1:m+1
			for j=1:n+1
				p(i+j-1)=p(i+j-1)+cef(m,i)*cef(n,j);
			end
		end
		p;
		mn=length(p);
		py=[p.*[mn:-1:1].^(-1),0];
		py1=py;
		py2=py;
		for i=1:mn+1
			py2(i)=py(i)*(-1)^(mn-i+1);
		end
		A(m,n)=sum(py1-py2);
		A(n,m)=A(m,n);
	end
end
A


function pn = Legen_pw(n)
```

```
% Legendre Polynomials
%   pn = power coefficients
%   n  = order of Legendre polynomials
pbb=[1]; if n==0, pn=pbb; break ; end
pb=[1,0]; if n==1, pn=pb; break; end
for i=2:n;
        pn=((2*i-1)*[pb,0] - (i-1)*[0,0,pbb])/i;
        pbb=pb; pb=pn;
end
```

```
cef =

    1.0000         0         0         0         0         0
    1.5000         0   -0.5000         0         0         0
    2.5000         0   -1.5000         0         0         0
    4.3750         0   -3.7500         0    0.3750         0
    7.8750         0   -8.7500         0    1.8750         0


A =

    0.6667         0         0         0         0
         0    0.4000         0         0         0
         0         0    0.2857         0         0
         0         0         0    0.2222         0
         0         0         0         0    0.1818
```

6. (Problem 5.19) Evaluate the following improper integral accurately up to the sixth decimal place by the extended trapezoidal rule :

$$I = \int_{-\infty}^{+\infty} \frac{\exp\left(-x^2\right)}{1+x^2} dx$$

Using the extended trapezoidal rule,

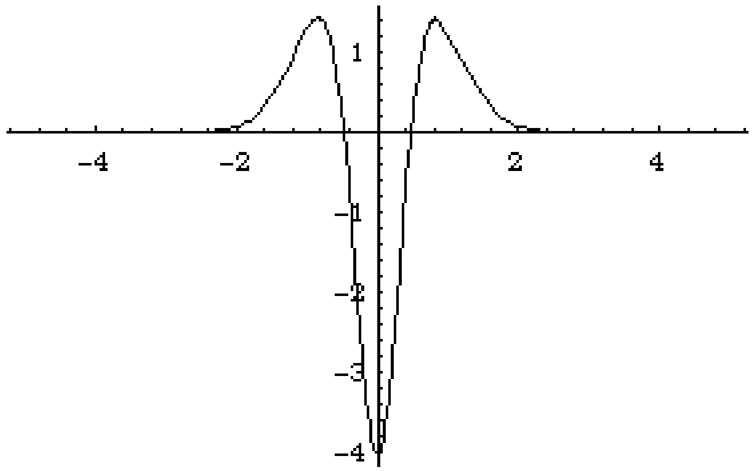
$$I = \int_{-\infty}^{+\infty} \frac{\exp\left(-x^2\right)}{1+x^2} dx \approx \frac{h}{2}\left(f_1 + 2 f_2 + ... + f_{n+1}\right)$$

where

$$f(x) = \frac{\exp\left(-x^2\right)}{1+x^2}$$

$$f''(x) = \frac{2\left(-2 + 7x^2 + 7x^4 + 2x^6\right)}{\left(1+x^2\right)^3} \exp\left(-x^2\right)$$

$$E \approx -\frac{(b-a)^3}{12n^2}\bar{f}'' \qquad \text{with} \qquad \bar{f}'' = \text{average of } f'' \text{ in } (a,b)$$



Noting that

$$f(\pm 5) = 5.34152 \times 10^{-13}$$

we can set up the integration interval $(0,5)$ because of symmetry of the function *f*, and also by using that

$$E \le \frac{(b-a)^3}{12n^2} \max_{x \in (a,b)} |f''(x)|,$$

we can set up the number of integration point *n* :

$$n^2 > \frac{2}{3} \times 5^3 \times 10^6 \quad \Rightarrow \quad n > 9129.$$

```
% Homework #4_6 : Problem 5.19
format long
I=[];
j=0;
for n=1:20
        j=j+1;
        a=5;
        h=a/n;
        x=0:h:5;
        f=exp(-x.^2)./(1+x.^2);
        I(j)=h*(sum(f)-(f(1)+f(length(f)))/2);
end
plot(I)
I '
```

```
ans =

   2.50000000000134
   1.25066567384075
   0.86076620945326
   0.72758235889289
   0.68761519596386
```

0.67618784416820
0.67293869187530
0.67201438126689
0.67175135011756
0.67167649190396
0.67165518678188
0.67164912316026
0.67164739739726
0.67164690622877
0.67164676643754
0.67164672665163
0.67164671532818
0.67164671210542
0.67164671118820
0.67164671092715

7. (Problem 5.21) Evaluate the following improper integrals accurately up to the sixth decimal place by the extended trapezoid rule, with exponential transformation given by Eq. (5.4.10) :

(a) $$I_1 = \int_0^1 \frac{\tan x}{x^{0.7}} dx$$

(b) $$I_2 = \int_0^1 \frac{\exp x}{\sqrt{1-x^2}} dx .$$

The exponential transformation

$$x = \frac{a+b+(b-a)\tanh z}{2} \qquad \text{and} \qquad \frac{dx}{dz} = \frac{b-a}{2\cosh^2 z}$$

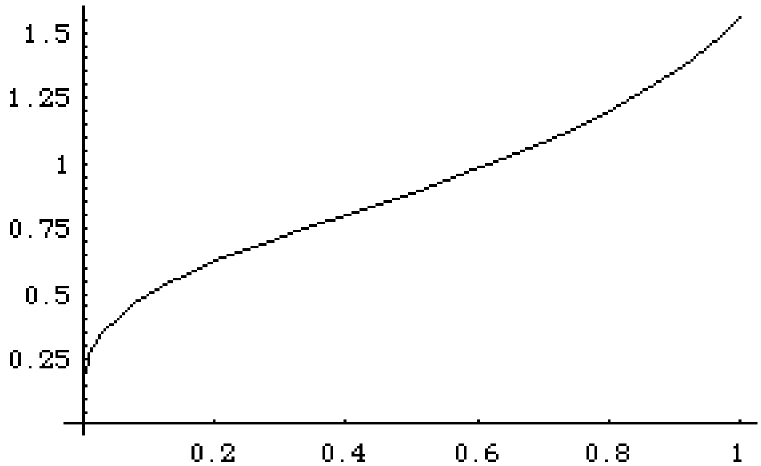
yields the relation

$$I = \int_a^b f(x)dx = \int_{-\infty}^{+\infty} f(x(z))\frac{dx}{dz}dz = \int_{-\infty}^{+\infty} f\left(\frac{a+b+(b-a)\tanh z}{2}\right)\frac{b-a}{2\cosh^2 z}dz = \int_{-\infty}^{+\infty} g(z)dz$$

where

$$g(z) = f\left(\frac{a+b+(b-a)\tanh z}{2}\right)\frac{b-a}{2\cosh^2 z}$$

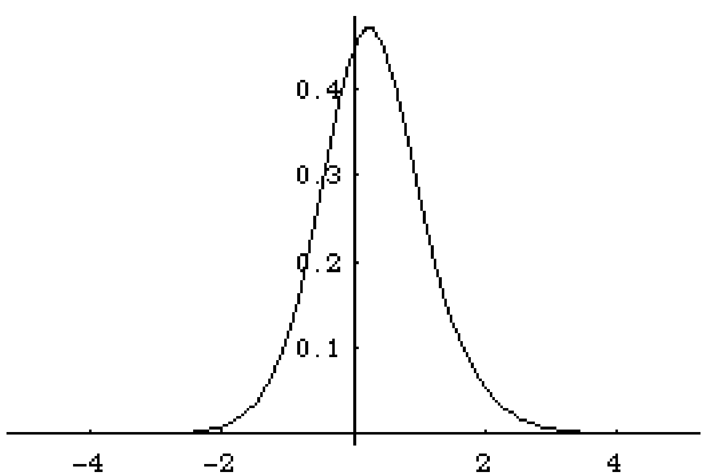
(a) The profile of the integrand is shown in the following :



In order to find out the profiles of the function transformed, the following MATHEMATICA script program may be applied :

```
a=0;
b=1;
x=(a+b+(b-a)*Tanh[z])/2;
dxdz=(b-a)/(2*Cosh[z]^2);
fa=Tan[x]/x^0.7;
g=Simplify[fa*dxdz]
Plot[g,{z,-5,5},PlotRange->All]
```

```
Out[43]=
                   2     1 + Tanh[z]
0.812252 Sech[z]  Tan[-----------]
                              2
---------------------------------
                           0.7
           (1 + Tanh[z])
```



Noting that

$$g(+10) = 1.02182 \times 10^{-11}$$

we can say if the interval (-10,10) is taken for the improper integral, it will be sufficient. The following value of the integral is obtained by using MATHEMATICA to check the result we shall obtain by MATLAB :

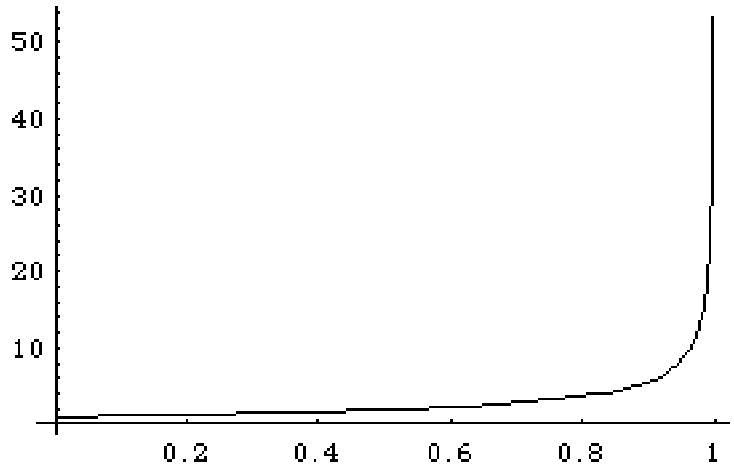
Out[67]=
0.906346

MATLAB program is made as follows :

```
% HW#4_7 : Problem 5.21
it=0;
Iit=[];
for n=10:10:100
        it=it+1;
        zmin=-10;
        zmax=10;
        h=(zmax-zmin)/n;
        z=zmin:h:zmax;
        a=0;
        b=1;
        x=(a+b+(b-a)*tanh(z))/2;
        dxdz=(b-a)./(2*cosh(z).^2);
        g=(tan(x)./(x.^0.7)).*dxdz;
        format long I
        I=h*(sum(g)-(g(1)+g(length(g)))/2);
        Iit(it)=I;
end
format long Iit
Iit'
```

```
ans =

   1.01803610145232
   0.90716888130764
   0.90635146646383
   0.90634597797740
   0.90634590519590
   0.90634590460080
   0.90634590462930
   0.90634590464961
   0.90634590466357
   0.90634590467357
```

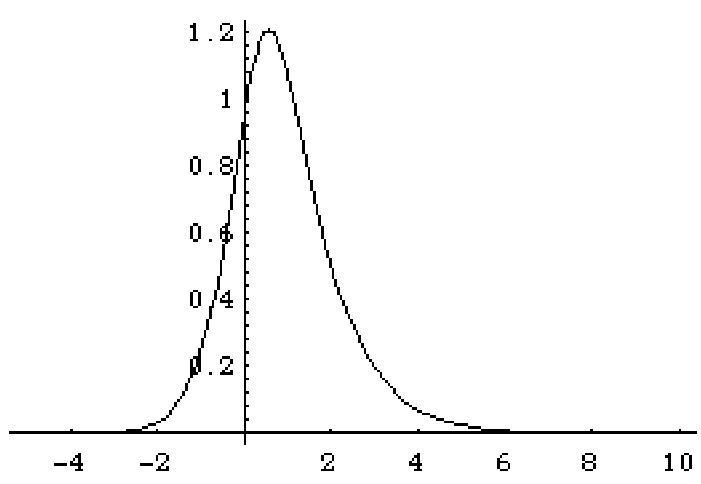
(b) Similarly, the profile of the integrand is given as follows :



and MATHMATICA program is used to find out the profile of the transformed function and the integrand after the exponential trasformation :

```
In[55]:=
a=0;
b=1;
x=(a+b+(b-a)*Tanh[z])/2;
dxdz=(b-a)/(2*Cosh[z]^2);
fa=Exp[x]/Sqrt[1-x^2];
g=Simplify[fa*dxdz]
Plot[g,{z,-5,10},PlotRange->All]
```

```
Out[60]=
 (1 + Tanh[z])/2        2
E                Sech[z]
--------------------------
                        2
            (1 + Tanh[z])
2 Sqrt[1 - --------------]
                  4
```



Using MATHEMATICA, we can integrate as follows :

Out[66]=
3.10438

Now, we shall use MATLAB to integrate the function in (b) :

```
% HW#4_7 : Problem 5.21
it=0;
Iit=[];
for n=10:10:100
        it=it+1;
%       zmin=-10;zmax=10;
        zmin=-5;zmax=10;
        h=(zmax-zmin)/n;
        z=zmin:h:zmax;
        a=0;
        b=1;
        x=(a+b+(b-a)*tanh(z))/2;
        dxdz=(b-a)./(2*cosh(z).^2);
        g=(exp(x)./sqrt(1-x.^2)).*dxdz;
%       g=(tan(x)./(x.^0.7)).*dxdz;
        format long I
        I=h*(sum(g)-(g(1)+g(length(g)))/2);
        Iit(it)=I;
end
format long Iit
Iit'
```

```
ans =

   3.06510854048172
   3.10425652839440
   3.10415143311500
   3.10415494305979
   3.10415643036235
   3.10415724125301
   3.10415773109577
   3.10415804938454
   3.10415826776853
   3.10415842406089
```