Solution of Homework #2, 1998 Fall

MEAM 501 Analytical Methods in Mechanics and Mechanical Engineering

One-dimensional Finite Element Method and Related Eigenvalue Problems *Let us consider axial vibration of an elastic bar, whose length is L while the axial rigidity is EA, shown in Fig. 1:*

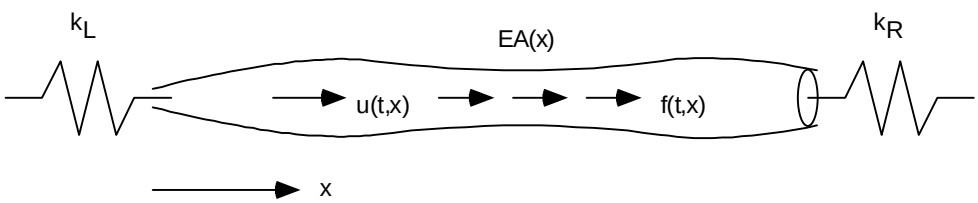


Figure 1 Vibration of an Elastic Bar in the Axial Direction

Suppose that the left and right end points are supported by two discrete springs whose spring constant is given by k_L and k_R. The equation of motion of this elastic bar is written as

$$\rho A\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) + f \qquad in \qquad (0,L)$$

where ρ *is the mass density, and the boundary condition is written by*

$$-EA\frac{\partial u}{\partial x} = -k_L u \qquad at \qquad x = 0 \qquad\qquad and \qquad\qquad EA\frac{\partial u}{\partial x} = -k_R u \qquad at \qquad x = L.$$

We shall apply the weighted residual method that is constructed by the finite element method to derive a discrete system of the axial vibration problem. To this end, let the domain (0,L) be decomposed into N_E number of finite elements Ω_e, e = 1,....,N_E, and let each finite element consist of M number nodes in which the axial displacement is assumed to be a M-1 degree polynomial:

$$u(t,x)=\sum_{j=1}^{M} u^{e}{}_{j}(t)N_{j}(s)=\{N_{1}(s) \quad ..... \quad N_{M}(s)\}\begin{Bmatrix} u^{e}{}_{1}(t) \\ : \\ u^{e}{}_{M}(t) \end{Bmatrix}=\mathbf{N}\mathbf{u}_{e}$$

$$x=\sum_{j=1}^{M} x^{e}{}_{j}N_{j}(s)=\{N_{1}(s) \quad ..... \quad N_{3}(s)\}\begin{Bmatrix} x^{e}{}_{1} \\ : \\ x^{e}{}_{3} \end{Bmatrix}=\mathbf{N}\mathbf{x}_{e}$$

$$N_{j}(s)=\prod_{\substack{k=1 \\ k\neq j}}^{M}\frac{x-x_{k}}{x_{j}-x_{k}}$$

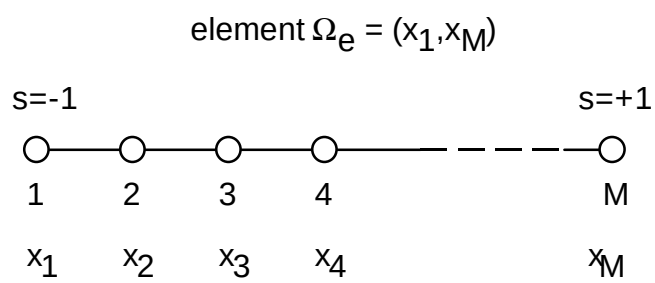


Figure X A Finite Element $W_e = (x_1, x_M)$

Noting that the weighted residual formulation of the equation of the motion and the boundary condition may be represented by the integral form

$$\int_{0}^{L}\left\{\mathrm{r}\,A\frac{\partial^{2}u}{\partial t^{2}}w+EA\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right\}dx+k_{L}u(t,0)w(t,0)+k_{R}u(t,L)w(t,L)=\int_{0}^{L} fwdx \quad , \quad \forall w$$

that is

$$\sum_{e=1}^{N_{E}}\int_{\Omega_{e}}\left\{\mathrm{r}\,A\frac{\partial^{2}u}{\partial t^{2}}w+EA\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right\}dx+k_{L}u(t,0)w(t,0)+k_{R}u(t,L)w(t,L)=\sum_{e=1}^{N_{e}}\int_{\Omega_{e}} fwdx \quad , \quad \forall w$$

the finite element approximation of the solution u and weighting function w in each finite element Ω_e *using the Lagrange polynomial, yields the following discrete problem*

$$\sum_{e=1}^{N_E}\sum_{i=1}^{M}\sum_{j=1}^{M} w^e{}_i \left\{ \left(\int_{\Omega_e} \rho A N_i N_j dx \right) \frac{d^2 u^e{}_j}{dt^2} + \left(\int_{\Omega_e} EA \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} dx u^e{}_j \right) \right\}$$
$$+ k_L u^1{}_1 w^1{}_1 + k_R u^{N_E}{}_M w^{N_E}{}_M = \sum_{e=1}^{N_e}\sum_{i=1}^{M} w^e{}_i \int_{\Omega_e} f N_i dx \quad , \quad \forall w$$

Here we have applied the finite element approximation of the weighting function w:

$$w(x) = \sum_{i=1}^{M} w^e{}_i N_i(s) = \mathbf{N}\mathbf{w}_e$$

in a finite element Ω_e*. Matrices*

$$\mathbf{M}_e = [m_{ij}] \quad , \quad m_{ij} = \int_{\Omega_e} \rho A N_i N_j dx$$

and

$$\mathbf{K}_e = [k_{ij}] \quad , \quad k_{ij} = \int_{\Omega_e} EA \frac{dN_i}{dx}\frac{dN_j}{dx} dx$$

are called the element mass and stiffness matrices, respectively. Modifying the element stiffness matrices of the first and last finite elements as follows:

$$\mathbf{K}_1 = \begin{bmatrix} k_{11} + k_L & \cdots\cdots & k_{1M} \\ \vdots & & \vdots \\ k_{M1} & \cdots\cdots & k_{MM} \end{bmatrix} \quad and \quad \mathbf{K}_{N_E} = \begin{bmatrix} k_{11} & \cdots\cdots & k_{1M} \\ \vdots & & \vdots \\ k_{M1} & \cdots\cdots & k_{MM} + k_R \end{bmatrix}$$

and defining the element generalized force vector $\mathbf{f}_e$ *by*

$$\mathbf{f}_e = \{f_i\} \quad , \quad f_i = \int_{\Omega_e} f N_i dx = \int_{x^e{}_1}^{x^e{}_M} f(x(s)) N_i(s) \left(\frac{x^e{}_M - x^e{}_1}{2} \right) ds$$

we can represent the discrete system as

$$\sum_{e=1}^{N_E} \mathbf{w}_e{}^T \left(\mathbf{M}_e \frac{d^2 \mathbf{u}_e}{dt^2} + \mathbf{K}_e \mathbf{u}_e \right) = \sum_{e=1}^{N_E} \mathbf{w}_e{}^T \mathbf{f}_e \qquad \forall \mathbf{w}_e$$

Defining the global generalized displacement ***u*** *whose restriction into a finite element* W_e *is given by the element generalized displacement vector* $\boldsymbol{u}_e$*, and similarly defining the global generalized weighting function* ***w****, we may write*

$$\mathbf{w}^T \left(\mathbf{M} \frac{d^2 \mathbf{u}}{dt^2} + \mathbf{K}\mathbf{u} \right) = \mathbf{w}^T \mathbf{f} \qquad \forall \mathbf{w}$$

that is

$$\mathbf{M} \frac{d^2 \mathbf{u}}{dt^2} + \mathbf{K}\mathbf{u} = \mathbf{f}$$

where the global generalized force vector ***f*** *is defined similarly as* ***u****.*

In order to find the mass and stiffness matries, you may use the following MATHEMATICA program:

```
(* Finite Element Method *)
(* for *)
(* Axial Vibration of an Elastic Bar *)
(* fem1 *)
(* Set up the Lagrange Polynomials in a Finite Element *)
```

```
M=3;
Nm=Table[1,{i,1,M}];
si=Table[-1+2*(i-1)/(M-1),{i,1,M}];
Block[{i,j},
        Do[Nmi=Nm[[i]];
        Do[Nmj=If[j==i,1,(s-si[[j]])/(si[[i]]-si[[j]])];
                Nmi=Nmi*Nmj,{j,1,M}];
        Nm[[i]]=Nmi,{i,1,M}]]
Nm
Plot[Release[Nm],{s,-1,1},AxesLabel->{"s","Lagrange Polynomials"}]
(* Compute the Element Mass and Stiffness Matrices *)
rA=1;
EA=1;
Me=Integrate[rA*Outer[Times,Nm,Nm],{s,-1,1}]
DNm=D[Nm,s];
Ke=Integrate[EA*Outer[Times,DNm,DNm],{s,-1,1}]
(* Set up the Nodal Coordinates and Element Connectivity of the Whole Structure *)
L=1;
nelx=5
nx=(M-1)*nelx+1
x=Table[(i-1)*L/(nx-1),{i,1,nx}]
ijk=Table[0,{nel,1,nelx},{i,1,M}];
Block[{i,nel},
        Do[ijk[[nel,i]]=(M-1)*(nel-1)+i,{nel,1,nelx},{i,1,M}]]
ijk
(* Forming the Global Mass and Stiffness Matrices by Assembling & Adding Boundary *)
 (* Condition *)
neq=nx;
Mg=Table[0,{i,1,neq},{j,1,neq}];
Kg=Table[0,{i,1,neq},{j,1,neq}];
Block[{i,j,nel},
        Do[le=x[[ijk[[nel,M]]]]-x[[ijk[[nel,1]]]]];
        Do[ijki=ijk[[nel,i]];
```

```
                Do[ijkj=ijk[[nel,j]];
                Mg[[ijki,ijkj]]=Mg[[ijki,ijkj]]+le*Me[[i,j]]/2;
                Kg[[ijki,ijkj]]=Kg[[ijki,ijkj]]+2*Ke[[i,j]]/le,
                {j,1,M}],{i,1,M}],{nel,1,nelx}]]
Mg
kL=1000;
kR=1000;
Kg[[1,1]]=Kg[[1,1]]+kL;
Kg[[neq,neq]]=Kg[[neq,neq]]+kR;
Kg
```

Now, we shall assume that you can obtain global mass and stiffness matrices for your choice of M and N_E (i.e. nelx in MATHEMATICA). Using such ***M*** *and* ***K****, solve the following problem for M = 5 and N_E = 2, together with EA = L = r A = 1.*

To solve the following problems, we shall use MATLAB instead of MATHEMATICA. That is, we shall apply the MATLAB programs :

```
% Homework #2, 1998 Fall
% MEAM 501   Analytical Methods in Mechanics and Mechanical Engineering
%
%   rA d^2u/dt^2 - d ( EA du/dx )/dx = f     in     (0,L)
%       u(0,x)=u0(x)   ,   du/dt (0,x) = v0(x)
%       EA du/dx = ku  at  x=0   ,   EA du/dx = -ku   at   x=L
%
%  m-1 degree Lagrange polynomials at each finite element
%  quadrature is the trapezoid rule
%  finite element method with Lagrange elements
%
____________________________________________________________________________________
%
```

```
rA=1;
EA=1;
L=1;
% f=exp(-2*t)*sin(2*pi*x/L);
% input data : element connectivities and nodal coordinates
%    nelx   = number of finite elements
%    nx    = number of nodes defining elements
%    ijk     = element connectivity
%    x      = nodal coordinates
nelx=2;
nx=3;
ijk=[1,1,2;2,2,3];
x=[0,0.5,1];
% number of integration points for the trapezoid rule ( 100 subintervals )
nint=100;
ds=2/nint;
s=-1:ds:1;
% forming the element mass and stiffness matrices
%    m     = number of nodes within an element
%    me    = element mass matrix
%    ke    = element stiffness matrix
m=5;
me=zeros(m);
ke=zeros(m);
for i=1:m
   Ni=lagpoly(s,i,m);
   dNi=dlagpoly(s,i,m);
   for j=i:m
      Nj=lagpoly(s,j,m);
      dNj=dlagpoly(s,j,m);
      meij=rA*Ni.*Nj;
      keij=EA*dNi.*dNj;
      me(i,j)=ds*sum(meij)-0.5*ds*(meij(1)+meij(nint+1));
```

```
      me(j,i)=me(i,j);
      ke(i,j)=ds*sum(keij)-0.5*ds*(keij(1)+keij(nint+1));
      ke(j,i)=ke(i,j)
   end
end
me
ke
% forming the global mass and stiffness matrices
%    mg    =  global mass matrix
%    kg     = global stiffness matrix
neq=(m-1)*2+1;
mg=zeros(neq);
kg=zeros(neq);
for nel=1:nelx
   xnel=x(nel+1)-x(nel);
   for i=1:m
      ijki=(m-1)*(nel-1)+i;
      for j=1:m
         ijkj=(m-1)*(nel-1)+j;
         mg(ijki,ijkj)=mg(ijki,ijkj)+me(i,j)*xnel/2;
         kg(ijki,ijkj)=kg(ijki,ijkj)+ke(i,j)*2/xnel;
      end
   end
end
mg
kg
%
% Problem 1 : Properties of the matrix kg
determinant=det(kg)
rankkg=rank(kg)
[U,S,V]=svd(kg)
% Problem 2 : Eigenvalues and Eigenvectors of kg
kg(1,1)=kg(1,1)+1000;
```

```
kg(neq,neq)=kg(neq,neq)+1000;
[X,R]=eig(kg)
% Problem 3 : Generalized Force Vector fg
fe=zeros(m,1);
for i=1:m
   Ni=lagpoly(s,i,m);
   fei=1*Ni;
   fe(i)=ds*sum(fei)-0.5*ds*(fei(1)+fei(nint+1));
end
fe
fg=zeros(neq,1);
fg(1:m,1)=fe*L/4;
fg(m:neq,1)=fg(m:neq,1)-fe*L/4;
fg
plot(fg)
xlabel('i')
ylabel('fg')
title('nodal forces')
fgeigen=(fg'*X)'
pause
% Problem 4 : Orthonomalization of X with respect to mg
Y=zeros(neq);
yi=X(:,1);
Y(:,1)=yi/sqrt(yi'*mg*yi);
for i=2:neq
   yi=X(:,i);
   for j=1:i-1
      yi=yi-(Y(:,j)'*mg*X(:,i))*Y(:,j);
   end
   Y(:,i)=yi/sqrt(yi'*mg*yi);
end
Y
check=zeros(neq);
```

```
for i=1:neq
   for j=1:neq
      check(i,j)=Y(:,i)'*mg*Y(:,j);
   end
end
check
% Problem 5 : Diagonalization of mg by the Householder Transformation
iteration=0;
error=1;
tolerance=0.00001;
mgd=mg;
while error>tolerance
   R=mgd;
   Q=eye(neq);
   for i=1:neq-1
      mi=R(:,i);
      if i>1, mi(1:i-1,1)=zeros(i-1,1);, end
      normmi=sqrt(mi'*mi);
      vi=mi;
      vi(i)=vi(i)-normmi;
      P=eye(neq)-2*vi*vi'/(vi'*vi);
      R=P*R;
      Q=Q*P;
   end
   mgn=R*Q;
   iteration=iteration+1;
   error=norm(mgn-mgd)/norm(mgn);
   mgd=mgn;
   if iteration>200, break, end
end
iteration
error
mgn
```

```
% Problem 6 : QR decomposition of mg
[Q,R]=qr(mg)
% Problem 7 : Generalized Eigenvalue Problem
kg(1,1)=kg(1,1)-1000;
kg(neq,neq)=kg(neq,neq)-1000;
[X,R]=eig(kg,mg)
%
% Problem 8 : Dynamical Response
%
kg(1,1)=kg(1,1)+1000;
kg(neq,neq)=kg(neq,neq)+1000;
[X,R]=eig(kg,mg)
% distributed force
fg=zeros(neq,1);
fe1=zeros(m,1);
for i=1:m
   Ni=lagpoly(s,i,m);
   fei=sin(2*pi*0.25*(1+s)).*Ni;
   fe1(i)=ds*sum(fei)-0.5*ds*(fei(1)+fei(nint+1));
end
fe1;
fe2=zeros(m,1);
for i=1:m
   Ni=lagpoly(s,i,m);
   fei=sin(2*pi*0.25*(3+s)).*Ni;
   fe2(i)=ds*sum(fei)-0.5*ds*(fei(1)+fei(nint+1));
end
fe2;
fg(1:m,1)=fe1*L/4;
fg(m:neq,1)=fg(m:neq,1)+fe2*L/4;
fg
fgeig=fg'*X
% initial displacement
```

```
u0e1=zeros(m,1);
for i=1:m
   Ni=lagpoly(s,i,m);
   fei=sin(pi*0.25*(1+s)).*Ni;
   u0e1(i)=ds*sum(fei)-0.5*ds*(fei(1)+fei(nint+1));
end
u0e1;
u0e2=zeros(m,1);
for i=1:m
   Ni=lagpoly(s,i,m);
   fei=sin(pi*0.25*(3+s)).*Ni;
   u0e2(i)=ds*sum(fei)-0.5*ds*(fei(1)+fei(nint+1));
end
u0e2;
u0g=zeros(neq,1);
u0g(1:m,1)=u0e1*L/4;
u0g(m:neq,1)=u0g(m:neq,1)+u0e2*L/4;
u0g
u0geig=u0g'*X
% two dominant eigenvectors
plot(X(:,8))
xlabel('i')
ylabel('eigenvector')
title('Dominant Eigenvector for the External Distributed Force')
pause
plot(X(:,9))
xlabel('i')
ylabel('eigenvector')
title('Dominant Eigenvector for the Initial Displacement')
pause
```

```
function li=lagpoly(x,i,n)
% (n-1) degree Lagrange polynomials evaluated at an arbitrary point x in (-1,1)
% n nodal points are equally distributed on [-1,1]
% The first node is at x = -1, while the last node is at x = +1
m=size(x,2);
xi=-1+2*(i-1)/(n-1);
Lij=ones(1,m);
for j=1:n
   xj=-1+2*(j-1)/(n-1);
   if j~=i, Lij=Lij.*(x-xj)/(xi-xj);, end
end
li=Lij;

function dli=dlagpoly(x,i,n)
% 1st derivative of the (n-1) degree Lagrange polynomials evaluated at an
arbitrary point x in (-1,1)
% n nodal points are equally distributed on [-1,1]
% The first node is at x = -1, while the last node is at x = +1
m=size(x,2);
xi=-1+2*(i-1)/(n-1);
dLij=zeros(1,m);
for k=1:n
   if k~=i
      xk=-1+2*(k-1)/(n-1);
      Lij=ones(1,m);
      for j=1:n
         xj=-1+2*(j-1)/(n-1);
         if j~=i & j~=k, Lij=Lij.*(x-xj)/(xi-xj);,end
      end
      dLij=dLij+Lij/(xi-xk);
   end
```

```
end
dli=dLij;
```

Computed Element Mass and Stiffness Matrices

```
me =

    0.1033    0.1041   -0.0612    0.0197   -0.0102
    0.1041    0.6321   -0.1354    0.0903    0.0197
   -0.0612   -0.1354    0.6603   -0.1354   -0.0612
    0.0197    0.0903   -0.1354    0.6321    0.1041
   -0.0102    0.0197   -0.0612    0.1041    0.1033


ke =

    2.6092   -3.6318    1.6217   -0.7840    0.1850
   -3.6318    8.8260   -7.5416    3.1315   -0.7840
    1.6217   -7.5416   11.8399   -7.5416    1.6217
   -0.7840    3.1315   -7.5416    8.8260   -3.6318
    0.1850   -0.7840    1.6217   -3.6318    2.6092
```

Computed Global Mass and Stiffness Matrices

```
mg =

  Columns 1 through 7

    0.0258    0.0260   -0.0153    0.0049   -0.0025         0         0
    0.0260    0.1580   -0.0339    0.0226    0.0049         0         0
   -0.0153   -0.0339    0.1651   -0.0339   -0.0153         0         0
    0.0049    0.0226   -0.0339    0.1580    0.0260         0         0
```

-0.0025	0.0049	-0.0153	0.0260	0.0516	0.0260	-0.0153
0	0	0	0	0.0260	0.1580	-0.0339
0	0	0	0	-0.0153	-0.0339	0.1651
0	0	0	0	0.0049	0.0226	-0.0339
0	0	0	0	-0.0025	0.0049	-0.0153

Columns 8 through 9

0	0
0	0
0	0
0	0
0.0049	-0.0025
0.0226	0.0049
-0.0339	-0.0153
0.1580	0.0260
0.0260	0.0258

kg =

Columns 1 through 7

10.4367	-14.5274	6.4867	-3.1361	0.7400	0	0
-14.5274	35.3041	-30.1665	12.5258	-3.1361	0	0
6.4867	-30.1665	47.3597	-30.1665	6.4867	0	0
-3.1361	12.5258	-30.1665	35.3041	-14.5274	0	0
0.7400	-3.1361	6.4867	-14.5274	20.8735	-14.5274	6.4867
0	0	0	0	-14.5274	35.3041	-30.1665
0	0	0	0	6.4867	-30.1665	47.3597
0	0	0	0	-3.1361	12.5258	-30.1665
0	0	0	0	0.7400	-3.1361	6.4867

Columns 8 through 9

0	0
0	0
0	0
0	0
-3.1361	0.7400
12.5258	-3.1361
-30.1665	6.4867
35.3041	-14.5274
-14.5274	10.4367

*1. For the case that $k_L = k_R = 0$, compute det(**K**), rank(**K**), and find the null space N(**K**) as well as the range of **K**, i.e., R(**K**).*

determinant =

1.6804e-005

rankkg =

8

U =

Columns 1 through 7

0.1062	-0.1158	0.2099	-0.2875	0.3564	-0.4306	0.4544
-0.3367	0.3619	-0.4047	0.4509	-0.2524	0.0415	0.2191
0.4660	-0.4878	0.0886	0.0826	-0.3961	0.4045	-0.0837

-0.3580	0.3430	0.3745	-0.4553	0.0164	0.3863	-0.3467
0.2451	0.0000	-0.5368	0.0000	0.5514	0.0000	-0.4864
-0.3580	-0.3430	0.3745	0.4553	0.0164	-0.3863	-0.3467
0.4660	0.4878	0.0886	-0.0826	-0.3961	-0.4045	-0.0837
-0.3367	-0.3619	-0.4047	-0.4509	-0.2524	-0.0415	0.2191
0.1062	0.1158	0.2099	0.2875	0.3564	0.4306	0.4544

Columns 8 through 9

-0.4675	0.3333
-0.4050	0.3333
-0.3026	0.3333
-0.1608	0.3333
0.0000	0.3333
0.1608	0.3333
0.3026	0.3333
0.4050	0.3333
0.4675	0.3333

S =

Columns 1 through 7

97.2281	0	0	0	0	0	0
0	92.4864	0	0	0	0	0
0	0	33.6950	0	0	0	0
0	0	0	26.3920	0	0	0
0	0	0	0	14.5173	0	0
0	0	0	0	0	8.5551	0
0	0	0	0	0	0	3.8378
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 9

0	0
0	0
0	0
0	0
0	0
0	0
0	0
0.9712	0
0	0.0000

V =

Columns 1 through 7

0.1062	-0.1158	0.2099	-0.2875	0.3564	-0.4306	0.4544
-0.3367	0.3619	-0.4047	0.4509	-0.2524	0.0415	0.2191
0.4660	-0.4878	0.0886	0.0826	-0.3961	0.4045	-0.0837
-0.3580	0.3430	0.3745	-0.4553	0.0164	0.3863	-0.3467
0.2451	0.0000	-0.5368	0.0000	0.5514	0.0000	-0.4864
-0.3580	-0.3430	0.3745	0.4553	0.0164	-0.3863	-0.3467
0.4660	0.4878	0.0886	-0.0826	-0.3961	-0.4045	-0.0837
-0.3367	-0.3619	-0.4047	-0.4509	-0.2524	-0.0415	0.2191
0.1062	0.1158	0.2099	0.2875	0.3564	0.4306	0.4544

Columns 8 through 9

-0.4675	0.3333
-0.4050	0.3333
-0.3026	0.3333

-0.1608	0.3333
0.0000	0.3333
0.1608	0.3333
0.3026	0.3333
0.4050	0.3333
0.4675	0.3333

From the singular value decomposition, the null space N is defined by the space spanned by the 9th column vector of V, that is, $\mathbf{v}_9^T = \{1,1,1,1,1,1,1,1,1\}/3$.

The range of the matrix K is the space spanned by the first 8 column vectors, $\mathbf{u}_1, ..., \mathbf{u}_8$, of U.

2. For the case that $k_L = k_R = 10000$, find the eigenvalues and eigenvectors of the global stiffness matrix.

X = (eigenvectors are the column vectors of the following matrix)

Columns 1 through 7

0.0030	-0.0015	0.0040	-0.0058	-0.0055	0.0103	-0.0100
0.3538	-0.1921	0.4341	-0.4999	-0.4092	0.3534	-0.3273
0.5010	-0.3541	0.3526	0.0018	0.1614	-0.4989	0.4735
0.3519	-0.4583	-0.1335	0.5001	0.3687	0.3551	-0.3690
0.0000	-0.5052	-0.5821	0.0000	-0.5840	0.0000	0.2547
-0.3519	-0.4583	-0.1335	-0.5001	0.3687	-0.3551	-0.3690
-0.5010	-0.3541	0.3526	-0.0018	0.1614	0.4989	0.4735
-0.3538	-0.1921	0.4341	0.4999	-0.4092	-0.3534	-0.3273
-0.0030	-0.0015	0.0040	0.0058	-0.0055	-0.0103	-0.0100

Columns 8 through 9

0.7070	0.7070

-0.0107	-0.0107
0.0052	0.0052
-0.0026	-0.0026
0.0000	0.0013
0.0026	-0.0026
-0.0052	0.0052
0.0107	-0.0107
-0.7070	0.7070

R = (the diagonals are the eigenvalues)

1.0e+003 *

Columns 1 through 7

0.0049	0	0	0	0	0	0
0	0.0012	0	0	0	0	0
0	0	0.0110	0	0	0	0
0	0	0	0.0227	0	0	0
0	0	0	0	0.0312	0	0
0	0	0	0	0	0.0901	0
0	0	0	0	0	0	0.0951
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 9

0	0
0	0
0	0
0	0
0	0

0	0
0	0
1.0107	0
0	1.0107

3. For the force $f(x)=\begin{cases}+1 & if \quad x\in(0,L/2)\\ -1 & if \quad x\in(L/2,L)\end{cases}$, *find its finite element approximation* $\mathbf{f}$ *corresponding to the choice of M and* N_E *in above. Then compute* $\mathbf{x}_i^T\mathbf{f}$, $i=1,....,neq$, *where* $\mathbf{x}_i$ *are the eigenvectors obtained in 2.*

fe = $\int_{-1}^{+1} N_i(s)ds \quad , \quad i=1,2,...,m=5$

0.1557
0.7108
0.2671
0.7108
0.1557

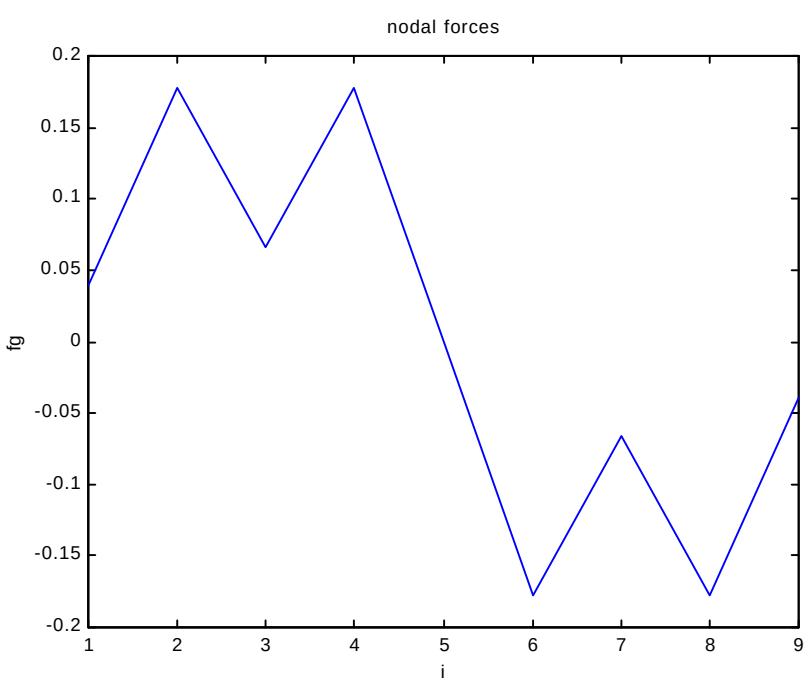
fg = global force vector

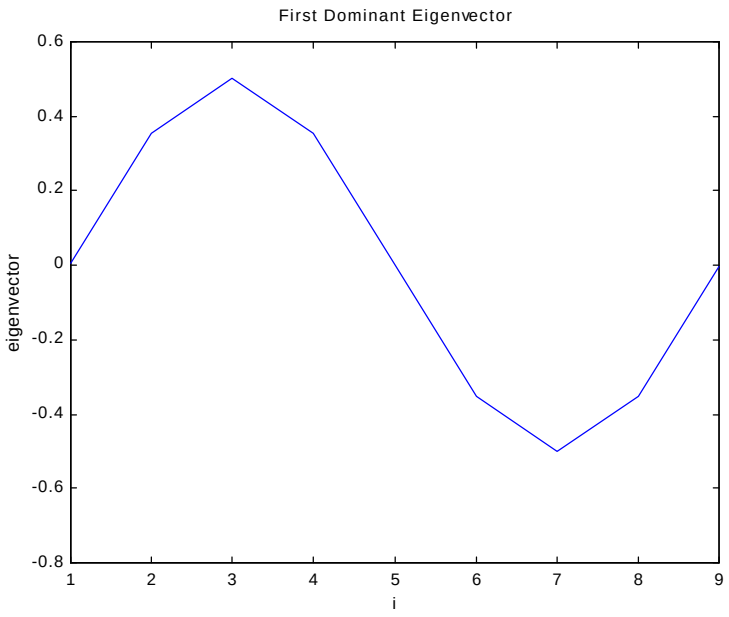
0.0389
0.1777
0.0668
0.1777
0.0000
-0.1777
-0.0668
-0.1777
-0.0389

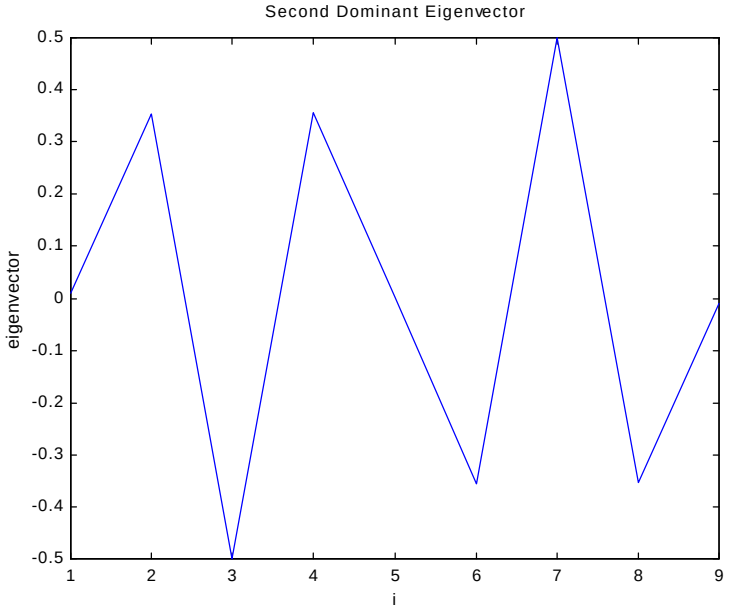
fgeigen = $\left(\mathbf{f}_g{}^T \mathbf{X}\right)^T$

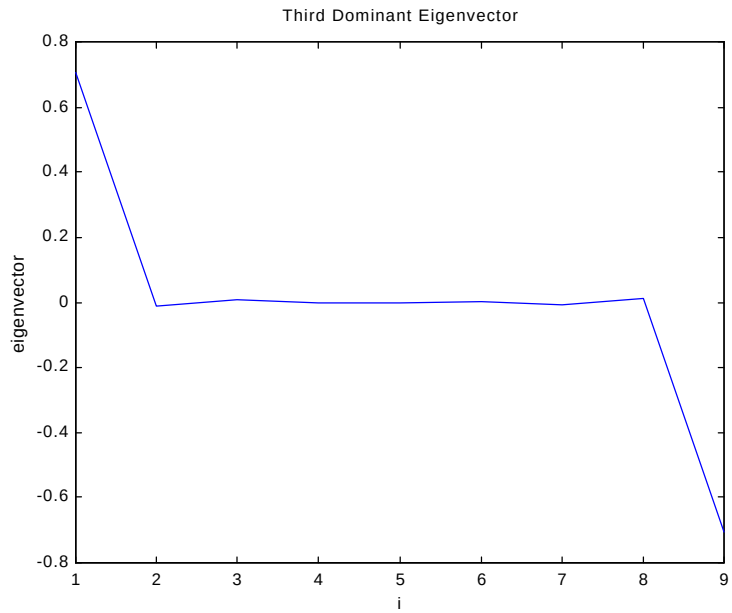
0.3179
0.0000
0.0000
-0.0001
0.0000
0.1860
0.0000
0.0510
0.0000

This implies that the first eigenvector is the dominant one, while the 6th and 8th eigenvectors are the second and third dominant eigenvectors.




First Dominant Eigenvector
eigenvector
i
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
1
2
3
4
5
6
7
8
9


Second Dominant Eigenvector
eigenvector
i
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
1
2
3
4
5
6
7
8
9



4. Orthonormalize the eigenvectors $\mathbf{x}_i$ *obtained in 2 with respect to the mass matrix* $\mathbf{M}$.

Applying the Gram-Schmidt orthonormalization process with respect to the global mass matrix $\mathbf{M}$:

$$\mathbf{y}_1 = \frac{\mathbf{x}_1}{\sqrt{\mathbf{x}_1{}^T \mathbf{M}\mathbf{x}_1}},$$

$$\mathbf{z}_i = \mathbf{x}_i - \sum_{j=1}^{i-1} \left(\mathbf{y}_j{}^T \mathbf{M}\mathbf{x}_i\right)\mathbf{y}_j \quad , \quad \mathbf{y}_i = \frac{\mathbf{z}_i}{\sqrt{\mathbf{z}_i{}^T \mathbf{M}\mathbf{z}_i}} \quad , \quad i = 2,\ldots,n$$

we have

Y =

Columns 1 through 7

0.0084	-0.0044	0.0116	-0.0157	-0.0162	0.0215	-0.0211
1.0008	-0.5446	1.2418	-1.3560	-1.1388	0.7009	-0.4435
1.4174	-1.0036	0.9965	0.0062	0.6324	-1.1272	0.9856
0.9954	-1.2991	-0.4140	1.3586	0.9151	0.7133	-1.0955
0.0000	-1.4320	-1.7124	0.0000	-2.6273	0.0000	3.4623
-0.9954	-1.2991	-0.4140	-1.3586	0.9151	-0.7133	-1.0955
-1.4174	-1.0036	0.9965	-0.0062	0.6324	1.1272	0.9856
-1.0008	-0.5446	1.2418	1.3560	-1.1388	-0.7009	-0.4435
-0.0084	-0.0044	0.0116	0.0157	-0.0162	-0.0215	-0.0211

Columns 8 through 9

4.8877	4.9865
-0.7520	-0.7629
0.3070	0.3751
0.0188	-0.1323
0.0000	0.9966
-0.0188	-0.1323
-0.3070	0.3751
0.7520	-0.7629
-4.8877	4.9865

5. Diagonalize the mass matrix $\boldsymbol{M}$ *by using the Householder transformation* $\boldsymbol{P}$.

iteration =

146

error =

9.9331e-006

mgn =

Columns 1 through 7

0.2325	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
0.0001	0.2249	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.1422	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.1373	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.1244	0.0001	0.0000
0.0000	0.0000	0.0000	0.0000	0.0001	0.1244	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0406
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

Columns 8 through 9

0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0000	0.0000
0.0203	0.0000
0.0000	0.0188

6. Make $\boldsymbol{QR}$ *decomposition of* $\boldsymbol{M}$.

Q =

Columns 1 through 7

-0.6437	0.6328	-0.3820	-0.0513	-0.1577	0.0982	-0.0425
-0.6491	-0.7365	-0.1538	0.1092	0.0208	-0.0130	0.0056
0.3813	-0.1825	-0.8828	-0.1917	-0.0594	0.0370	-0.0160
-0.1226	-0.0633	0.1581	-0.9667	0.1203	-0.0750	0.0324
0.0635	-0.1406	0.1614	-0.1188	-0.7960	0.4959	-0.2143
0	0	0	0	-0.4821	-0.8484	-0.1912
0	0	0	0	0.2832	0.0256	-0.9371
0	0	0	0	-0.0910	-0.1020	0.1563
0	0	0	0	0.0472	-0.0804	0.1089

Columns 8 through 9

-0.0054	0.0196
0.0007	-0.0026
-0.0020	0.0074
0.0041	-0.0150
-0.0272	0.0991
0.1057	-0.0112
-0.1912	0.0673
-0.9668	-0.1487
-0.1293	0.9812

R =

Columns 1 through 7

-0.0401	-0.1347	0.0979	-0.0484	-0.0073	0.0017	-0.0010
0	-0.0959	-0.0106	-0.0210	-0.0114	-0.0037	0.0021
0	0	-0.1425	0.0537	0.0262	0.0042	-0.0025
0	0	0	-0.1472	-0.0277	-0.0031	0.0018
0	0	0	0	-0.0540	-0.1083	0.0776

0	0	0	0	0	-0.1247	0.0300
0	0	0	0	0	0	-0.1519
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 9

0.0003	-0.0002
-0.0007	0.0004
0.0008	-0.0004
-0.0006	0.0003
-0.0375	-0.0058
-0.0358	-0.0106
0.0539	0.0208
-0.1474	-0.0250
0	0.0201

7. Solve the generalized eigenvalue problem $\mathbf{Kx} = \mathrm{l}\,\mathbf{Mx}$ *for the case that* $k_L = k_R = 0$.

X =

Columns 1 through 7

0.5430	0.6674	0.5305	0.4746	-0.4570	-0.4560	0.4486
-0.1432	-0.2061	-0.1973	-0.1636	0.0191	-0.1588	0.3147
0.1290	0.1085	0.0000	-0.3132	0.4313	0.3170	0.0000
-0.1432	-0.0196	0.1973	0.3871	0.0191	0.4079	-0.3147
0.5430	0.0000	-0.5305	0.0000	-0.4570	0.0000	-0.4486
-0.1432	0.0196	0.1973	-0.3871	0.0191	-0.4079	-0.3147
0.1290	-0.1085	0.0000	0.3132	0.4313	-0.3170	0.0000
-0.1432	0.2061	-0.1973	0.1636	0.0191	0.1588	0.3147
0.5430	-0.6674	0.5305	-0.4746	-0.4570	0.4560	0.4486

Columns 8 through 9

0.4472	0.3333
0.4131	0.3333
0.3162	0.3333
0.1712	0.3333
0.0000	0.3333
-0.1712	0.3333
-0.3162	0.3333
-0.4131	0.3333
-0.4472	0.3333

R =

1.0e+003 *

Columns 1 through 7

1.5146	0	0	0	0	0	0
0	1.0609	0	0	0	0	0
0	0	0.6791	0	0	0	0
0	0	0	0.2780	0	0	0
0	0	0	0	0.1591	0	0
0	0	0	0	0	0.0893	0
0	0	0	0	0	0	0.0395
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 9

0	0

0	0
0	0
0	0
0	0
0	0
0	0
0.0099	0
0	0.0000

8. Find the dynamical response of the bar at x = 0.5 for the case that

$$k_L = k_R = 10000,\ f(t,x) = e^{-2t}\sin\left(2\pi\frac{x}{L}\right),\ u_0(x) = 0.1\sin\left(\pi\frac{x}{L}\right),\ v_0(0) = 0$$

In order solve the above problem, you may have many questions. Since I have made this homework problem at the first time, description need not be very clear, and you may have difficulty to understand the problem. Please feel free to ask any questions.

Noting that MATLAB generalized eigenvalue solver yields the eigenvectors normalized themselves, that is, $\mathbf{x}_i^T\mathbf{x}_j = \delta_{ij}$. Thus, modal analysis implies the following decomposed problems :

$$m_i\frac{d^2U_i}{dt^2} + k_iU_i = \mathbf{x}_i^T\mathbf{f} \quad , \quad m_iU_i(0) = \mathbf{x}_i^T\mathbf{u}_0 \quad , \quad m_i\frac{dU_i}{dt}(0) = \mathbf{x}_i^T\mathbf{v}_0 \quad , \quad i = 1,2,.....,n$$

where mi and ki are diagonals of the matrices $\mathbf{X}^T\mathbf{MX}$ and $\mathbf{X}^T\mathbf{KX}$.

X =

Columns 1 through 7

0.6897	0.6974	0.0098	0.0166	-0.0117	-0.0074	0.0048
-0.1060	-0.1078	0.0235	0.3240	-0.4373	-0.4999	0.4594
0.0527	0.0443	-0.1447	-0.5320	0.3592	0.0035	0.3544
-0.0193	0.0026	0.2773	0.3342	0.1846	0.5000	-0.1795
0.1411	0.0000	-0.8961	0.0000	-0.5395	0.0000	-0.5120
-0.0193	-0.0026	0.2773	-0.3342	0.1846	-0.5000	-0.1795
0.0527	-0.0443	-0.1447	0.5320	0.3592	-0.0035	0.3544
-0.1060	0.1078	0.0235	-0.3240	-0.4373	0.4999	0.4594
0.6897	-0.6974	0.0098	-0.0166	-0.0117	0.0074	0.0048

Columns 8 through 9

0.0031	0.0016
0.3552	0.1923
0.4992	0.3538
0.3530	0.4616
0.0000	0.4996
-0.3530	0.4616
-0.4992	0.3538
-0.3552	0.1923
-0.0031	0.0016

R =

1.0e+004 *

Columns 1 through 7

5.0693	0	0	0	0	0	0
0	4.8605	0	0	0	0	0
0	0	0.1057	0	0	0	0
0	0	0	0.0407	0	0	0

0	0	0	0	0.0276	0	0
0	0	0	0	0	0.0167	0
0	0	0	0	0	0	0.0089
0	0	0	0	0	0	0
0	0	0	0	0	0	0

Columns 8 through 9

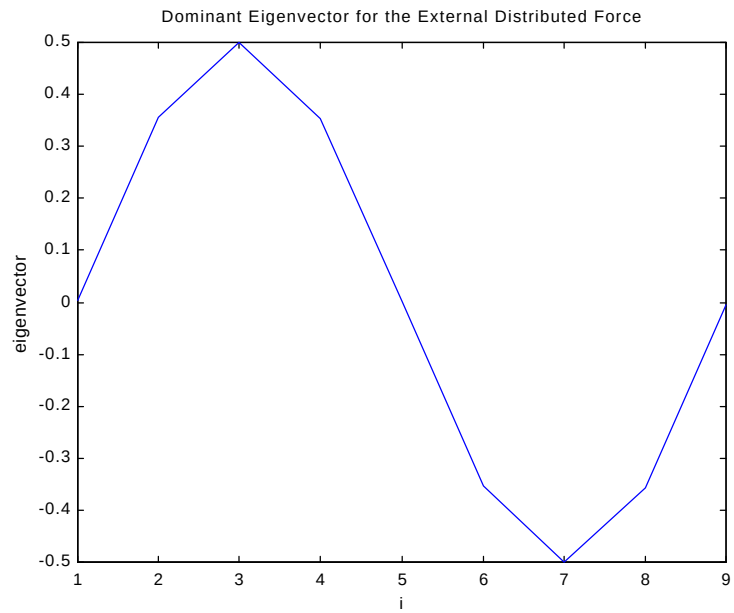
0	0
0	0
0	0
0	0
0	0
0	0
0	0
0.0039	0
0	0.0010

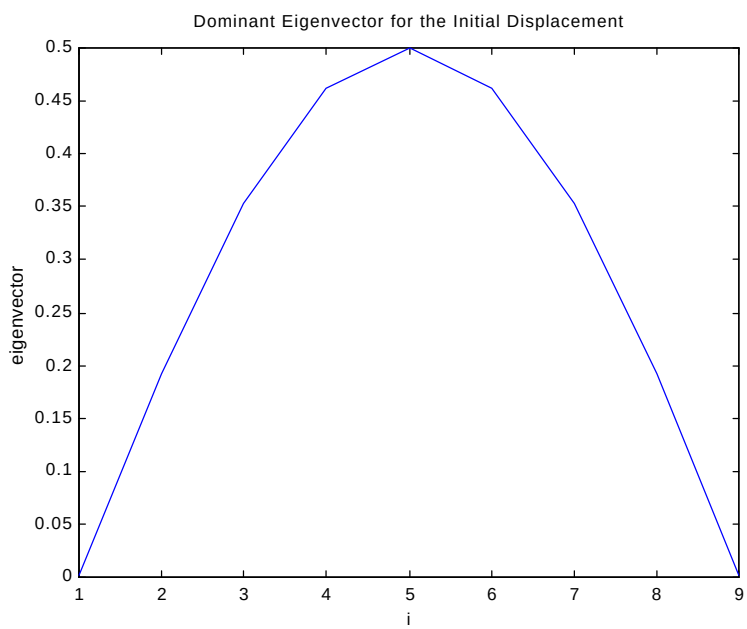
fg =

```
    0.0066
    0.0940
    0.1171
    0.0940
    0.0000
   -0.0940
   -0.1171
   -0.0940
   -0.0066
```

fgeig =

```
  Columns 1 through 7

    0.0000   -0.0001    0.0000   -0.0006    0.0000    0.0007    0.0000

  Columns 8 through 9

    0.2500    0.0000

u0g =

    0.0011
    0.0623
    0.0572
    0.1567
    0.0819
    0.1567
    0.0572
    0.0623
    0.0011

u0geig =

  Columns 1 through 7

   -0.0001    0.0000    0.0000    0.0000    0.0003    0.0000   -0.0004

  Columns 8 through 9

    0.0000    0.2500
```





Picking up the two dominant eigenvectors $\mathbf{x}_8$ and $\mathbf{x}_9$, we have the following two single problems :

$$0.125\frac{d^2U_8}{dt^2}+4.9158U_8=1.0001e^{-2t} \quad ,$$

$$0.125\,U_8(0)=0 \quad , \quad 0.125\frac{dU_8}{dt}(0)=0$$

and

$$0.125\frac{d^2U_9}{dt^2}+1.2289U_9=0 \quad ,$$

$$0.125U_9(0)=0.1*0.49995 \quad , \quad 0.125\frac{dU_9}{dt}(0)=0$$

Solving these two problems in U_8 and U_9:

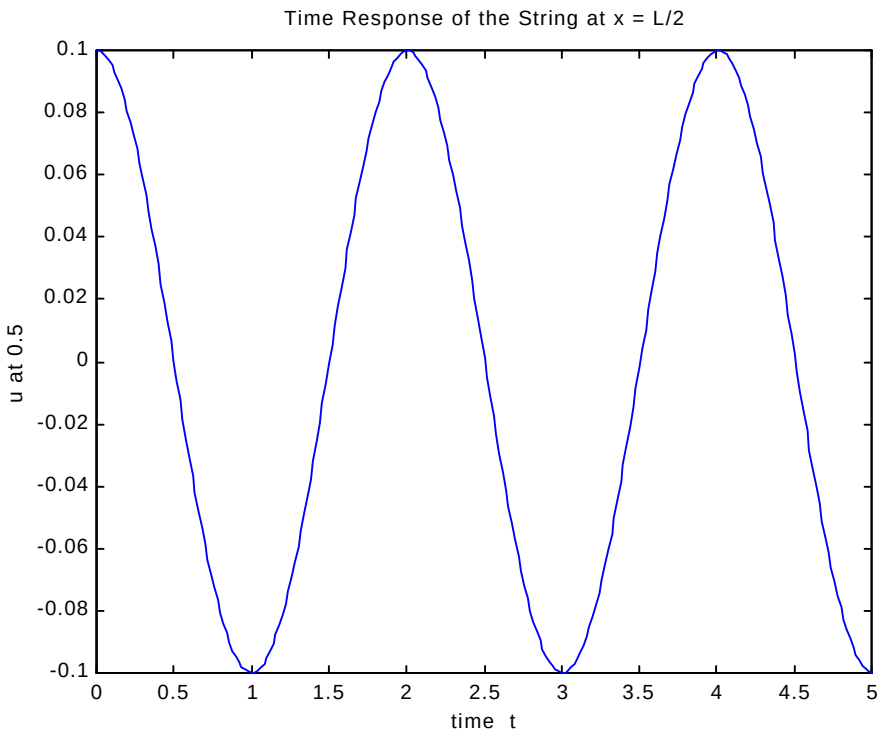
$$U_8(t)=\frac{\frac{1.0001}{0.125}}{4+\frac{4.9158}{0.125}}e^{-2t} \qquad and \qquad U_9(t)=0.20002\cos\left(\sqrt{\frac{1.2289}{0.125}}t\right)$$

an approximated time response can be obtained as

$$\mathbf{u}_{app}(t)=U_8(t)\mathbf{x}_8+U_9(t)\mathbf{x}_9$$

and then the response at x = 0.5 L can be obtained by

$$u_{app}(t)=U_8(t)x_{85}+U_9(t)x_{95}=0.4996U_9(t)=0.4996*0.20002\cos\left(\sqrt{\frac{1.2289}{0.125}}t\right).$$


Time Response of the String at x = L/2
0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
u at 0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
time t