Homework #2

MEAM 501 Fall 1998

Due Date October 20

One-dimensional Finite Element Method and Related Eigenvalue Problems Let us consider axial vibration of an elastic bar, whose length is L while the axial rigidity is EA, shown in Fig. 1:

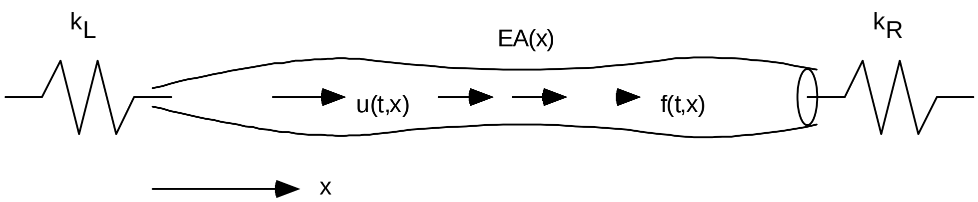

Figure 1 Vibration of an Elastic Bar in the Axial Direction

Suppose that the left and right end points are supported by two discrete springs whose spring constant is given by k_L and k_R. The equation of motion of this elastic bar is written as

$$\rho A \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x} \right) + f \quad in \quad (0, L)$$

where ρ is the mass density, and the boundary condition is written by

$$-EA\frac{\partial u}{\partial x} = -k_L u \quad at \quad x = 0 \qquad and \qquad EA\frac{\partial u}{\partial x} = -k_R u \quad at \quad x = L.$$

We shall apply the weighted residual method that is constructed by the finite element method to derive a discrete system of the axial vibration problem. To this end, let the domain $(0,L)$ be decomposed into N_E number of finite elements Ω_e, $e = 1,....,N_E$, and let each finite element consist of M number nodes in which the axial displacement is assumed to be a M-1 degree polynomial:

$$u(t,x)=\sum_{j=1}^{M} u^{e}{}_{j}(t)N_{j}(s)=\{N_1(s) \quad ..... \quad N_M(s)\}\begin{Bmatrix} u^{e}{}_{1}(t) \\ : \\ u^{e}{}_{M}(t) \end{Bmatrix}=\mathbf{N}\mathbf{u}_{e}$$

$$x=\sum_{j=1}^{M} x^{e}{}_{j}N_{j}(s)=\{N_1(s) \quad ..... \quad N_3(s)\}\begin{Bmatrix} x^{e}{}_{1} \\ : \\ x^{e}{}_{3} \end{Bmatrix}=\mathbf{N}\mathbf{x}_{e}$$

$$N_j(s)=\prod_{\substack{k=1\\ k\neq j}}^{M}\frac{x-x_k}{x_j-x_k}$$

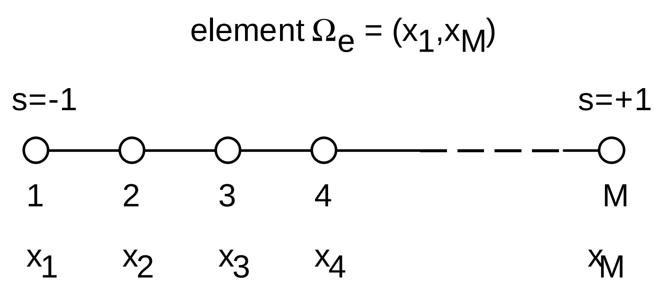

Figure X A Finite Element Ω_e = (x_1, x_M)

Noting that the weighted residual formulation of the equation of the motion and the boundary condition may be represented by the integral form

$$\int_0^L\left\{\mathrm{r}\,A\frac{\partial^2 u}{\partial t^2}w+EA\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right\}dx+k_L u(t,0)w(t,0)+k_R u(t,L)w(t,L)=\int_0^L fwdx \quad , \quad \forall w$$

that is

$$\sum_{e=1}^{N_E}\int_{\Omega_e}\left\{\mathrm{r}\,A\frac{\partial^2 u}{\partial t^2}w+EA\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right\}dx+k_L u(t,0)w(t,0)+k_R u(t,L)w(t,L)=\sum_{e=1}^{N_e}\int_{\Omega_e} fwdx \quad , \quad \forall w$$

the finite element approximation of the solution u and weighting function w in each finite element Ω_e using the Lagrange polynomial, yields the following discrete problem

$$\sum_{e=1}^{N_E}\sum_{i=1}^{M}\sum_{j=1}^{M} w^e{}_i \left\{ \left(\int_{\Omega_e} \mathrm{r} A N_i N_j dx \right) \frac{d^2 u^e{}_j}{dt^2} + \left(\int_{\Omega_e} EA \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} dx u^e{}_j \right) \right\}$$
$$+ k_L u^1{}_1 w^1{}_1 + k_R u^{N_E}{}_M w^{N_E}{}_M = \sum_{e=1}^{N_e}\sum_{i=1}^{M} w^e{}_i \int_{\Omega_e} f N_i dx \quad , \quad \forall w$$

Here we have applied the finite element approximation of the weighting function w:

$$w(x) = \sum_{i=1}^{M} w^e{}_i N_i(s) = \mathbf{N}\mathbf{w}_e$$

in a finite element Ω_e. Matrices

$$\mathbf{M}_e = [m_{ij}] \quad , \quad m_{ij} = \int_{\Omega_e} \mathrm{r} A N_i N_j dx$$

and

$$\mathbf{K}_e = [k_{ij}] \quad , \quad k_{ij} = \int_{\Omega_e} EA \frac{dN_i}{dx}\frac{dN_j}{dx} dx$$

are called the element mass and stiffness matrices, respectively. Modifying the element stiffness matrices of the first and last finite elements as follows:

$$\mathbf{K}_1 = \begin{bmatrix} k_{11} + k_L & ..... & k_{1M} \\ : & & : \\ k_{M1} & ..... & k_{MM} \end{bmatrix} \quad and \quad \mathbf{K}_{N_E} = \begin{bmatrix} k_{11} & ..... & k_{1M} \\ : & & : \\ k_{M1} & ..... & k_{MM} + k_R \end{bmatrix}$$

and defining the element generalized force vector $\mathbf{f}_e$ by

$$\mathbf{f}_e = \{f_i\} \quad , \quad f_i = \int_{\Omega_e} f N_i dx = \int_{x^e_1}^{x^e_M} f(x(s)) N_i(s) \left(\frac{x^e_M - x^e_1}{2} \right) ds$$

we can represent the discrete system as

$$\sum_{e=1}^{N_E} \mathbf{w}_e{}^T \left(\mathbf{M}_e \frac{d^2 \mathbf{u}_e}{dt^2} + \mathbf{K}_e \mathbf{u}_e \right) = \sum_{e=1}^{N_E} \mathbf{w}_e{}^T \mathbf{f}_e \qquad \forall \mathbf{w}_e$$

Defining the global generalized displacement **u** whose restriction into a finite element Ω_e is given by the element generalized displacement vector $\mathbf{u}_e$, and similarly defining the global generalized weighting function **w**, we may write

$$\mathbf{w}^T \left(\mathbf{M} \frac{d^2 \mathbf{u}}{dt^2} + \mathbf{K}\mathbf{u} \right) = \mathbf{w}^T \mathbf{f} \qquad \forall \mathbf{w}$$

that is

$$\mathbf{M} \frac{d^2 \mathbf{u}}{dt^2} + \mathbf{K}\mathbf{u} = \mathbf{f}$$

where the global generalized force vector **f** is defined similarly as **u**.

In order to find the mass and stiffness matries, you may use the following MATHEMATICA program:

```
(* Finite Element Method *)
(* for *)
(* Axial Vibration of an Elastic Bar *)
(* fem1 *)
(* Set up the Lagrange Polynomials in a Finite Element *)
```

```
M=3;
Nm=Table[1,{i,1,M}];
si=Table[-1+2*(i-1)/(M-1),{i,1,M}];
Block[{i,j},
	Do[Nmi=Nm[[i]];
	Do[Nmj=If[j==i,1,(s-si[[j]])/(si[[i]]-si[[j]])];
		Nmi=Nmi*Nmj,{j,1,M}];
	Nm[[i]]=Nmi,{i,1,M}]]
Nm
Plot[Release[Nm],{s,-1,1},AxesLabel->{"s","Lagrange Polynomials"}]
(* Compute the Element Mass and Stiffness Matrices *)
rA=1;
EA=1;
Me=Integrate[rA*Outer[Times,Nm,Nm],{s,-1,1}]
DNm=D[Nm,s];
Ke=Integrate[EA*Outer[Times,DNm,DNm],{s,-1,1}]
(* Set up the Nodal Coordinates and Element Connectivity of the Whole Structure *)
L=1;
nelx=5
nx=(M-1)*nelx+1
x=Table[(i-1)*L/(nx-1),{i,1,nx}]
ijk=Table[0,{nel,1,nelx},{i,1,M}];
Block[{i,nel},
	Do[ijk[[nel,i]]=(M-1)*(nel-1)+i,{nel,1,nelx},{i,1,M}]]
ijk
(* Forming the Global Mass and Stiffness Matrices by Assembling & Adding Boundary *)
 (* Condition *)
neq=nx;
Mg=Table[0,{i,1,neq},{j,1,neq}];
Kg=Table[0,{i,1,neq},{j,1,neq}];
Block[{i,j,nel},
	Do[le=x[[ijk[[nel,M]]]]-x[[ijk[[nel,1]]]];
	Do[ijki=ijk[[nel,i]];
```

```
        Do[ijkj=ijk[[nel,j]];
        Mg[[ijki,ijkj]]=Mg[[ijki,ijkj]]+le*Me[[i,j]];
        Kg[[ijki,ijkj]]=Kg[[ijki,ijkj]]+Ke[[i,j]]/le,
        {j,1,M}],{i,1,M}],{nel,1,nelx}]]
Mg
kL=1000;
kR=1000;
Kg[[1,1]]=Kg[[1,1]]+kL;
Kg[[neq,neq]]=Kg[[neq,neq]]+kR;
Kg
```

Now, we shall assume that you can obtain global mass and stiffness matrices for your choice of M and N_E (i.e. nelx in MATHEMATICA). Using such $\mathbf{M}$ and $\mathbf{K}$, solve the following problem for M = 5 and N_E = 2, together with $EA = L = \rho A = 1$.

1. For the case that $k_L = k_R = 0$, compute det($\mathbf{K}$), rank($\mathbf{K}$), and find the null space $N(\mathbf{K})$ as well as the range of $\mathbf{K}$, i.e., $R(\mathbf{K})$.
2. For the case that $k_L = k_R = 10000$, find the eigenvalues and eigenvectors of the global stiffness matrix.
3. For the force $f(x) = \begin{cases} +1 & if \quad x \in (0, L/2) \\ -1 & if \quad x \in (L/2, L) \end{cases}$, find its finite element approximation $\mathbf{f}$ corresponding to the choice of M and N_E in above. Then compute $\mathbf{x}_i{}^T\mathbf{f}$, $i = 1,\dots,neq$, where $\mathbf{x}_i$ are the eigenvectors obtained in 2.
4. Orthonormalize the eigenvectors $\mathbf{x}_i$ obtained in 2 with respect to the mass matrix $\boldsymbol{M}$.
5. Diagonalize the mass matrix $\mathbf{M}$ by using the Householder transformation $\mathbf{P}$.
6. Make $\mathbf{QR}$ decomposition of $\mathbf{M}$.
7. Solve the generalized eigenvalue problem $\mathbf{Kx} = \lambda\,\mathbf{Mx}$ for the case that $k_L = k_R = 0$.
8. Find the dynamical response of the bar at $x = 0.5$ for the case that

$$k_L = k_R = 10000,\ f(t,x) = e^{-2t}\sin\left(2\pi\frac{x}{L}\right),\ u_0(x) = 0.1\sin\left(\pi\frac{x}{L}\right),\ v_0(0) = 0$$

In order solve the above problem, you may have many questions. Since I have made this homework problem at the first time, description need not be very clear, and you may have difficulty to understand the problem. Please feel free to ask any questions.