Note on the Tangent and Normal of a Curve

(01/24/00)

Let a curve C is defined in space, and let s be the coordinate along the curve. The position vector of an arbitrary point P of the curve C is given by

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$$

where $\{\mathbf{e}_x \quad \mathbf{e}_y \quad \mathbf{e}_z\}$ are the unit vectors along the *x*, *y*, and *z* axes, respectively. Definition of the derivative yields a tangent vector to the curve C:

$$\frac{d\mathbf{r}}{ds} = \lim_{\Delta s \to 0} \frac{\mathbf{r}|_{s+\Delta s} - \mathbf{r}|_s}{\Delta s} = \frac{dx}{ds}\mathbf{e}_x + \frac{dy}{ds}\mathbf{e}_x + \frac{dz}{ds}\mathbf{e}_x$$

while the differential *ds* along the curve C is given by

$$ds = \sqrt{dx^2 + dy^2 + dz^2} \ .$$

Thus the length of the curve C may be computed by

$$L = \int_C ds = \int_{\{x_0 \quad y_0 \quad z_0\}}^{\{x \quad y \quad z\}} \sqrt{dx^2 + dy^2 + dz^2} \ .$$

Furthermore, if ***t*** is the unit tangent vector to the curve C, it satisfies the relation

$$\mathbf{t} \bullet \mathbf{t} = 1 \quad \Rightarrow \quad 2\mathbf{t} \bullet \frac{d\mathbf{t}}{ds} = 0 \quad \Rightarrow \quad \mathbf{t} \perp \frac{d\mathbf{t}}{ds}$$

that is, the vector $\frac{d\mathbf{t}}{ds}$ is normal to the tangent vector **t**. Thus, the normal vector to the curve C can be defined by the first derivative of the unit tangent vector **t** of the curve C.

Example A function *f* is defined by $y = f(x) = x - x^2$. In this case, since

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

the length of the curve formed by the function *f* is defined by

$$L = \int_C ds = \int_{x_0}^{x} \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx\ .$$

while the unit tangent vector of the curve C is nothing but

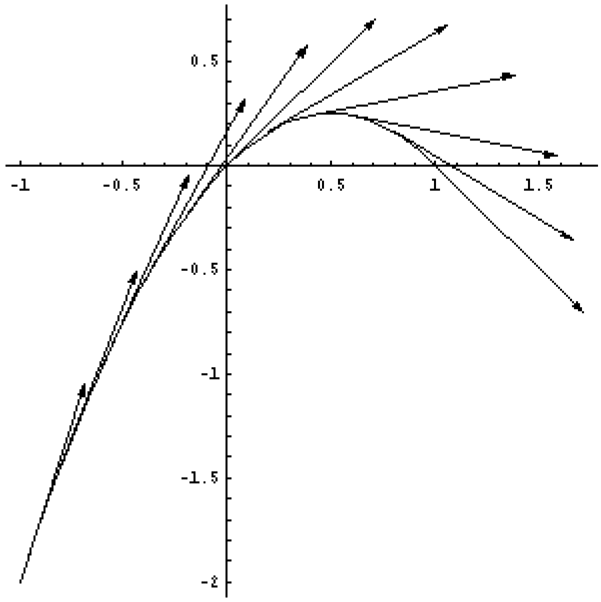
$$\mathbf{t} = \frac{dx}{ds}\mathbf{e}_x + \frac{dy}{ds}\mathbf{e}_y = \frac{dx}{ds}\mathbf{e}_x + \frac{dx}{ds}\frac{dy}{dx}\mathbf{e}_y = \frac{\mathbf{e}_x + \frac{df}{dx}\mathbf{e}_y}{\sqrt{1 + \left(\frac{df}{dx}\right)^2}}\ .$$

Using the following MATHEMATICA script program

```
f=x-x^2;
df=D[f,x];
tv={1,df}/Sqrt[1+df^2]
dtv=D[tv,x];
nv=Simplify[dtv/Sqrt[dtv.dtv]]
Simplify[tv.nv]
<<Graphics`PlotField`
tvlist=Table[{{x,f},tv},{x,-1,1,0.2}];
g1=ListPlotVectorField[tvlist,AspectRatio->1]
g2=Plot[f,{x,-1,1},AspectRatio->1]
Show[g1,g2]
nvlist=Table[{{x,f},nv},{x,-1,1,0.2}];
g3=ListPlotVectorField[nvlist,AspectRatio->1]
Show[g2,g3]
```

We can find the unit tangent and normal vectors.

Tangent Vector



Normal Vector