

Solutions of Homework #5, 1998 Fall
MEAM 501 Analytical Methods in Mechanics and Mechanical Engineering

1. Consider a constrained minimization problem

$$\min_{\mathbf{v} \in K} F(\mathbf{v}) \quad , \quad F(\mathbf{v}) = \frac{1}{2} \mathbf{v}^T \mathbf{A} \mathbf{v} - \mathbf{v}^T \mathbf{b} \quad , \quad \mathbf{v} \in \mathbf{R}^n \quad , \quad \mathbf{A} \in \mathbf{R}^{n \times n} \quad , \quad \mathbf{b} \in \mathbf{R}^n$$

$$K = \{ \mathbf{v} \in \mathbf{R}^n : \mathbf{B} \mathbf{v} - \mathbf{g} \leq \mathbf{0} \} \quad , \quad \mathbf{B} \in \mathbf{R}^{m \times n} \quad , \quad \mathbf{g} \in \mathbf{R}^m$$

where a matrix \mathbf{A} is symmetric, that is, $\mathbf{A}^T = \mathbf{A}$, and the constrained set K is non-empty.

(1) Find the necessary condition that an element $\mathbf{u} \in K$ is a minimizer of the functional F on the constrained set K .

$$dF(\mathbf{u})(\mathbf{v} - \mathbf{u}) = \lim_{a \rightarrow 0} \frac{F(\mathbf{u} + a(\mathbf{v} - \mathbf{u})) - F(\mathbf{u})}{a} = (\mathbf{v} - \mathbf{u})^T \mathbf{A} \mathbf{u} - (\mathbf{v} - \mathbf{u})^T \mathbf{b} \geq 0 \quad , \quad \forall \mathbf{v} \in K$$

(2) Obtain the Lagrangian L to this constrained minimization problem, and set up the "equivalent" unconstrained problem on the primal variable \mathbf{v} by considering the necessary condition of the problem obtained by the Lagrange multiplier method.

Lagrangian L is defined by

$$L(\mathbf{v}, \boldsymbol{\mu}) = F(\mathbf{v}) - \boldsymbol{\mu}^T (\mathbf{B} \mathbf{v} - \mathbf{g}) = \frac{1}{2} \mathbf{v}^T \mathbf{A} \mathbf{v} - \mathbf{v}^T \mathbf{b} - \boldsymbol{\mu}^T (\mathbf{B} \mathbf{v} - \mathbf{g}) \quad , \quad \boldsymbol{\mu} \in \mathbf{R}^m \leq \mathbf{0}$$

and we shall consider the problem

$$\min_{\mathbf{v}} \max_{\boldsymbol{\mu} \leq \mathbf{0}} L(\mathbf{v}, \boldsymbol{\mu})$$

Suppose that $(\mathbf{u}, \boldsymbol{\mu}) \in \mathbf{R}^n \times \mathbf{R}^m$, $\boldsymbol{\mu} \leq \mathbf{0}$ is a solution of the min-max problem. Then we have

$$(1) \quad L(\mathbf{u}, \boldsymbol{\mu}) \leq L(\mathbf{v}, \boldsymbol{\mu}) \quad , \quad \forall \mathbf{v} \in \mathbf{R}^n$$

$$(2) \quad L(\mathbf{u}, \boldsymbol{\mu}) \leq L(\mathbf{u}, \boldsymbol{\mu}') \quad , \quad \forall \boldsymbol{\mu}' \in \mathbf{R}^m \leq \mathbf{0}$$

From (1), we have the following necessary condition

$$dL(\mathbf{u}, \boldsymbol{\mu})(\mathbf{v} - \mathbf{u}) = \lim_{a \rightarrow 0} \frac{\partial}{\partial a} L(\mathbf{u} + a(\mathbf{v} - \mathbf{u}), \boldsymbol{\mu}) = (\mathbf{v} - \mathbf{u})^T (\mathbf{A} \mathbf{u} - \mathbf{B}^T \boldsymbol{\mu} - \mathbf{b}) \geq 0 \quad , \quad \forall \mathbf{v} \in \mathbf{R}^n$$

Since \mathbf{v} is arbitrary, and since we can take $\mathbf{v} = \mathbf{u} \pm d\mathbf{u}$, we have

$$\pm \mathbf{d}\mathbf{u}^T (\mathbf{A}\mathbf{u} - \mathbf{B}^T \boldsymbol{\mu} - \mathbf{b}) \geq 0, \quad \forall \mathbf{d}\mathbf{u} \in \mathbf{R}^n$$

that is

$$\mathbf{A}\mathbf{u} - \mathbf{B}^T \boldsymbol{\mu} = \mathbf{b} \quad \text{in} \quad \mathbf{R}^n$$

Similarly, from the second inequality relation (2), we have

$$dL(\mathbf{u}, \boldsymbol{\mu}) = \lim_{a \rightarrow 0} \frac{\partial}{\partial a} L(\mathbf{u}, \boldsymbol{\mu} + a(\boldsymbol{\mu} - \boldsymbol{\mu}^*)) = -(\boldsymbol{\mu} - \boldsymbol{\mu}^*)^T (\mathbf{B}\mathbf{u} - \mathbf{g}) \leq 0, \quad \forall \boldsymbol{\mu} \in \mathbf{R}^m \leq \mathbf{0}$$

that is

$$(\boldsymbol{\mu} - \boldsymbol{\mu}^*)^T (\mathbf{B}\mathbf{u} - \mathbf{g}) \geq 0, \quad \forall \boldsymbol{\mu} \in \mathbf{R}^m \leq \mathbf{0}$$

Combining these, the necessary condition of the Lagrange multiplier formulation becomes

$$(a) \quad \mathbf{A}\mathbf{u} - \mathbf{B}^T \boldsymbol{\mu} = \mathbf{b} \quad \text{in} \quad \mathbf{R}^n$$

$$(b) \quad (\boldsymbol{\mu} - \boldsymbol{\mu}^*)^T (\mathbf{B}\mathbf{u} - \mathbf{g}) \geq 0, \quad \forall \boldsymbol{\mu} \in \mathbf{R}^m \leq \mathbf{0}$$

The second inequality yields also the KKT condition

$$\mathbf{B}\mathbf{u} - \mathbf{g} \leq \mathbf{0}, \quad \boldsymbol{\mu} \leq \mathbf{0}, \quad \boldsymbol{\mu}^T (\mathbf{B}\mathbf{u} - \mathbf{g}) = 0$$

(3) Solve the problem for \mathbf{A} , \mathbf{B} , and \mathbf{g} obtained by the following MATLAB program.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 2 & 1 \end{bmatrix}, \quad \mathbf{g} = \begin{bmatrix} 5 \\ 10 \\ 8 \end{bmatrix}$$

Based on the result in (2), we need to solve

$$\begin{aligned} \mathbf{A}\mathbf{u} - \mathbf{B}^T \boldsymbol{\mu} &= \mathbf{b} \quad \text{in } \mathbf{R}^n \\ (\boldsymbol{\mu} - \boldsymbol{\mu}^*)^T (\mathbf{B}\mathbf{u} - \mathbf{g}) &\geq 0, \quad \forall \boldsymbol{\mu} \in \mathbf{R}^m \leq \mathbf{0} \end{aligned}$$

The second inequality can be modified as

$$\begin{aligned} (\boldsymbol{\mu} - \boldsymbol{\mu}^*)^T \{ -\boldsymbol{\mu}^* + \mathbf{a}(\mathbf{B}\mathbf{u} - \mathbf{g}) \} &\geq 0, \quad \forall \boldsymbol{\mu} \in \mathbf{R}^m \leq \mathbf{0} \\ &= \min \{ \mathbf{0}, -\mathbf{a}(\mathbf{B}\mathbf{u} - \mathbf{g}) \} \end{aligned}$$

Thus, we need to solve the following nonlinear problem:

$$\begin{aligned} \mathbf{A}\mathbf{u} &= \mathbf{b} + \mathbf{B}^T \boldsymbol{\mu}^* \\ &= \min \{ \mathbf{0}, -\mathbf{a}(\mathbf{B}\mathbf{u} - \mathbf{g}) \} \end{aligned}$$

This form suggests the following iteration method:

$$\begin{aligned} \mathbf{A}\mathbf{u}^{(i+1)} &= \mathbf{b} + \mathbf{B}^T \boldsymbol{\mu}^{(i)} \\ \boldsymbol{\mu}^{(i+1)} &= \min \{ \mathbf{0}, \boldsymbol{\mu}^{(i)} - \mathbf{a}(\mathbf{B}\mathbf{u}^{(i+1)} - \mathbf{g}) \} \end{aligned}$$

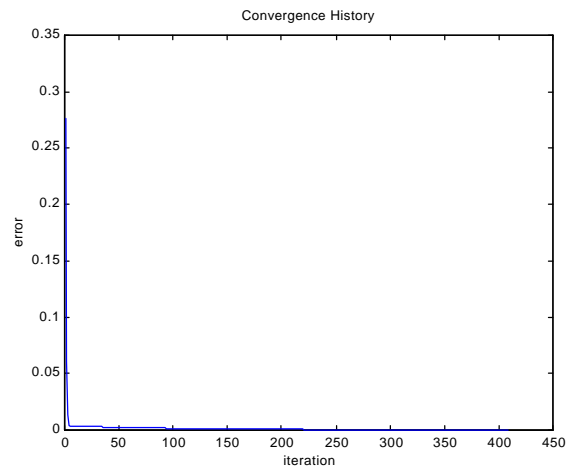
for assuming $\boldsymbol{\mu}^{(0)} = \mathbf{0}$. Implementing this iteration scheme as in the following MATLAB program, we can solve constrained minimization problem.

```
% HW#5 1998 Fall
n=10;
m=3;
A=zeros(n);
for i=1:n
    A(i,i)=2;
    if i>1, A(i-1,i)=-1;, end
    if i<n, A(i+1,i)=-1;, end
end
A
f=ones(n,1)
B=[1,2,3,2,1,0,0,0,0,0;0,1,2,3,4,3,2,1,0,0;0,0,0,0,1,2,3,4,2,1]
g=[5;10;8]
iter=0;
tole=10^(-4);
err=1;
errc=[];
imax=500;
alfa=0.001;
lam=zeros(m,1);
while err>tole
    flam=f+B'*lam;
    x=A\flam;
```

```

lamn=lam-alfa*(B*x-g);
for j=1:m
    if lamn(j,1)>0, lamn(j,1)=0;, end
end
iter=iter+1;
err=norm(lamn-lam);
errc(iter)=err;
lam=lamn;
if iter>imax, break, end
end
plot(errc)
xlabel('iteration')
ylabel('error')
title('Convergence History')
iter
err
x
lam

```



iter =

408

err =

9.9395e-005

x =

0.5992
0.6001
0.4521
0.6052
0.7059
0.7312
0.5621
0.4808
0.7697
0.7196

lam =

-0.4016
-0.0481
-0.3305

B*x-g =

0.0721
-0.0578
0.0366

As we can see, convergence cannot be obtained easily. To overcome this, we shall combine the Penalty method with the Lagrange multiplier method:

$$\begin{aligned} L_e(\mathbf{v}, \mu) &= F(\mathbf{v}) - \mu^T (\mathbf{B}\mathbf{v} - \mathbf{g}) - \frac{1}{2} e \mu^T \mu \\ &= \frac{1}{2} \mathbf{v}^T \mathbf{A} \mathbf{v} - \mathbf{v}^T \mathbf{b} - \mu^T (\mathbf{B}\mathbf{v} - \mathbf{g}) - \frac{1}{2} e \mu^T \mu, \quad \mu \in \mathbf{R}^m \leq \mathbf{0} \end{aligned}$$

This modification yields

$$\begin{aligned} \mathbf{A}\mathbf{u} &= \mathbf{b} + \mathbf{B}^T \\ (\mu - \frac{1}{e})^T (e + \mathbf{B}\mathbf{u} - \mathbf{g}) &\geq 0, \quad \forall \mu \leq \mathbf{0} \end{aligned}$$

that is,

$$\begin{aligned} \mathbf{A}\mathbf{u} &= \mathbf{b} + \mathbf{B}^T \\ (\mu - \frac{1}{e})^T \left(e + \frac{1}{e} (\mathbf{B}\mathbf{u} - \mathbf{g}) \right) &\geq 0, \quad \forall \mu \leq \mathbf{0} \end{aligned}$$

Solving this, we have

$$\begin{aligned} \mathbf{A}\mathbf{u} &= \mathbf{b} + \mathbf{B}^T \\ &= \min \left\{ \mathbf{0}, -\frac{1}{e} (\mathbf{B}\mathbf{u} - \mathbf{g}) \right\} = -\frac{1}{e} (\mathbf{B}\mathbf{u} - \mathbf{g})_+ \end{aligned}$$

These nonlinear equations can be solved by the following iteration method:

$$\begin{aligned}\mathbf{A}\mathbf{u}^{(i+1)} &= \mathbf{b} - \mathbf{B}^T \frac{1}{e} \left\{ (\mathbf{B}\mathbf{u}^{(i+1)} - \mathbf{g}) - (\mathbf{B}\mathbf{u}^{(i+1)} - \mathbf{g})_- \right\} \\ \Rightarrow \left(\mathbf{A} + \frac{1}{e} \mathbf{B}^T \mathbf{B} \right) \mathbf{u}^{(i+1)} &= \mathbf{b} + \mathbf{B}^T \frac{1}{e} (\mathbf{g} + (\mathbf{B}\mathbf{u}^{(i)} - \mathbf{g})_-) \\ \mathbf{u}^{(i+1)} &= \min \left\{ \mathbf{0}, -\frac{1}{e} (\mathbf{B}\mathbf{u}^{(i+1)} - \mathbf{g}) \right\} = -\frac{1}{e} (\mathbf{B}\mathbf{u}^{(i+1)} - \mathbf{g})_+ \quad , \quad i = 0, 1, \dots\end{aligned}$$

where $\mathbf{l}^{(0)} = \mathbf{0}$ and $f = f_+ + f_-$. This can be translated into the following MATLAB program

% Augmented Lagrangian = Penalty Formulation

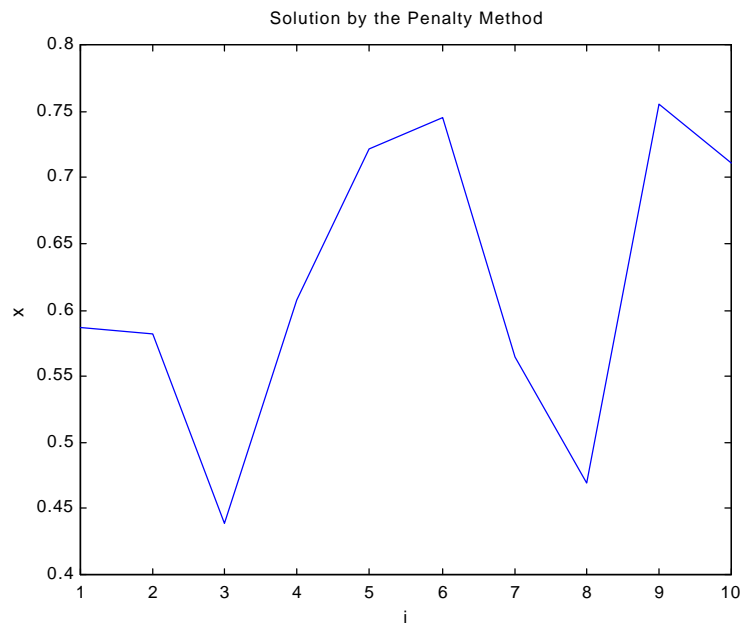
```
iter=0;
tole=10^(-4);
err=1;
errc=[];
imax=10;
ep=0.01;
lam=zeros(m,1);
lamm=lam;
while err>tole
    x=(A+B'*B/ep)\(f+B'*(g+lamm)/ep);
    lamn=-(B*x-g)/ep;
    for j=1:m
        if lamn(j,1)>0, lamn(j,1)=0;, end
    end
    lamm=-(B*x-g)/ep-lamn;
    iter=iter+1;
    err=norm(lamn-lam);
    errc(iter)=err;
    lam=lamn;
    if iter>imax, break, end
end
iter
err
x
lam
plot(x)
title('Solution by the Penalty Method')
xlabel('i')
ylabel('x')
```

Then, after two iteration, we have the following solution:

```
x =
    0.5863
    0.5822
    0.4389
    0.6078
    0.7210
    0.7452
    0.5639
    0.4697
    0.7555
    0.7105
```

lam =

-0.4095
-0.0417
-0.3346



Using the toolbox OPTIMIZATION, we have

EDU » [x,lam]=qp(A,-f,B,g)

x =

0.5857
0.5814
0.4382
0.6077
0.7214
0.7456
0.5638
0.4691
0.7548
0.7100

lam =

0.4099
0.0415
0.3347

2. Consider a functional

$$F(v) = \frac{1}{2} \int_0^L \left(1 + \frac{1}{2} \sin\left(m\pi \frac{x}{L}\right) \right) \left(\frac{dv}{dx} \right)^2 dx - \int_0^L v dx$$

on an admissible space V such that its element is continuous and piecewise continuously differentiable in the interval $(0, L)$, and its value at $x = 0$ is zero.

- (1) Find the first variation of the functional F , and the necessary condition of the minimizer u of the functional F on the admissible space V .

$$\begin{aligned} dF(u)(v) &= \lim_{a \rightarrow 0} \frac{\partial}{\partial a} F(u + av) \\ &= \int_0^L \left(1 + \frac{1}{2} \sin\left(m\pi \frac{x}{L}\right) \right) \frac{du}{dx} \frac{dv}{dx} dx - \int_0^L v dx \\ &= \int_0^L \left[-\frac{d}{dx} \left\{ \left(1 + \frac{1}{2} \sin\left(m\pi \frac{x}{L}\right) \right) \frac{du}{dx} \right\} - 1 \right] v dx + \left[\left(1 + \frac{1}{2} \sin\left(m\pi \frac{x}{L}\right) \right) \frac{du}{dx} v \right]_{x=0}^{x=L} \\ &= \int_0^L \left[-\frac{d}{dx} \left\{ \left(1 + \frac{1}{2} \sin\left(m\pi \frac{x}{L}\right) \right) \frac{du}{dx} \right\} - 1 \right] v dx + \left[\left(1 + \frac{1}{2} \sin\left(m\pi \frac{x}{L}\right) \right) \frac{du}{dx} v \right]_{x=0}^{x=L} \\ &= 0 \quad , \quad \forall v \quad s.t. \quad v(0) = 0 \end{aligned}$$

Therefore, the necessary condition becomes

$$\begin{aligned} -\frac{d}{dx} \left\{ \left(1 + \frac{1}{2} \sin\left(m\pi \frac{x}{L}\right) \right) \frac{du}{dx} \right\} &= 1 \quad in \quad (0, L) \\ \left(1 + \frac{1}{2} \sin\left(m\pi \frac{x}{L}\right) \right) \frac{du}{dx} &= 0 \quad at \quad x = L \\ u &= 0 \quad at \quad x = 0 \end{aligned}$$

or

$$\int_0^L \left(1 + \frac{1}{2} \sin\left(m\pi \frac{x}{L}\right) \right) \frac{du}{dx} \frac{dv}{dx} dx = \int_0^L v dx \quad , \quad \forall v \quad s.t. \quad v(0) = 0$$

- (2) Solve the minimization problem by the Ritz method by using the polynomial basis functions for $m = 10$. Number of terms must be determined appropriately so as to yield sufficient accuracy in Ritz's approximation.

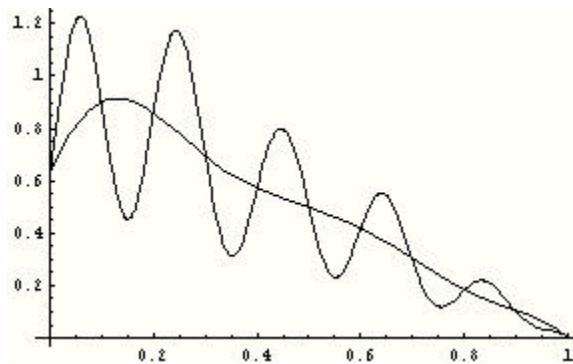
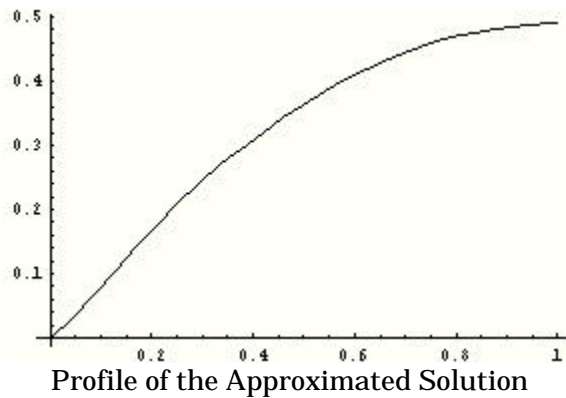
```
In[1]:=
m=10;
L=1;
ax=1+Sin[m*Pi*x/L]/2;
```



```

Plot[ax,{x,0,L}]
n=9;
bf=Table[x^i,{i,1,n}];
dbf=D[bf,x];
kg=Table[0,{i,1,n},{j,1,n}];
fg=Table[0,{i,1,n}];
Do[kgij=ax*dbf[[i]]*dbf[[j]];
  kg[[i,j]]=NIntegrate[kgij,{x,0,L}];
  kg[[j,i]]=kg[[i,j]],
  {i,1,n},{j,1,n}];
kg;
Do[fgi=1*bf[[i]];
  fgi[[i]]=NIntegrate[fgi,{x,0,L}],
  {i,1,n}];
fg;
ug=LinearSolve[kg,fg];
ugx=ug.bf;
Plot[ugx,{x,0,L},PlotRange->All]
dugx=ug.dbf;
sigx=ax*dugx;
Plot[{dugx,sigx},{x,0,L},PlotRange->All]
Fmin=0.5*ug.kg.ug-ug.fg

```



Profiles of the First Derivative of the Approximation and its Multiplication by the Oscillated Coefficient

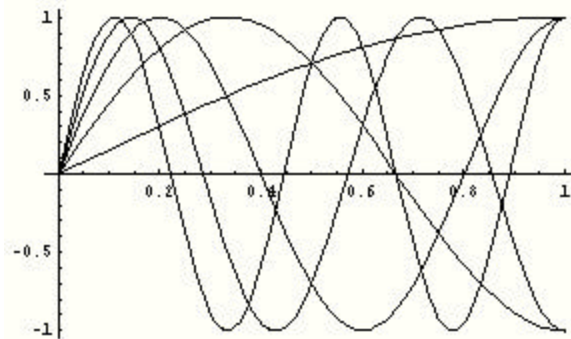
It is clear that the polynomial based Ritz method cannot provide good accurate approximation of the minimizer of the functional.

- (3) Solve the minimization problem by the Ritz method by using the trigonometric basis functions for $m = 10$.

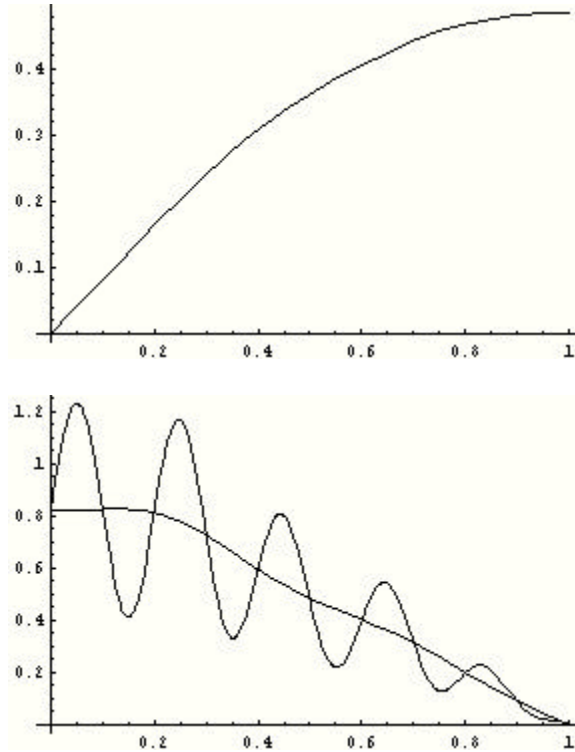
```

n=5;
bf=Table[Sin[(2*i-1)*Pi*x/(2*L)],{i,1,n}];
Plot[Release[bf],{x,0,L}]
dbf=D[bf,x];
kg=Table[0,{i,1,n},{j,1,n}];
fg=Table[0,{i,1,n}];
Do[kgij=ax*dbf[[i]]*dbf[[j]];
  kg[[i,j]]=NIntegrate[kgij,{x,0,L}];
  kg[[j,i]]=kg[[i,j]],
  {i,1,n},{j,i,n}]
kg;
Do[fgi=1*bf[[i]];
  fg[[i]]=NIntegrate[fgi,{x,0,L}],
  {i,1,n}]
fg;
ug=LinearSolve[kg,fg]
ugx=ug.bf;
Plot[ugx,{x,0,L},PlotRange->All]
dugx=D[ugx,x];
sigx=ax*dugx;
Plot[{dugx,sigx},{x,0,L},PlotRange->All]
Fmin=0.5*ug.kg.ug-ug.fg

```



Basis Functions



(4) Solve the minimization problem by the Ritz method by using the basis functions :

$$f_1(x) = \frac{x}{L}$$

$$f_2(x) = y\left(\frac{x}{L}\right), \quad y(s) = \begin{cases} 2s & \text{if } 0 \leq s \leq \frac{1}{2} \\ 2-2s & \text{if } \frac{1}{2} \leq s \leq 1 \\ 0 & \text{if } s < 0 \text{ or } s > 1 \end{cases}$$

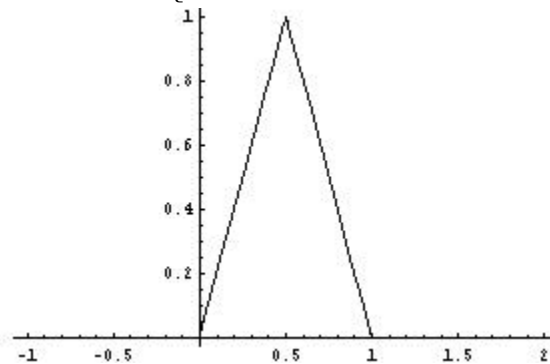


Figure X Function Profile of $y(s)$

$y[s_] := \text{If}[s < 0, 0, \text{If}[s < 1/2, 2*s, \text{If}[s < 1, 2-2*s, 0]]]$

$$\begin{aligned}
f_3(x) &= y \left(2 \frac{x}{L} \right) \\
f_4(x) &= y \left(2 \frac{x}{L} - 1 \right) \\
f_5(x) &= y \left(4 \frac{x}{L} \right) \\
f_6(x) &= y \left(4 \frac{x}{L} - 1 \right) \\
&\dots \\
f_{2^j+k}(x) &= y \left(2^j \frac{x}{L} - k + 1 \right) \quad , \quad k = 1, \dots, 2^j \quad , \quad j = 0, 1, 2, \dots \\
&\dots
\end{aligned}$$

for $m = 10$. That is, find the coefficients c_i , $i = 1, 2, \dots, n$ of an approximation of the minimizer u : $u(x) \approx \sum_{i=1}^n c_i f_i(x)$.

```

In[13]:=
mw[x_]:=If[x<0,0,If[x<0.5,2*x,If[x<1,2*(1-x),0]]];
Plot[mw[x],{x,-1,2}]
jmax=5;
n=Sum[2^j,{j,0,jmax}]+1
bf=Table[0,{i,1,n}];
bf[[1]]=x;
bf[[2]]=mw[x];
Do[nj=1+Sum[2^k,{k,0,j-1}];
  Do[bf[[nj+k]]=mw[2^j*x-k+1],{k,1,2^j}],{j,1,jmax}]
bf;
dbf=D[bf,x];
kg=Table[0,{i,1,n},{j,1,n}];
fg=Table[0,{i,1,n}];
Do[kgij=ax*dbf[[i]]*dbf[[j]];
  kg[[i,j]]=NIntegrate[kgij,{x,0,L}];
  kg[[j,i]]=kg[[i,j]],
  {i,1,n},{j,1,n}]
kg;
Do[fgi=1*bf[[i]];
  fgi[[i]]=NIntegrate[fgi,{x,0,L}],
  {i,1,n}]
fg;
ug=LinearSolve[kg,fg]
ugx=ug.bf;
Plot[ugx,{x,0,L},PlotRange->All]
dugx=D[ugx,x];
sigx=ax*dugx;

```

```
Plot[{dugx,sigx},{x,0,L},PlotRange->All]  
Fmin=0.5*ug.kg.ug-ug.fg
```

