Partial Solutions of Review Problems for Midterm Examination
MEAM501, Fall 1998

1. Weighted Residual Method

Introducing an arbitrary weighting function w, we have

$$\int_0^L w\left(-\frac{d}{dx}\left(a\frac{du}{dx}\right)+u\frac{du}{dx}+ku\right)dx=\int_0^L wfdx \quad , \quad \forall w$$
$$and$$
$$w(0)u(0)=w(0)u_0 \quad , \quad \forall w(0)$$

If we apply the integration by parts to the first term, we have

$$\int_0^L w\left(-\frac{d}{dx}\left(a\frac{du}{dx}\right)+u\frac{du}{dx}+ku\right)dx=\int_0^L wfdx \quad , \quad \forall w$$
$$-\left[wa\frac{du}{dx}\right]_{x=0}^{x=L}+\int_0^L\left(\frac{dw}{dx}a\frac{du}{dx}+wu\frac{du}{dx}+wku\right)dx=\int_0^L wfdx \quad , \quad \forall w$$

If we have additional boundary condition

$$a\frac{du}{dx}=0 \quad at \quad x=L$$

we may have the weighted residual formulation

$$\int_0^L\left(\frac{dw}{dx}a\frac{du}{dx}+wu\frac{du}{dx}+wku\right)dx=\int_0^L wfdx \quad , \quad \forall w\ s.t.\ w(0)=0$$
$$and$$
$$u=u_0 \quad at \quad x=0$$

2. Weighted Residual Method

Starting from the given boundary value problem

$$
\begin{aligned}
&-\frac{d}{dx}\left(a\frac{du}{dx}\right)+ku=f \quad , \quad x\in(0,b)\cup(b,L)\\
&a\frac{du}{dx}\bigg|_{x=b-}-a\frac{du}{dx}\bigg|_{x=b+}=P\\
&and\\
&u(0)=u_0 \quad , \quad a\frac{du}{dx}\bigg|_{x=L}=0
\end{aligned}
$$

we have

$$
\begin{aligned}
&\int_0^b w\left(-\frac{d}{dx}\left(a\frac{du}{dx}\right)+ku\right)dx+\int_b^L w\left(-\frac{d}{dx}\left(a\frac{du}{dx}\right)+ku\right)dx=\int_0^b wfdx+\int_b^L wfdx \quad , \quad \forall w\\
&-\left[wa\frac{du}{dx}\right]_{x=0}^{x=b}-\left[wa\frac{du}{dx}\right]_{x=b}^{x=L}+\int_0^L\left(\frac{dw}{dx}a\frac{du}{dx}+wku\right)dx=\int_0^L wfdx \quad , \quad \forall w\\
&-wa\frac{du}{dx}\bigg|_{x=b-}+wa\frac{du}{dx}\bigg|_{x=b+}+\int_0^L\left(\frac{dw}{dx}a\frac{du}{dx}+wku\right)dx=\int_0^L wfdx \quad , \quad \forall w \;\; s.t. \;\; w(0)=0\\
&\int_0^L\left(\frac{dw}{dx}a\frac{du}{dx}+wku\right)dx=\int_0^L wfdx+Pw(b) \quad , \quad \forall w \;\; s.t. \;\; w(0)=0.
\end{aligned}
$$

Thus, a weighted residual formulation becomes

$$
\begin{aligned}
&\int_0^L\left(\frac{dw}{dx}a\frac{du}{dx}+wku\right)dx=\int_0^L wfdx+Pw(b) \quad , \quad \forall w \;\; s.t. \;\; w(0)=0\\
&and\\
&u(0)=u_0
\end{aligned}
$$

Now approximation

$$
u(x)=\sum_{j=1}^{n}u_j\mathrm{f}_j(x) \quad , \quad w(x)=\sum_{i=1}^{n}w_i\mathrm{y}_i(x) \qquad with \qquad \mathrm{y}_i(0)=0
$$

we have

$$\sum_{i=1}^{n} w_i \sum_{j=1}^{n}\left\{\int_0^L\left(a\frac{d\psi_i}{dx}\frac{d\phi_j}{dx}+k\psi_i\phi_j\right)dx\right\}u_j = \sum_{i=1}^{n} w_i\left(\int_0^L \psi_i(x) f dx + P\psi_i(b)\right) \quad , \quad \forall w_i$$

$$and$$

$$\sum_{j=1}^{n} u_j \phi_j(0) = u_0$$

that is

$$\sum_{j=1}^{n}\left\{\int_0^L\left(a\frac{d\psi_i}{dx}\frac{d\phi_j}{dx}+k\psi_i\phi_j\right)dx\right\}u_j = \int_0^L \psi_i(x) f dx + P\psi_i(b) \quad , \quad \forall i = 1,2,\ldots,n$$

$$and$$

$$\sum_{j=1}^{n} u_j \phi_j(0) = u_0$$

Defining

$$k_{ij} = \int_0^L\left(a\frac{d\psi_i}{dx}\frac{d\phi_j}{dx}+k\psi_i\phi_j\right)dx \quad , \quad f_i = \int_0^L \psi_i(x) f dx + P\psi_i(b)$$

we have the matrix equation

$$\mathbf{Ku} = \mathbf{f}$$

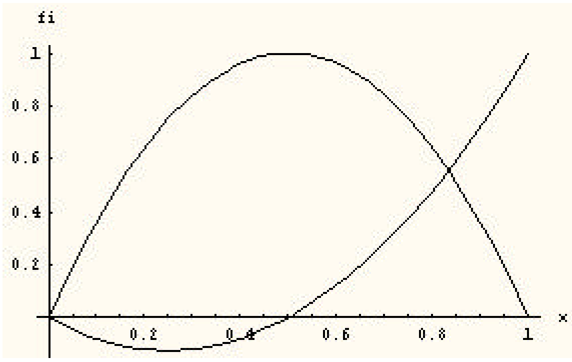
$$with$$

$$\sum_{i=1}^{n} \phi_i(0) u_i = u_0 \quad i.e. \quad \boldsymbol{\phi}_0^T \mathbf{u} = u_0$$

3. Eigenvalue Problem (2-by-2 Matrix System from Problem 2)

(1) MATHEMATICA

```
fi={1-4*(x-0.5)^2,2*x*(x-0.5)};
Plot[Release[fi],{x,0,1},PlotRange->All,AxesLabel->{"x","fi"}]
```




```
dfi=D[fi,x];
Kg=Integrate[Outer[Times,dfi,dfi]+Ou
ter[Times,fi,fi],{x,0,1}];
MatrixForm[Kg]
fg=Integrate[fi,{x,0,1}]+fi/.x->0.5;
MatrixForm[fg]
L=Eigenvalues[Kg]
X=Eigenvectors[Kg]
ug=LinearSolve[Kg,fg]
Plot[ug.fi,{x,0,1},PlotRange-
>All,AxesLabel->{"x","u"}]
uc=Table[X[[i]].ug,{i,1,2}]
p1=Plot[uc[[1]]*X[[1]].fi,{x,0,1},PlotRange->All,AxesLabel->{"x","u1"}]
p2=Plot[uc[[2]]*X[[2]].fi,{x,0,1},PlotRange->All,AxesLabel->{"x","u2"}]
Show[{p1,p2}]
```

(1) 2-by-2 Stiffness Matrix **K** and 2 component **f**

$$\begin{pmatrix} 5.86667 & -2.6 \\ -2.6 & 2.46667 \end{pmatrix} \qquad \begin{pmatrix} 1.66667 \\ 0.166667 \end{pmatrix}$$

(2) Eigenvalues {7.27311, 1.06022} and Eigenvectors of **K** :

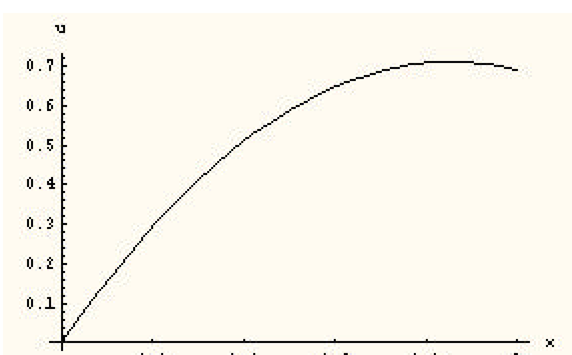
{{0.879559, -0.475789}, {0.475789, 0.879559}}

(3) Solution of the static equilibrium **u**

{0.589337, 0.688761}

that is

$$u(x) = 0.589337 f_1(x) + 0.688761 f_2(x).$$



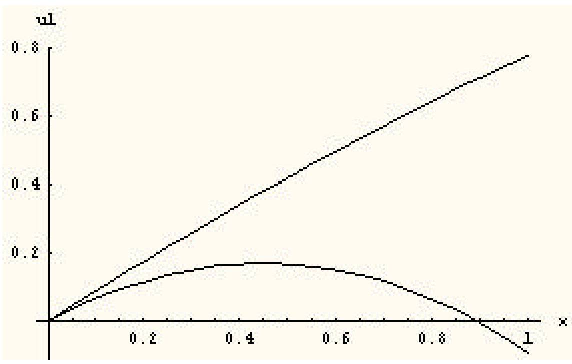
(4) Representation of u in terms of eigenvectors

{0.190652, 0.886206}

that is

$$\mathbf{u} = 0.190652\mathbf{x}_1 + 0.886206\mathbf{x}_2$$

(5) The second mode is dominant.



4. These should be answered by yourselves.