Final Examination, 1998 Fall

MEAM 501 Analytical Methods in Mechanics and Mechanical Engineering

Name Noboru Kikuchi

1. *What is the Legendre polynomial in an interval (-1,1)? Give its definition and major properties.*

The Legendre polynomials $L_i(x)$ are orthogonal polynomials obtained from the polynomial basis functions $1, x, x^2, x^3 ....., x^n, ...$ with respect to the inner product

$$(f,g) = \int_{-1}^{1} f(x) g(x) dx \, .$$

The polynomial $L_i(x)$ is the *i* degree polynomial and it possesses the *i* number of distinct roots in the interval (-1,1).

2. *For a set of given* $n+1$ *points* $x_1 < x_2 < x_3 .... < x_n < x_{n+1}$ *in the interval* (a,b), *define the Lagrange polynomials* $L_i(x)$, *and state their major properties. As a special cases, give the Lagrange polynomials (a) for n = 1, and* $x_1 = -1$, $x_2 = +1$, *and (b) for n = 2, and* $x_1 = -1$, $x_2 = 0$, $x_3 = +1$, *and compute* $\sum_{i=1}^{n+1} L_i(x)$.

Lagrange polynomials are n degree polynomials defined by

$$L_i(x) = \frac{(x-x_1)...(x-x_{i-1})(x-x_{i+1})...(x-x_{n+1})}{(x_i-x_1)...(x_i-x_{i-1})(x_i-x_{i+1})...(x_i-x_{n+1})} = \prod_{\substack{j=1 \\ j \neq i}}^{n+1} \frac{x-x_j}{x_i-x_j}$$

such that

$$L_i\left(x_j\right)=\begin{cases}1 & if \quad i=j\\ 0 & if \quad i\neq j\end{cases}.$$

Therefore, they can be used for interpolation of a function. Furthermore, we have

(a)

$$L_1(x)=\frac{x-x_2}{x_1-x_2}=\frac{1}{2}(1-x) \quad , \quad L_2(x)=\frac{x-x_1}{x_2-x_1}=\frac{1}{2}(1+x)$$

$$L_1(x)+L_2(x)=\frac{x-x_2}{x_1-x_2}+\frac{x-x_1}{x_2-x_1}=1$$

(b)

$$L_1(x)=\frac{1}{2}x(x-1) \quad , \quad L_2(x)=1-x^2 \quad , \quad L_3(x)=\frac{1}{2}x(x+1)$$

$$L_1(x)+L_2(x)+L_3(x)=\frac{1}{2}x(x-1)+1-x^2+\frac{1}{2}x(x+1)=1$$

3. *Suppose that a curve is defined by four data,* $f(0), \frac{df}{dx}(0), f(1), and \frac{df}{dx}(1)$, *which are the values of the function and its first derivative at the two end points of the interval (0,1). Obtain a cubic polynomial* $f(s)$ *that is determined by these four data. It is noted that such a cubic polynomial is called the Hermite cubic polynomial.*

$$f(s)=\{1 \quad s \quad s^2 \quad s^3\}\begin{Bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{Bmatrix} \quad , \quad f'(s)=\{0 \quad 1 \quad 2s \quad 3s^2\}\begin{Bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{Bmatrix}$$

$$\begin{Bmatrix}f(0)\\ f'(0)\\ f(1)\\ f'(1)\end{Bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&1\\0&1&2&3\end{bmatrix}\begin{Bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix}c_1\\ c_2\\ c_3\\ c_4\end{Bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&1\\0&1&2&3\end{bmatrix}^{-1}\begin{Bmatrix}f(0)\\ f'(0)\\ f(1)\\ f'(1)\end{Bmatrix}$$

$$f(s)=\{1 \quad s \quad s^2 \quad s^3\}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 1 & 1 & 1 & 1\\ 0 & 1 & 2 & 3\end{bmatrix}^{-1}\begin{Bmatrix} f(0)\\ f'(0)\\ f(1)\\ f'(1)\end{Bmatrix}$$

$$=\{1-3s^2+2s^3 \quad s-2s^2s^3 \quad 3s^2-2s^3 \quad -s^2+s^3\}\begin{Bmatrix} f(0)\\ f'(0)\\ f(1)\\ f'(1)\end{Bmatrix}$$

4. *Define the Bezier spline with* $n+1$ *control points* $x_1, x_2, \ldots, x_{n+1}$.

$$x=\sum_{i=1}^{n+1} x_i B_i^n(x) \quad , \quad B_i^n(x)=\frac{1}{(i-1)!(n-i+1)!}x^{i-1}(1-x)^{n-i+1}$$

5. *The position vector of an arbitrary point P of a curve C on a two dimensional plane is given by a parametric form* $\mathbf{r}(\mathsf{x})=\begin{Bmatrix} x(\mathsf{x})\\ y(\mathsf{x})\end{Bmatrix}$, *where* x *is a parametric coordinate in* $(0,1)$. *Suppose that a coordinate s is defined along the curve, and let s be zero at the point defined by* $\mathsf{x}=0$, *while its value is set as the total length L of the curve at the other end of the curve defined by* $\mathsf{x}=1$. *(a) Establish the relation between s and* x. *(b) How to compute the total length of the curve? (c) How to define the unit tangent vector* ***t****? (d) What is the unit normal vector* ***n****? (e) State a way to calculate the curvature?*

(a) Noting that $ds=\sqrt{d\mathbf{r}\bullet d\mathbf{r}}=\sqrt{\left(\frac{dx}{d\mathsf{x}}\right)^2+\left(\frac{dy}{d\mathsf{x}}\right)^2}\,d\mathsf{x}$, we have the relation

$$s=\int_0^{\mathsf{x}} ds=\int_0^{\mathsf{x}}\sqrt{\left(\frac{dx}{d\mathsf{x}}\right)^2+\left(\frac{dy}{d\mathsf{x}}\right)^2}\,dx$$

(b) $L = \int_0^1 ds = \int_0^1 \sqrt{\left(\frac{dx}{d\chi}\right)^2 + \left(\frac{dy}{d\chi}\right)^2}\, dx$

(c) A unit tangent vector is given by

$$\mathbf{t} = \frac{d\mathbf{r}}{ds} = \frac{d\chi}{ds}\frac{d\mathbf{r}}{d\chi} = \frac{\frac{dx}{d\chi}\mathbf{e}_x + \frac{dy}{d\chi}\mathbf{e}_y}{\sqrt{\left(\frac{dx}{d\chi}\right)^2 + \left(\frac{dy}{d\chi}\right)^2}}$$

, $\mathbf{e}_x, \mathbf{e}_y$ are the unit vectors along the x and y axes.

(d) Noting that the tangent vector **t** is unit, we have

$$\mathbf{t} \bullet \mathbf{t} = 1 \quad \Rightarrow \quad \frac{d\mathbf{t}}{ds} \bullet \mathbf{t} = 0 \quad \Rightarrow \quad \frac{d\mathbf{t}}{ds} \perp \mathbf{t} \quad \Rightarrow \mathbf{n} = \frac{\frac{d\mathbf{t}}{ds}}{\left\| \frac{d\mathbf{t}}{ds} \right\|} = \frac{\frac{d\mathbf{t}}{ds}}{\sqrt{\frac{d\mathbf{t}}{ds} \bullet \frac{d\mathbf{t}}{ds}}}$$

(e) Curvature is defined by

$$\frac{d\mathbf{t}}{ds} \perp \mathbf{t} \quad \Rightarrow \quad \frac{d\mathbf{t}}{ds} = \kappa \mathbf{n} \quad \Rightarrow \kappa = \left\| \frac{d\mathbf{t}}{ds} \right\| = \sqrt{\frac{d\mathbf{t}}{ds} \bullet \frac{d\mathbf{t}}{ds}}$$

6. State the Gauss-Legendre quadarature.

Integration of a function f defined on the interval (-1,+1) is approximated by the summation of a linear combination of the weights w_i and the values of the function at the quadrature points x_i which are the roots of the n degree Legendre polynomial $L_n(x)$ defined on the interval (-1,+1):

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

This quadrature can integrate $2n-1$ degree polynomials exactly.

7. What is the necessary condition of the following minimization problem:

$$\min_{\mathbf{x}\in\mathbf{R}^n} \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{b} \quad , \quad \mathbf{A}\in\mathbf{R}^{n\times n}, \mathbf{b}\in\mathbf{R}^n .$$

$$dF(\mathbf{x})(\mathbf{y}) = \lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}F(\mathbf{x}+\alpha\mathbf{y}) = \lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}\left\{\frac{1}{2}(\mathbf{x}+\alpha\mathbf{y})^T\mathbf{A}(\mathbf{x}+\alpha\mathbf{y}) - (\mathbf{x}+\alpha\mathbf{y})^T\mathbf{b}\right\}$$
$$= \frac{1}{2}\mathbf{y}^T\mathbf{A}\mathbf{x} + \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{y} - \mathbf{y}^T\mathbf{b} = \mathbf{y}^T\left\{\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)\mathbf{x} - \mathbf{b}\right\} = 0 \quad , \quad \forall\mathbf{y}$$

therefore, the necessary condition becomes the matrix equation

$$\frac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)\mathbf{x} = \mathbf{b}$$

8. Define the Lagrangian L to the following constrained minimization problem

$$\min_{\substack{\mathbf{x}\in\mathbf{R}^2 \\ x_1+2x_2\le 1}} \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{b} \quad , \quad \mathbf{x} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \mathbf{A} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \mathbf{b} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$$

,and find the necessary condition of the mini-max problem associated to the Lagrangian L defined.

$$L(\mathbf{x},\lambda) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} - \mathbf{x}^T\mathbf{b} - \lambda\left(\begin{bmatrix}1 & 2\end{bmatrix}\mathbf{x} - 1\right)$$
$$= \frac{1}{2}\begin{Bmatrix}x_1 & x_2\end{Bmatrix}\begin{bmatrix}2 & -1\\ -1 & 3\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} - \begin{Bmatrix}x_1 & x_2\end{Bmatrix}\begin{Bmatrix}1\\ 2\end{Bmatrix} - \lambda\left(\begin{bmatrix}1 & 2\end{bmatrix}\begin{Bmatrix}x_1\\ x_2\end{Bmatrix} - 1\right)$$

From $L(\mathbf{x},\lambda) \le L(\delta\mathbf{x},\lambda)$, $\forall\delta\mathbf{x}$ we have

$$\delta\mathbf{x}^T\left\{\mathbf{A}\mathbf{x} - \mathbf{b} - \lambda\begin{Bmatrix}1\\ 2\end{Bmatrix}\right\} = 0 \quad , \quad \forall\delta\mathbf{x} \qquad \Rightarrow \qquad \mathbf{A}\mathbf{x} - \lambda\begin{Bmatrix}1\\ 2\end{Bmatrix} = \mathbf{b}$$

From $L(\mathbf{x},\delta\lambda) \le L(\mathbf{x},\lambda)$, $\forall\delta\lambda \le 0$, we have

$$(\delta\lambda - \lambda)\left(\begin{bmatrix}1 & 2\end{bmatrix}\mathbf{x} - 1\right) \ge 0 \quad , \quad \forall\delta\lambda \le 0$$

Combining these two, we have the necessary condition of the mini-max problem for the Lagrangian:

$$\mathbf{A}\mathbf{x} - \lambda\begin{Bmatrix}1\\ 2\end{Bmatrix} = \mathbf{b}$$
$$(\delta\lambda - \lambda)\left(\begin{bmatrix}1 & 2\end{bmatrix}\mathbf{x} - 1\right) \ge 0 \quad , \quad \forall\delta\lambda \le 0$$

9. Find the first variation of the functional F defined by

$$F(v) = \frac{1}{2}\int_0^L\left\{EA\left(\frac{dv}{dx}\right)^2 + kv^2 - 2fv\right\}dx + \frac{1}{2}k_0\left\{v\left(\frac{L}{2}\right)\right\}^2 - P_L v(L)$$

for given functions $EA, k,$ *and* f , *and constants* $k_0, P_L,$ *and* L.

$$
\begin{aligned}
\delta F(u)(v) &= \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} F(u+\alpha v) \\
&= \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \left[\begin{aligned} &\frac{1}{2}\int_0^L \left\{ EA\left(\frac{du}{dx} + \alpha \frac{dv}{dx}\right)^2 + k(u+\alpha v)^2 - 2f(u+\alpha v) \right\} dx \\ &+ \frac{1}{2} k_0 \left\{ u\left(\frac{L}{2}\right) + \alpha v\left(\frac{L}{2}\right) \right\}^2 - P_L \{u(L) + \alpha v(L)\} \end{aligned} \right] \\
&= \int_0^L \left\{ EA \frac{du}{dx}\frac{dv}{dx} + kuv - 2fv \right\} dx + k_0 u\left(\frac{L}{2}\right) v\left(\frac{L}{2}\right) - P_L v(L)
\end{aligned}
$$

10. If a minimization problem

$$
\min_{v \in K} F(v) \quad , \quad K = \{ v \in V \mid v(0) = 1 \}
$$

where V is a linear space of all piecewise continuously differentiable functions defined on the interval (0,L), is considered for the functional F defined in Problem 9, find the necessary condition.

$$
\begin{aligned}
\delta F(u)(v-u) &= \int_0^L \left\{ EA \frac{du}{dx} \frac{d(v-u)}{dx} + ku(v-u) - 2f(v-u) \right\} dx \\
&+ k_0 u\left(\frac{L}{2}\right)\left(v\left(\frac{L}{2}\right) - u\left(\frac{L}{2}\right)\right) - P_L (v(L) - u(L)) \geq 0 \quad , \quad \forall v \in K
\end{aligned}
$$

Noting that

$$
v = u \pm \delta u \quad , \quad \forall \delta u \quad s.t.\ \delta u(0) = 0
$$

we have

$$\mathrm{d}F(u)(\mathrm{d}u) = \int_0^L \left\{ EA \frac{du}{dx}\frac{d\mathrm{d}u}{dx} + ku\mathrm{d}u - 2\,f\mathrm{d}u \right\} dx + k_0 u\left(\frac{L}{2}\right)\mathrm{d}u\left(\frac{L}{2}\right) - P_L \mathrm{d}u(L) = 0$$

$$,\quad \forall \mathrm{d}u \quad s.t.\; \mathrm{d}u(0) = 0$$