Midterm Examination : 1998 Fall
MEAM 501 Analytical Methods in Mechanics and Mechanical Engineering

1. (15 Points) Formulate the following boundary value problem

$$-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+h(x)\frac{du}{dx}+k(x)u = f(x) \quad , \quad x\in(0,b)\cup(b,L)$$
$$a(b_-)\frac{du}{dx}(b_-)-a(b_+)\frac{du}{dx}(b_+)=P$$
$$u(0)=u_0 \quad , \quad a(L)\frac{du}{dx}(L)=v_0$$

by the weighted residual method and by applying the integration by parts rule, where $g(b_-)=\lim_{\varepsilon\to0}g(b-\varepsilon)$, $g(b_+)=\lim_{\varepsilon\to0}g(b+\varepsilon)$, b in a point inside of the interval (0,L), and P is a given number.

$$\int_0^b w\left\{-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+h(x)\frac{du}{dx}+k(x)u\right\}dx+\int_b^L w\left\{-\frac{d}{dx}\left(a(x)\frac{du}{dx}\right)+h(x)\frac{du}{dx}+k(x)u\right\}dx$$
$$=\int_0^b wf(x)dx+\int_b^L wf(x)dx=\int_0^L wf(x)dx$$

$$-\left[wa(x)\frac{du}{dx}\right]_0^b+\int_0^b\left\{a(x)\frac{dw}{dx}\frac{du}{dx}+h(x)w\frac{du}{dx}+k(x)wu\right\}dx$$
$$-\left[wa(x)\frac{du}{dx}\right]_b^L+\int_b^L\left\{a(x)\frac{dw}{dx}\frac{du}{dx}+h(x)w\frac{du}{dx}+k(x)wu\right\}dx=\int_0^L wf(x)dx$$

$$-w(b_-)a(b_-)\frac{du}{dx}(b_-)+w(0)a(0)\frac{du}{dx}(0)-w(L)a(L)\frac{du}{dx}(L)+w(b_+)a(b_+)\frac{du}{dx}(b_+)$$
$$+\int_0^L\left\{a(x)\frac{dw}{dx}\frac{du}{dx}+h(x)w\frac{du}{dx}+k(x)wu\right\}dx=\int_0^L wf(x)dx$$

$$\int_0^L\left\{a(x)\frac{dw}{dx}\frac{du}{dx}+h(x)w\frac{du}{dx}+k(x)wu\right\}dx=\int_0^L wf(x)dx+w(b)P+w(L)v_0 \quad , \quad \forall w \quad s.t. \quad w(0)=0$$

Here we assume that an arbitrary weighting function is continuous on the domain (0,L).

2. (25 Points) Consider the matrix equation

$$\begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 2.5 \\ 3.5 \end{Bmatrix} \quad \textit{that is} \quad \mathbf{Ax} = \mathbf{b}$$

(1) Find the eigenvalues λ of the matrix **A** together with the eigenvectors ***y***.

$$\det\begin{bmatrix} 1.5-\lambda & 0.5 \\ 0.5 & 1.5-\lambda \end{bmatrix} = (1.5-\lambda)^2 - 0.5^2 = (2-\lambda)(1-\lambda) = 0 \quad \Rightarrow \quad \lambda = 1 \quad \textit{and} \quad \lambda = 2$$

For $\lambda = 1$

$$\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

For $\lambda = 2$

$$\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

(2) Solve the matrix equation $\mathbf{Ax} = \mathbf{b}$.

$$\mathbf{x} = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}^{-1} \begin{Bmatrix} 2.5 \\ 3.5 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

(3) Represent the solution **x** as a linear combination of the two eigenvectors of the 2-by-2 matrix **A**.

$$\mathbf{x} = (\mathbf{x}, \mathbf{y}_1)\mathbf{y}_1 + (\mathbf{x}, \mathbf{y}_2)\mathbf{y}_2 = -\frac{1}{\sqrt{2}}\mathbf{y}_1 + \frac{3}{\sqrt{2}}\mathbf{y}_3$$

(4) Find which mode (i.e. eigenvector) is dominant.

Second eigenvector is dominant.

1. (60 Points) Answer to the following questions:

(1) For a *m*-by-*n* matrix **A**, state the definition of the singular value decomposition of **A**. For the matrix **A** defined by

```
A =

     3     2     1    -3
     4     1     2     4
     2     1     3    -1
```

the singular value decomposition is made by MATLAB :

```
EDU t [U,S,V]=svd(A)

U =

    0.4232   -0.7303   -0.5362
    0.7753    0.5981   -0.2027
    0.4688   -0.3299    0.8194


S =

    6.7701         0         0         0
         0    5.1038         0         0
         0         0    1.7656         0


V =

    0.7841   -0.0898   -0.4421   -0.4262
    0.3088   -0.2336   -0.2581    0.8851
    0.4993   -0.1026    0.8589    0.0492
    0.2013    0.9627   -0.0123    0.1803
```

Using the above result of the singular value decomposition, what is the rank of **A**, what is the range of **A**, and what is the null space of **A** ?

(2) When are two vectors $\mathbf{u} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$ *and* $\mathbf{v} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$ orthogonal and unit (that is, orthonormal) ? If not, orthonormalize them by the Gram-Schmidt process.

(3) What is an orthogonal matrix **Q** ? Show that if a matrix **Q** defined by $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \ldots\ldots, \mathbf{q}_n]$ where $\mathbf{q}_i$ are orthonormal, **Q** is an orthogonal matrix.

(4) What is the Householder transformation **P** defined by a vector $\mathbf{v} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$? State the property of **P**.

(5) What is the QR decomposition of a square matrix **A** ? State QR algorithm to find the eigenvalues and eigenvectors of a symmetric square matrix **A**.