

A Quadratic Program based Control Synthesis under Spatiotemporal Constraints and Non-vanishing Disturbances

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Abstract—In this paper, we study the effect of non-vanishing disturbances on the stability of fixed-time stable (FxTS) systems. We present a new result on FxTS, which allows a positive term in the time derivative of the Lyapunov function with the aim to model bounded, non-vanishing disturbances in system dynamics. We characterize the neighborhood to which the system trajectories converge, as well as the convergence time. Then, we use the new FxTS result and formulate a quadratic program (QP) that yields control inputs which drive the trajectories of a class of nonlinear, control-affine systems to a goal set in the presence of control input constraints and non-vanishing, bounded disturbances in the system dynamics. We consider an overtaking problem on a highway as a case study, and discuss how to both set up the QP and decide when to start the overtake maneuver in the presence of sensing errors.

I. INTRODUCTION

Control design for systems with input and state constraints is not a trivial task. *Spatio-temporal* specifications typically impose spatial constraints that require the system trajectories to be in a safe set at all times, and temporal constraints impose convergence of the system trajectories to a goal set within a given time. Incorporating safety-related constraints on the system states can be achieved via control barrier functions (CBF) [1]. For requirements involving convergence of the system states to a desired location or a set, approaches using control Lyapunov functions (CLF) [2]–[4] are very popular. Many authors have used CLFs in control design either via Sontag’s formula [5], [6], or in an optimization framework [2], [7] to guarantee convergence of closed-loop system trajectories to a given goal point or a goal set.

For concurrent safety and convergence guarantees, a combination of CLFs and CBFs in the control synthesis can be used [1], [5], where the CLF guarantees convergence while the CBF guarantees safety of the state trajectories. The authors in [8] utilize Lyapunov-like barrier functions to guarantee asymptotic tracking of a time-varying output trajectory, while the system output always remains inside a given set. Casting control synthesis problems as quadratic programs has gained popularity recently due to ease of implementation on real-time systems [9], [10]. The fact that CLF and CBF conditions are linear in the control input enables the use of QPs for problems involving spatiotemporal specifications [1]–[3]. The authors in [11] use CBF to encode signal-temporal logic (STL) specifications and formulate a QP to compute the control input. Most of the aforementioned work is concerned with designing control laws so that reaching a desired location or a desired goal set is achieved as time goes to infinity, i.e., asymptotically. Based on the notion of

fixed-time stability (FxTS) [12], the authors in [13] define a Fixed-Time CLF to guarantee convergence of the state trajectories to the origin within a fixed time, as opposed to asymptotic or exponential convergence. Practically, it is also important to design robust controllers against uncertainties and disturbances in the system dynamics to account for unmodelled dynamics and sensing errors. Robust CBFs guarantee forward-invariance of safe sets [9], [14], [15]. Typically, the safe set is *contracted* by a small amount that depends upon the Lipschitz constants of the CBF and the bound on the considered disturbance.

In the presence of non-vanishing disturbances, typically only boundedness of the trajectories in a neighborhood of the nominal equilibrium (or set) can be guaranteed (see, e.g., [16, Section 9.2]). In this paper, we consider bounded, non-vanishing disturbances in the dynamics of a (nominal) system with a FxTS equilibrium, and guarantee that the system trajectories converge to a neighborhood of the nominal equilibrium point within a fixed time. We characterize the size of this neighborhood and the convergence time as a function of the bound on the considered disturbances. Then, in conjunction with robust CBFs, we formulate a QP to compute a control input that renders the safe set forward invariant, and drives the closed-loop trajectories to a neighborhood of a desired goal set within a fixed time, in the presence of control input constraints. Finally, we consider a two-lane overtake scenario where an *Ego* car is required to overtake a *Lead* car while maintaining a safe distance from it, within an available time-window dictated by the presence of an *Oncoming* car in the overtake lane. We assume that the position and velocity of the other cars are available to the *Ego* car within some bounded error to model sensing uncertainties, and that the control inputs are subject to some bounded actuation error. Then, utilizing the new robust FxTS result, we formulate a systematic way of deciding for the *Ego* car whether executing an overtake is safe or not. When safe, the developed QP formulation produces the controller for the *Ego* car to safely perform the overtake maneuver in the available time frame.

The paper is organized as follows. Section II provides the foundations for Set Invariance and Fixed-Time Stability (FxTS), and introduces preliminary results on Robust FxT CLFs and Robust CBFs. In Section III an overtaking problem is used to motivate the Robust FxT-CLF-CBF-QP framework, while Section IV discusses the simulation results. We end with conclusion and directions for future work in Section V.

II. MATHEMATICAL PRELIMINARIES

Subsequently, \mathbb{R} denotes the set of real numbers and \mathbb{R}_+ the set of non-negative real numbers. We use $\|\cdot\|$ to denote the Euclidean norm. We write ∂S for the boundary of the closed set S , $\text{int}(S)$ for its interior. The Lie derivative of a

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function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ along a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is denoted as $L_f V(x) \triangleq \frac{\partial V}{\partial x} f(x)$.

A. Forward Invariance of Safe Set

Consider the control-affine system

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad x(0) = x_0, \quad (1)$$

where $x \in \mathbb{R}^n$, $u \in \mathcal{U} \subset \mathbb{R}^m$ are the state and the control input vector, respectively, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are continuous functions. Here, \mathcal{U} denotes the set of admissible control inputs. Define a safe set $S_s = \{x \mid h_s(x) \geq 0\}$, where $h_s : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuously differentiable function.

We present a necessary and sufficient condition, known as Nagumo's Theorem, for guaranteeing forward invariance of the safe set S_s , i.e., safety of the system trajectories.

Lemma 1. *Let the solution of (1) exist and be unique in forward time. Then, the set S_s is forward-invariant for the closed-loop trajectories of (1) with $x(0) \in S_s$ if and only if there exists $u \in \mathcal{U}$ such that $L_f h(x) + L_g h(x)u \geq 0$ for all $x \in \partial S_s$, where $\partial S_s \triangleq \{x \mid h_s(x) = 0\}$ is the boundary of the safe set S_s .*

B. Fixed-Time Stability

Next, we review the notion of fixed-time stability. Consider the nonlinear system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0, \quad (2)$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous with $f(0) = 0$. The origin is said to be an FxTS equilibrium of (2) if it is Lyapunov stable and *fixed-time convergent*, i.e., for all $x(0) \in \mathbb{R}^n$, the system trajectories satisfy $\lim_{t \rightarrow T} x(t) = 0$, where $T < \infty$ is independent of $x(0)$ [12]. Lyapunov conditions for FxTS is given as follows.

Theorem 1 ([12]). *Suppose there exists a positive definite function $V : \mathbb{R}^n \rightarrow \mathbb{R}$ such that*

$$\dot{V}(x) \leq -aV(x)^p - bV(x)^q, \quad (3)$$

holds along the trajectories of (2) with $a, b > 0$, $0 < p < 1$ and $q > 1$. Then, the origin of (2) is FxTS with a settling time $T \leq T_b$ where

$$T_b = \frac{1}{a(1-p)} + \frac{1}{b(q-1)}. \quad (4)$$

Building upon the *nominal* system (1), we now consider the *perturbed* system, given as

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) + \phi(x(t)), \quad x(0) = x_0 \quad (5)$$

where f, g are as in (1), and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an added, unmatched disturbance, possibly non-vanishing, which is assumed to be bounded. We denote the upper bound as $\|\phi\|_\infty \triangleq \sup_{x \in \mathcal{D}_0} \|\phi(x)\|$, where $\mathcal{D}_0 \subseteq \mathbb{R}^n$ is a neighborhood of the origin. The additive disturbance ϕ can model uncertainties in the parameters used in the control design; modelling uncertainties; external perturbations to the dynamics, such as wind; and actuation errors, for example a power surge, to name a few. Next, we present a new result on robustness of the trajectories around nominal FxTS equilibria against a class of bounded, non-vanishing disturbances.

C. Robust FxT CLF

We extend the result in Theorem 1 by introducing a positive constant in the upper bound of the time derivative of the Lyapunov candidate, V . We label V a robust FxT-CLF.

Theorem 2. *Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable, positive definite, proper function, satisfying*

$$\dot{V} \leq -c_1 V^{a_1} - c_2 V^{a_2} + c_3, \quad (6)$$

with $c_1, c_2 > 0$, $c_3 \in \mathbb{R}$, $a_1 = 1 + \frac{1}{\mu}$, $a_2 = 1 - \frac{1}{\mu}$ for some $\mu > 1$, along the trajectories of (2). Then, there exists a neighborhood D of the origin such that for all $x(0) \in \mathbb{R}^n \setminus D$, the trajectories of (2) reach D in a fixed time T , where

$$D = \begin{cases} \{x \mid V(x) \leq k \left(\frac{c_3 + \sqrt{c_3^2 - 4c_1 c_2}}{2c_1} \right)^\mu\}; & c_3 \geq 2\sqrt{c_1 c_2}, \\ \{x \mid V(x) \leq \frac{c_3}{2\sqrt{c_1 c_2}}\}, & 0 < c_3 < 2\sqrt{c_1 c_2}, \\ \{0\}, & c_3 \leq 0, \end{cases} \quad (7)$$

$$T \leq \begin{cases} \frac{\mu}{c_1(b-a)} \log \left(\frac{kb-b}{kb-a} \right); & c_3 \geq 2\sqrt{c_1 c_2}, \\ \frac{\mu}{c_1 k_1} \left(\frac{\pi}{2} - \tan^{-1} k_2 \right), & 0 < c_3 < 2\sqrt{c_1 c_2}, \\ \frac{\mu\pi}{2\sqrt{c_1 c_2}}, & c_3 \leq 0, \end{cases} \quad (8)$$

where $k > 1$, a, b are the solutions of $\gamma(s) = c_1 s^2 - c_3 s + c_2 = 0$, $k_1 = \sqrt{\frac{4c_1 c_2 - c_3^2}{4c_1^2}}$, and $k_2 = -\frac{c_3}{\sqrt{4c_1 c_2 - c_3^2}}$.

Proof. Note that for $c_3 \leq 0$, we obtain (3) from (6), and so FxTS of the origin is guaranteed for all $x \in \mathbb{R}^n$. Thus, we focus on when $c_3 > 0$, for which sufficiently small values of V cause the right hand side of (6) to become positive. The proof follows from [17, Lemma 2]. Consider (6) and let $c_1 V^{a_1} + c_2 V^{a_2} > c_3$. Re-write the inequality to obtain

$$\int_{V(x(0))}^{V(x(T))} \frac{1}{-c_1 V^{a_1} - c_2 V^{a_2} + c_3} dV \geq \int_0^T dt = T, \quad (9)$$

where T is the time when the system trajectories first reach the domain D . It is easy to show that for each of the cases listed in the Theorem statement, $c_1 V^{a_1} + c_2 V^{a_2} > c_3$ and thus the right-hand side of (6) is negative for all $x \notin D$. Now, to show that the system trajectories converge to D in fixed-time, we compute upper bounds on T .

For the case when $c_3 < 2\sqrt{c_1 c_2}$, part (i) in [17, Lemma 2] provides an upper bound on the left-hand side of (9) for $x \notin D = \{x \mid V(x) \leq \frac{c_3}{2\sqrt{c_1 c_2}}\}$. Similarly, for the case $c_3 \geq 2\sqrt{c_1 c_2}$, part (ii) of the Lemma provides upper bounds on the left-hand side of (9). Thus, we obtain the domains D and the bounds on convergence times T for the various cases directly. Since for all three cases, $T < \infty$ and is independent of the initial conditions, we have that the system trajectories reach the set D within a fixed time T . \square

Next, we use Theorem 2 to show robustness of a FxTS origin against a class of non-vanishing, bounded, additive disturbance in the system dynamics.

Corollary 1. *Assume that there exists $u(t) \in \mathcal{U}$, where \mathcal{U} is a set of admissible control inputs, such that the origin for the nominal system (1) is fixed-time stable, and that there exists a Lyapunov function V satisfying conditions of Theorem 1. Additionally, assume that there exists $L > 0$ such that $\|\frac{\partial V}{\partial x}\| \leq L$ for all $x \in \mathcal{D}_0 \subseteq \mathbb{R}^n$. Then, there exists $D \subset \mathbb{R}^n$ such that for all $x(0) \in \mathcal{D}_0 \setminus D$, the trajectories of (5) reach the set D in a fixed time.*

Proof. The time derivative of V along the system trajectories of (5) reads

$$\dot{V} = \frac{\partial V}{\partial x} [f(x) + g(x)u + \phi(x)] \leq -aV^p - bV^q + L\|\phi\|_\infty.$$

Hence, using Theorem 2, we obtain that there exists $D \subset \mathbb{R}^n$ such that all solutions starting outside D reach the set D in a fixed time T , where the set D and the convergence time T is a function of a, b, p, q, L and $\|\phi\|_\infty$. \square

In Theorem 2, (7) characterizes an estimate, D , of a neighborhood of the equilibrium to where system trajectories are guaranteed to converge within a fixed-time, T , as convergence to the equilibrium cannot be guaranteed, and (8) provides an upper bound on T independent of $x(0) \in D$.

D. Robust CBF

Next, we review the notion of a robust CBF, to guarantee forward invariance of a safe set, in the presence of a class of additive, non-vanishing disturbances. Here, we assume that $S_s \subset \mathcal{D}_0$.

Lemma 2. *The set S_s is forward-invariant for the closed-loop trajectories of (5) if*

$$\sup_{u \in \mathcal{U}} \{L_f h_s(x) + L_g h_s(x)u\} \geq - \left\| \frac{\partial h_s(x)}{\partial x} \right\| \|\phi\|_\infty, \quad (10)$$

holds for all $x \in \partial S_s \cap \mathcal{D}_0$.

Proof. The time derivative of h_s along the trajectories of (5) reads

$$\dot{h}_s = L_f h_s(x) + L_g h_s(x)u + \frac{\partial h_s(x)}{\partial x} \phi(x).$$

For $x \in \partial S_s \cap \mathcal{D}_0$, we have that $h_s(x) = 0$ and $\|\phi(x)\| \leq \|\phi\|_\infty$. Using (10), we obtain that there exists a $u \in \mathcal{U}$ such that $\dot{h}_s \geq 0$. Thus, using Lemma 1, we have that forward invariance of set S is guaranteed. \square

Thus, condition (10) can be used to guarantee forward invariance of a safe set S_s in the presence of a class of additive, non-vanishing disturbances. Next, we take up a case study, and discuss how to use the robust FxT-CLF and robust CBF in a QP framework to compute a control input so that conditions (6), (10) hold along the closed-loop trajectories.

III. CASE STUDY: OVERTAKE PROBLEM

In this section, we introduce a framework for computing overtake control via a FxT-CLF-CBF QP subject to bounded, non-vanishing, additive disturbances.

A. Problem Formulation

We consider an Ego car starting behind a slowly-moving, Lead car on a two-lane undivided highway, where the Ego car seeks to overtake the Lead car in a safe, timely manner whilst avoiding oncoming traffic (see Figure 1). The combined effort to achieve lane keeping, obstacle avoidance, and goal-reaching makes this problem challenging.

For each vehicle, we select the model of a kinematic bicycle in an inertial frame, introduced in [18] and adapted for automobile highway merging in [19]. We use subscripts e, l, oc to denote the Ego, the Lead and Oncoming car. The motion of the cars is modelled as:

$$\dot{q}_i = \begin{bmatrix} v_i \cos(\theta_i) \\ v_i \sin(\theta_i) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \omega_i \\ a_i \end{bmatrix} + \phi_i, \quad (11)$$

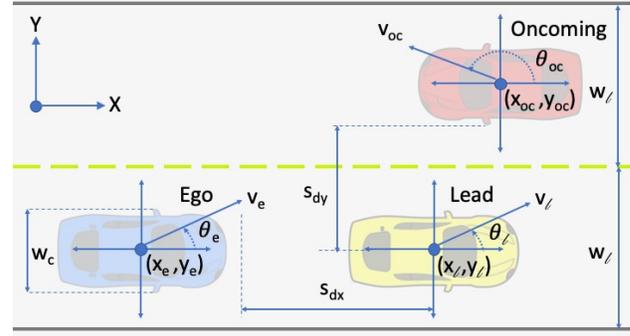


Fig. 1. Problem setup for the overtake problem.

where $q_i = [x_i \ y_i \ \theta_i \ v_i]^T$ is the state vector of car $i \in \{e, l, oc\}$, x_i is the longitudinal position, y_i is the transverse position, θ_i is the heading angle, v_i is the velocity, ω_i is the angular control input, and a_i is the longitudinal control input (measured as a fraction of $M_i g$, where M_i is the vehicle mass and $g = 9.81 \text{ m/s}^2$). The disturbance in each car's dynamics, ϕ_i , takes into account modelling error and external perturbations such as wind or road grade. We assume that the disturbance ϕ_i is bounded, i.e., if $\hat{q}_l^e, \hat{q}_{oc}^e$ denote the states of the Lead and Oncoming car as estimated by the Ego car, then there exists $\epsilon > 0$ such that $\|\hat{q}_j^e(t) - q_j(t)\| \leq \epsilon$ for all $t \geq 0, j \in \{l, oc\}$. Consistent with the discussion in the previous section, we define $\|\phi\|_\infty = \epsilon$.

The control input $u_i \in \mathbb{R}^2$ for car i consists of ω_i and a_i . Notably, our adjustment to the dynamics of [18] is such that θ describes the full steering dynamics, $\theta = \frac{v \tan(\beta)}{l_v}$, where β is the steering angle in rad and l_v is the length of the vehicle in m. This is reasonable due to the small angle approximation, which we expect to hold in our problem, and from which we obtain that $\tan(\beta) \approx \beta$, such that $\theta \approx \frac{v\beta}{l_v}$. Additionally, the model imposes the no-slip condition, and vehicle volumes are considered for safety.

The overtake problem considered in the case study is formally stated below.

Problem 1. *Given $q_e(0), q_l(0), q_{oc}(0)$ determine if overtaking the Lead car is safe, i.e., if there exist vehicle state and control trajectories, $q_e(t), q_l(t), q_{oc}(t), u_e(t)$, where $u_e(t) \in \mathcal{U} = \{(\omega, a) \mid \omega_m \leq \omega \leq \omega_M, a_m \leq a \leq a_M\}$, such that $\|x_e(t) - x_i(t)\| > s_{dx}, \|y_e(t) - y_i(t)\| > s_{dy}$ for $i \in \{l, oc\}$ and $t \in [0, T]$, where T is the upper bound on time required to complete the overtake. If safe, design a control input, $u_e(t) \in \mathcal{U}$ for all $x_e(0) < x_l(0) - s_{dx}, y_e(0) = y_l(0), x_l(0) < x_{oc}(0), \theta_e(0) = \theta_l(0) = 0, \theta_{oc}(0) = \pi$, and $v_e(0), v_l(0), v_{oc}(0) > 0$, so that the closed-loop trajectories of the Ego car overtake the Lead car.*

We divide the Problem 1 into the following sub-problems:

- 1) Determine when an overtake is safe to initiate;
- 2) Steer Ego Vehicle safely into overtaking lane;
- 3) Advance Ego Vehicle safely past Lead Vehicle;
- 4) Steer Ego Vehicle safely back into original lane.

We use $h_{s,1}$ to encode that the Ego vehicle maintains all wheels within road limits even with bounded steering capabilities, and $h_{s,2}$, that the Ego vehicle respects a safety-zone around the Lead Vehicle. The CBFs $h_{s,i}(q)$ are defined

as follows:

$$h_{s,1} = (y_e - e_1)(y_e - e_2), \quad (12)$$

$$h_{s,2} = 1 - \left(\frac{x_l - x_e}{s_{dx}} \right)^2 - \left(\frac{y_l - y_e}{s_{dy}} \right)^2, \quad (13)$$

where $s_{dx} = v_e \tau \cos \theta_e + l_c$, $s_{dy} = w_l - \frac{w_c}{2}$ are the semi-major axes for the x and y coordinates respectively, and e_1, e_2 are parameters which define the edge of the road in the y coordinate. Here, $\tau = 1.8$ sec is the time headway¹. Specifically, we define $e_1 = \frac{w_c}{2} + v_e \omega_{max} (1 - \cos \theta_e)$, and $e_2 = (2w_l - \frac{w_c}{2}) - v_e \omega_{max} (1 - \cos \theta_e)$, where $w_l = 3$ m is the width of a lane² and $w_c = 2.27$ m and $l_c = 5.05$ m are the width and length of a car³.

To capture the convergence requirement in each of the sub-problems 2) – 4), we define goal sets $S_{G_j} = \{q \mid V(q - q_j^g) \leq 0\}$, where $q_j^g = [x_j^g \ y_j^g \ \theta_j^g \ v_j^g]^T$ denotes the goal location for the j -th sub-problem, $j \in \{2, 3, 4\}$, and we define $\bar{q}_j = q - q_j^g$. We use a CLF $V: \mathbb{R}^4 \rightarrow \mathbb{R}$ to encode the convergence requirement, defined as

$$V(\bar{q}) = K(k_x \bar{x}^2 + k_v \bar{v}^2 + k_{xv} \bar{x} \bar{v} + k_y \bar{y}^2 + k_\theta \bar{\theta}^2 + k_{y\theta} \bar{y} \bar{\theta} - 1) \quad (14)$$

where K is a constant selected during our parameter selection phase, $k_x, k_y, k_\theta, k_v, k_{xv}, k_{y\theta}$ are constant gains chosen such that $k_{xv}^2 \leq 4k_x k_v$ and $k_{y\theta}^2 \leq 4k_y k_\theta$, which influence the size and shape of the goal subspace, and $\bar{x} = x_e - x_j^g, \bar{y} = y_e - y_j^g, \bar{\theta} = \theta_e - \theta_j^g$ and $\bar{v} = v_e - v_j^g$. Note that the convergence requirement defining the goal location, and thus the CLF V , changes for each sub-problem; nevertheless, V is positive definite w.r.t to its K -sublevel set in each case. Consider the following inequalities.

$$L_f h_{s_1}(x) + L_g h_{s_1}(x)u \geq 0, \quad (15)$$

$$L_f h_{s_2}(x) + L_g h_{s_2}(x)u \geq 0, \quad (16)$$

which when true at the boundary of the sets S_{s_1} and S_{s_2} guarantee forward invariance of the respective sets. We need the following viability assumption before we can proceed with our main results.

Assumption 1. *There exists a control input $u \in \mathcal{U}$ such that*

- 1) for all $q \in \partial S_{s_1} \cap \partial S_{s_2}$, both (15) and (16) holds;
- 2) for all $q \in \partial S_{s_1}$ (respectively, $q \in \partial S_{s_2}$), (15) (respectively, (16)) holds.

Furthermore, $S_{G_j} \cap S_{s_1} \cap S_{s_2} \neq \emptyset, \forall j \in \{2, 3, 4\}$.

Assumption 1 guarantees that the viable set is non-empty, and that there exists a control input on the boundary of the safe set as well as their intersection, that can keep the system trajectories inside the intersection of the safe sets. We now introduce a QP formulation to solve Problem 1:

$$\min_{u, \delta_1, \delta_2, \delta_3} \frac{1}{2} u^T u + p_1 \delta_1^2 + p_2 \delta_2^2 + p_3 \delta_3^2 + q_1 \delta_1 \quad (17a)$$

$$\text{s.t. } A_u u \leq b_u \quad (17b)$$

$$L_f V(q_e) + L_g V(q_e)u \leq \delta_1 - \alpha_1 \max\{0, V(q_e)\}^{\gamma_1} - \alpha_2 \max\{0, V(q_e)\}^{\gamma_2} \quad (17c)$$

$$L_f h_{s_1}(q_e) + L_g h_{s_1}(q_e)u \geq -\delta_2 h_{s_1}(q_e) \quad (17d)$$

$$L_f h_{s_2}(q_e) + L_g h_{s_2}(q_e)u \geq -\delta_3 h_{s_2}(q_e) \quad (17e)$$

¹ $\tau = 1.8$ sec comes from the "half-the-speedometer" rule, as in [20].

²Taken from <https://tinyurl.com/knzhwje>

³Taken from <https://tinyurl.com/y2rr375y>

where (17a), quadratic in the decision variables, models a minimum-norm controller with relaxation variables $\delta_1, \delta_2, \delta_3$ and $p_1, p_2, p_3, q_1 \geq 0, \gamma_1 = 1 + \frac{1}{\mu}, \gamma_2 = 1 - \frac{1}{\mu}$, where $\mu > 1$, and $\alpha_i = \frac{\pi \mu}{2T}$ for $i = \{1, 2\}$. The constraints, all of which are linear in the decision variables, accomplish the following: (17b) enforces input constraints, (17c) provides the FxT convergence guarantee, and (17d) and (17e) provide safety guarantees. Our formulation, specifically (17c), utilizes the result of Theorem 2 in order to guarantee fixed-time convergence for any δ_1 . Moreover, we discuss the relationship between this δ_1 term and an upper limit on the class of additive, bounded, non-vanishing disturbances considered in Problem 1.

Next, we discuss the feasibility of the QP (17).

Lemma 3. *Under Assumption 1, the QP (17) is feasible for all $q \in (S_{s_1} \cap S_{s_2}) \setminus S_G$.*

Proof. Let $q \notin S_G$, and consider the three cases $q \in \text{int}(S_{s_1}) \cap \text{int}(S_{s_2}), q \in \partial S_{s_1}$ and $q \in \partial S_{s_2}$, separately.

In the first case, we have that $h_{s_1}, h_{s_2}, V \neq 0$. Choose any u that satisfies (17b). With this choice of u , one can choose $\delta_1, \delta_2, \delta_3$ so that (17c)-(17e) hold with equality. This is possible since functions V, h_{s_1}, h_{s_2} are non-zero. Thus, for all $q \in (\text{int}(S_{s_1}) \cap \text{int}(S_{s_2})) \setminus S_G$, there exists a solution to (17). Per Assumption 1, for all $q \in \partial S_{s_1}$, there exists a control input $u \in \mathcal{U}$ such that (17d) holds with any δ_2 (since $h_{s_1}(q) = 0$ for $q \in \partial S_{s_1}$, the choice of δ_2 does not matter). Thus, using any u that satisfies (17d), one can define δ_1 and δ_3 so that (17c) and (17e) hold with equality. Similarly, one can construct a solution for the case when $q \in \partial S_{s_2}$, and $q \in \partial S_{s_1} \cap \partial S_{s_2}$. Thus, the QP (17) is feasible for all $q \in (S_{s_1} \cap S_{s_2}) \setminus S_G$. \square

We are now ready to present our main result.

Theorem 3. *Let the solution to the QP (17) be denoted as $z^*(\cdot) = (u^*(\cdot), \delta_1^*(\cdot), \delta_2^*(\cdot), \delta_3^*(\cdot))$. Assume that $\|\phi\|_\infty \leq \frac{\delta_1^*}{\|\frac{\partial V}{\partial q}\|}$ for all q_e , i.e. $L_\phi V \leq \delta_1^*(q)$. If the solution $z^*(\cdot)$ is continuous on $(S_{s_1} \cap S_{s_2}) \setminus S_G$, then under the effect of the control input $u(q_e) = u^*(q_e)$, the closed-loop trajectories of (5) reach a neighborhood D of the goal set S_g in fixed-time T , and satisfy $q_e(t) \in S_{s_1} \cap S_{s_2}$ for all $t \geq 0$, where the neighborhood D and time of convergence are given by (7) and (8), with $c_1 = \alpha_1, c_2 = \alpha_2$ and $c_3 = 2 \max \delta_1^*$.*

Proof. The proof for the unperturbed case is immediate. The constraint (17c) ensures that the conditions of Theorem 2 are satisfied and therefore convergence to the neighborhood D is achieved in fixed-time, T , for the nominal system $\dot{q} = f(q) + g(q)u$. For the perturbed system, $\dot{q} = f(q) + g(q)u + \phi(q)$, we have that $\dot{V} = L_f V + L_g V u + L_\phi V \leq -\alpha_1 V^{\gamma_1} - \alpha_2 V^{\gamma_2} + \delta_1^*$, which may be rewritten as $\dot{V} = L_f V + L_g V u \leq -\alpha_1 V^{\gamma_1} - \alpha_2 V^{\gamma_2} + 2 \max \delta_1^*$. Thus, we have that the closed-loop trajectories of $\dot{q} = f(q) + g(q)u + \phi(q)$ reach D in fixed-time T , given by (7) and (8), respectively, with $c_1 = \alpha_1, c_2 = \alpha_2, c_3 = 2 \max \delta_1^*$. \square

Remark 1. *Comparing (17c) and Theorem 2 yields an observation that δ_1^* in the solution of (17) is analogous to c_3 in (6). However, in the context of solving Problem 1, (17) must be point-wise in the state space. It follows, therefore, that by considering $\max \delta_1^*$ over the solution set of (17) we can use $c_3 = 2 \max \delta_1^*$ to obtain a useful, albeit conservative,*

estimate for the settling time to a neighborhood, D , of the goal set, S_g .

Remark 2. This method does not estimate the disturbance term, ϕ ; rather, it determines a tolerable upper bound such that FxTS to a neighborhood D of a goal set S_G is preserved, and provides characterizations of D and the convergence time, T .

Next, we introduce a method for conditioning the parameters in (17) such that $2 \max_q \delta_1(q)^* \leq 2\sqrt{\alpha_1 \alpha_2}$. We then use $c_3 = 2 \max_q \{\delta_1^*\}$ to compute a conservative estimate on settling time for the Ego Vehicle during each segment of Problem 1. We use the sum total of these time estimates to compute an unsafe overtaking horizon, i.e. $(v_e \cos \theta_e - \hat{v}_{oc} \cos \hat{\theta}_{oc}) T_{est}$. If an Oncoming Vehicle is inside of the overtaking horizon (nearer to the Ego Vehicle than the horizon), then the Ego Vehicle does not begin its overtake.

IV. SIMULATION RESULTS AND DISCUSSION

A. Simulation Parameters

The CLF gains in (14) are fixed as: $k_x = \frac{1}{60^2} \text{m}^{-2}$, $k_y = 100 \text{m}^{-2}$, $k_\theta = 400 \text{rad}^{-2}$, $k_v = 1 (\text{m/s})^{-2}$, $k_{xv} = 0.05 \sqrt{k_x k_v} = \frac{1}{1200} \text{m}^{-2} \text{s}$, $k_{y\theta} = 0.5 \sqrt{k_y k_\theta} = 100 (\text{rad m})^{-1}$ so that the goal set is defined as $C_g: \|\bar{x}\| \leq 60 \text{ m}$, $\|\bar{y}\| \leq 0.1 \text{ m}$, $\|\bar{\theta}\| \leq 0.05 \text{ rad}$, $\|\bar{v}\| \leq 1 \text{ m/s}$. The physical boundaries of the road are set to be $y = 0$ and $y = 2w_l$ respectively. The input constraints are given as $|\omega| \leq \frac{\pi}{18} \text{ rad}$ and $|a| \leq 0.25g \text{ ms}^{-2}$. We used a time-step of $dt = 0.001 \text{ sec}$. Other simulation parameters are: $\mu = 5$, which leads to $\gamma_1 = 1.2$ and $\gamma_2 = 0.8$, as well $p_1 = 1200$, $p_2 = 1$, and $q_1 = 1000$. We define $\theta_g = \tan^{-1}(\frac{y_g - y_e}{x_g - x_e})$ and set $v_g = 25$ as soon as it is safe to overtake. The final states of one segment are used as initial states to the subsequent segment.

The following discussion will outline in greater detail the setup for each subproblem.

1) *Ego Vehicle Identify Opportunity to Perform Overtake:* The initial states of the Ego ($q_e(0)$), Lead ($q_l(0)$), and Oncoming ($q_{oc}(0)$) are chosen as $q_e(0) = [-\tau v_l(0) \frac{w_l}{2} 0 v_l(0)]^T$, $q_l(0) = [\tau v_l(0) \frac{w_l}{2} 0 v_l(0)]^T$, and $q_{oc}(0) = [x_e(0) + 2v_l(0)t_p \quad 2w_l - \frac{w_l}{2} \quad -\pi \quad 25]^T$ where $v_l(0)$ and t_p , the time until the Oncoming Vehicle passes by the Ego Vehicle, are chosen a priori. The goal state, q_g , is defined as an evolving function of q_l where the goal location is chosen as $x_g = x_l - 1.5\tau v_l + 50$, $y_g = y_l$, and $v_g = v_l$ until an overtake maneuver is safe to initiate.

2) *Ego Vehicle Merge into Overtaking Lane:* We define y_g and v_g in this segment as: $y_g = y_l + w_l$, $v_g = 25$. The upper bound on settling time, T , is set to $T = 10 \text{ sec}$.

3) *Ego Vehicle Move a Safe Distance beyond Lead Vehicle:* The x_g coordinate is modified to be: $x_g = x_l + 1.5\tau v_l + 50$, and $T = \frac{2\tau v_l(0)}{v_g - v_l(0)} + 4 \text{ sec}$ to adjust for an increase in safe following distance at increased initial velocities.

4) *Ego Vehicle Merge back into Original Lane:* We set $y_g = y_l$, and $T = 6 \text{ sec}$.

B. Results

In accordance with Theorem 2, we desire to choose parameters such that it is guaranteed that $2 \max \delta_1 = c_3 < 2\sqrt{c_1 c_2}$, where $c_i = \alpha_i = \frac{\pi \mu}{2T}$ for $i \in \{1, 2\}$ for our nominal

simulation. The initial conditions chosen are $v_l(0) = 17$ and $t_p = 2$. Thus, we varied K from 10^{-5} to 1, T from 13.15 to 30.65, ω_{max} from 0.0175 to 14.45, and a_{max} from 0.245 to 245.25, and compared c_3 with $2\sqrt{c_1 c_2}$ for the trials. We selected the final values as $K = 0.0001$, $T = 27.65$, and $u_{max} = [0.1745 \quad 2.45]^T$. As such, we selected $c_3 = 2 \max \delta_1^* = 0.638$. To model the perturbation, we chose a zero-mean, Gaussian normal distribution with $3\sigma = \|\phi\|_\infty$ and saturated at $\pm \|\phi\|_\infty$, where σ is the standard deviation with $\|\phi\|_\infty = 3.99$.

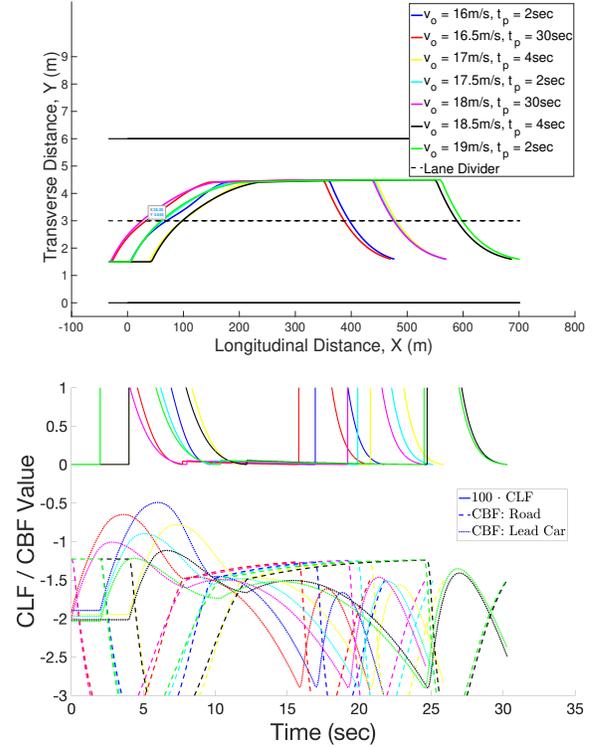


Fig. 2. State trajectories and CLF / CBF evolution respectively (top to bottom) of the Ego Vehicle during simulated scenarios using 7 different initial conditions. True CBF values have been negated for better visuals.

Figure 2 plots the paths traced by the Ego vehicle for various initial conditions $q_e(0)$. With the selected parameters for the QP (17), it is clear from the figure that for all chosen initial conditions the Ego car maneuvered and converged within the fixed-time windows and safety constraints were obeyed at all times. Additionally, in the case where the Oncoming Vehicle was scheduled to pass the Ego Vehicle at $t_p = 30 \text{ sec}$, the Ego Vehicle appropriately made the decision to execute the overtake immediately.

Finally, 10 evenly spaced upper bounds on $\phi(q)$, from $0.1\|\phi\|_\infty$ to $1.0\|\phi\|_\infty$ are considered and the overtake maneuver is simulated. Figure 3 shows that for 100 trials of the perturbed simulation, 10 for each disturbance bound, the solutions of the individual sub-problems converged within the finite-time window. In Figure 4 we display the results for one such simulation. The two following observations are notable: 1) for $t_p = 30$ the safety estimator computed that Oncoming Vehicle was inside of the overtaking horizon, and as such decided not to initiate the overtake until after it passed; 2) consistent with (8), as the disturbance bound grew so did the overtaking horizon - notably, when $t_p = 34$, the

safety estimator computed that for $\|\phi\|_\infty = 0.4, 1.6$, the Ego Vehicle could complete the overtake safely, whereas at larger disturbance bounds the decision to withhold the overtake was made until the Oncoming vehicle had passed safely by. Meanwhile, the controller satisfied the safety requirement for all trials.

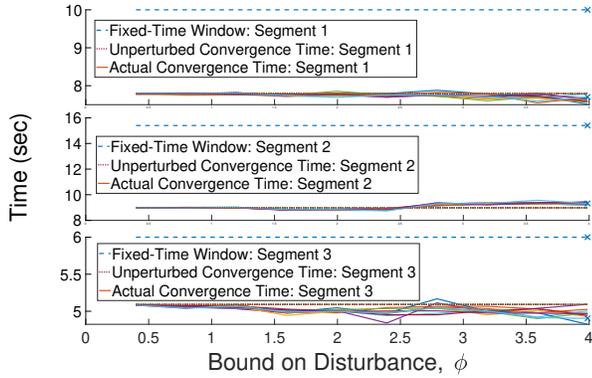


Fig. 3. Time of convergence to goal set versus upper bound on disturbance term, ϕ , for nominal and perturbed solutions, shown in conjunction with the fixed-time prescription per subproblem.

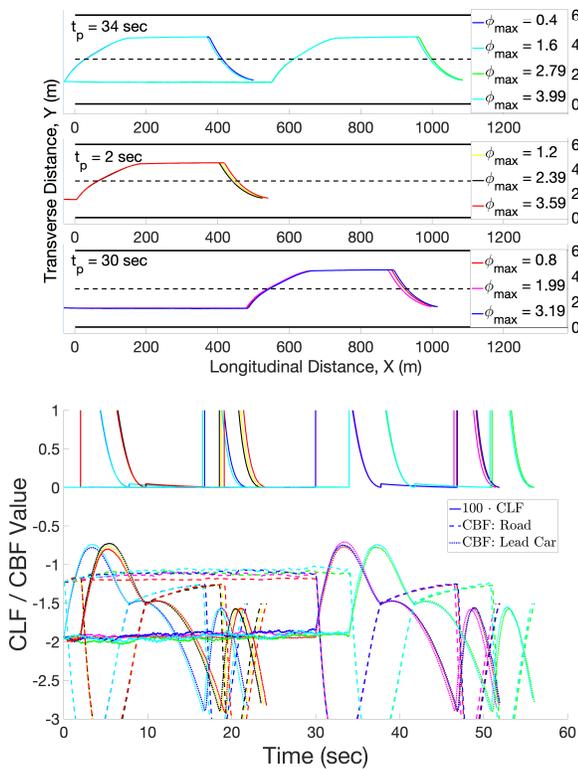


Fig. 4. State trajectories and CLF / CBF evolution respectively (top to bottom) of the Ego Vehicle during simulated scenarios using 1 initial condition and 10 different disturbance bounds. True CBF values have been negated for better visuals.

V. CONCLUSION

In this study on robust control synthesis using CLF- and CBF-based techniques we introduced a new approach for driving a dynamical system subject to spatiotemporal and

input constraints to a neighborhood of a goal set in fixed-time despite bounded, additive, non-vanishing disturbances. We provided theoretical guarantees of FxT convergence for such a system whose control is computed by a FxT-CLF-CBF QP provided that disturbances do not exceed a quantified bound. Next, we outlined a procedure for conditioning the QP and selecting parameters such that FxT convergence to a neighborhood of a goal set is guaranteed for any initial condition, and presented definitions for such a neighborhood. We then demonstrated the procedure on an overtake problem and highlighted the efficacy of the method with repeated simulated trials. In the future, we plan to explore reducing the conservativeness of this approach by considering an estimate of the non-vanishing disturbance term via online adaptation and/or learning based techniques.

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