Finite-Time Stabilization of Switched Systems with Unstable Modes ⋆

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Abstract: In this work, we study finite-time stability of switched systems in the presence of unstable modes. We present sufficient conditions in terms of multiple Lyapunov functions for the origin of the system to be finite time stable. More specifically, we show that even if the value of the Lyapunov function increases in between two switches, i.e., if there are unstable modes in the system, finite-time stability can still be guaranteed if the finite time convergent mode is active for a long enough cumulative time duration. Then, we present a method on the synthesis of a finite-time stabilizing switching signal. As a case study, we design a finite-time stable output feedback controller for a linear switched system, in which only one of the modes is both controllable and observable. We present one numerical example to demonstrate the efficacy of the proposed method.

Keywords: Switching stability and control; Control of switched systems; Finite-Time Stability; Multiple Lyapunov Functions; Output feedback control.

1. INTRODUCTION

Stability of switched systems has been analyzed by many researchers in the past, and is typically studied using either a *common* Lyapunov function, or *multiple* Lyapunov functions. The book Liberzon (2003) discusses necessity and sufficiency of the existence of a common Lyapunov function for all subsystems of a switched system for asymptotic stability under arbitrary switching. The authors in Ishii and Francis (2002) study linear switched systems with dwell-time using a common quadratic control Lyapunov function (CQLF) and state-space partitioning. In the review article Shorten et al. (2007), the authors study the stability of switched linear systems and linear differential inclusions. They present sufficient conditions for the existence of CQLFs and discuss converse Lyapunov results for switched systems. In Branicky (1998), the author introduces the concept of multiple Lyapunov functions to analyze stability of switched systems; since then, a lot of work has been done on the stability of switched systems using multiple Lyapunov functions Zhao and Hill (2008); Zhao et al. (2012); Lin and Antsaklis (2009). In Zhao and Hill (2008), the authors relax the non-increasing condition on the Lyapunov functions by introducing the notion of generalized Lyapunov functions. They present necessary and sufficient conditions for stability of switched systems under arbitrary switching. In Zhao et al. (2012), the authors introduce the concept of Multiple Linear Copositive Lyapunov functions (ML-CLFs) and give sufficient conditions for exponential stability of Switched Positive Linear Systems (SPLS) in terms of feasibility of Linear Matrix Inequalities (LMI). The authors in Zhao et al. (2017) use discontinuous multiple Lyapunov functions in order to guarantee stability of *slowly* switched systems, where the stable subsystems are required to switch slower (i.e., stay active for a longer duration) as compared to unstable subsystems.

In contrast to AS, which pertains to convergence as time goes to infinity, finite-time Stability (FTS) [1] is a concept that requires convergence of solutions in finite time. FTS is a well-studied concept, motivated in part from a practical viewpoint due to properties such as convergence in finite time, as well as robustness with respect to disturbances. In the seminal work Bhat and Bernstein (2000), the authors introduce necessary and sufficient conditions in terms of Lyapunov function for continuous, autonomous systems to exhibit FTS, with focus on continuous-time autonomous systems.

FTS of switched/hybrid systems has gained popularity in the last few years. The authors in Liu et al. (2017) consider the problem of designing a controller for a linear switched system under delay and external disturbance with finite- and fixed-time convergence. In Li and Sanfelice (2013), the authors design a hybrid observer and show finite-time convergence in the presence of unknown, constant bias. The authors in Li and Sanfelice (2019) present conditions in terms of a common Lyapunov function for FTS of hybrid systems. They require the value of the Lyapunov function to be decreasing during the continuous flow and non-increasing at the discrete jumps. The authors in Ríos et al. (2015) design an FTS state-observer for switched systems via a sliding-mode technique. In Orlov (2004),

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[1] With slight abuse of notation, we use FTS to denote the phrase "finite-time stability" or "finite-time stable", depending on the context.

the authors introduce the concept of a *locally* homogeneous system, and show FTS of switched systems with uniformly bounded uncertainties. More recently, Zhang (2018) studies FTS of homogeneous switched systems by introducing the concept of hybrid homogeneous degree, and relating negative homogeneity with FTS. In Fu et al. (2015), the authors consider systems in strict-feedback form with positive powers and design a controller as well as a switching law so that the closed-loop system is FTS. In Bejarano et al. (2011), the authors design an FTS observer for switched systems with unknown inputs. They assume that each linear subsystem is *strongly* observable, and that the first switching occurs after an *a priori* known time. In contrast, in the current paper we do not assume that the subsystems are homogeneous or in strict feedback form, and present conditions in terms of multiple Lyapunov functions for FTS of the origin.

In this paper, we consider a general class of switched systems, and develop sufficient conditions for FTS of the origin of the switched system in terms of multiple Lyapunov functions. *To the best of authors' knowledge, this is the first work considering FTS of switched systems using multiple Lyapunov functions*. The main contributions are summarized as follows.

FTS of switched systems: We first define the notion of FTS for switched systems so that it does not restrict each mode of the switched system to be FTS in itself. More specifically, we relax the requirement in Zhao and Hill (2008); Li and Sanfelice (2019) that the Lyapunov function is strictly decreasing during the continuous flow; instead, we allow the multiple Lyapunov functions to increase during the continuous flow and only require that these increments are bounded. In this respect, we allow the switched system to have unstable modes while still guaranteeing FTS.

Switching-signal design for FTS, and applications: We use the proposed multiple Lyapunov function framework to design a switching signal so that the origin of the resulting switched system is FTS. As an application, we study the problem of a switched linear control system with objective of stabilizing the origin using output feedback for the case when only one of the subsystems (or modes) is controllable and observable. We design an FTS observer based FTS controller so that the origin of the resulting closed-loop switched systems is finite-time stable.

2. PRELIMINARIES

We denote by $\|\cdot\|$ the Euclidean norm of vector $(\cdot)$, $|\cdot|$ the absolute value if $(\cdot)$ is scalar and the length if $(\cdot)$ is a time interval. The set of non-negative reals is denoted by $\mathbb{R}_+$, set of non-negative integers by $\mathbb{Z}_+$ and set of positive integers by $\mathbb{N}$. We denote by $\text{int}(S)$ the interior of the set S, and by t^- and t^+ the time just before and after the time instant t, respectively.

Definition 1. (**Class-$\mathcal{GK}$ function**): A function $\alpha : D \to \mathbb{R}_+$, $D \subset \mathbb{R}_+$, is called a class-$\mathcal{GK}$ function if it is increasing, i.e., for all $x > y \geq 0$, $\alpha(x) > \alpha(y)$, and right continuous at the origin with $\alpha(0) = 0$.

Definition 2. (**Class-$\mathcal{GK}_\infty$ function**): A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is called a class-$\mathcal{GK}_\infty$ function if it is a class-$\mathcal{GK}$ function, and $\lim_{r\to\infty} \alpha(r) = \infty$.

Note that the class-$\mathcal{GK}$ (respectively, $\mathcal{GK}_\infty$) functions have similar composition properties as those of class-$\mathcal{K}$ (respectively, class-$\mathcal{K}_\infty$) functions, e.g., for $\alpha_1, \alpha_2 \in \mathcal{GK}$ and $\alpha \in \mathcal{K}$, we have:

- $\alpha_1 \circ \alpha_2 \in \mathcal{GK}$ and $\alpha_1 + \alpha_2 \in \mathcal{GK}$;
- $\alpha \circ \alpha_1 \in \mathcal{GK}$, $\alpha_1 \circ \alpha \in \mathcal{GK}$ and $\alpha_1 + \alpha \in \mathcal{GK}$.

Consider the system:

$$\dot{y}(t) = f(y(t)), \tag{1}$$

where $y \in \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ is continuous on an open neighborhood $D \subseteq \mathbb{R}^n$ of the origin and $f(0) = 0$. The origin is said to be an FTS equilibrium of (1) if it is Lyapunov stable and *finite-time convergent*, i.e., for all $y(0) \in \mathcal{N} \setminus \{0\}$, where $\mathcal{N}$ is some open neighborhood of the origin, $\lim_{t\to T} y(t) = 0$, where $T = T(y(0)) < \infty$ Bhat and Bernstein (2000). The authors also presented Lyapunov conditions for FTS of the origin of (1):

Theorem 1. (Bhat and Bernstein (2000)). Suppose there exists a continuous function $V\colon D \to \mathbb{R}$ such that the following holds:
(i) V is positive definite
(ii) There exist real numbers $c > 0$ and $\alpha \in (0, 1)$, and an open neighborhood $\mathcal{V} \subseteq D$ of the origin such that

$$\dot{V}(y) \leq -cV(y)^\alpha, \; y \in \mathcal{V} \setminus \{0\}. \tag{2}$$

Then the origin is an FTS equilibrium for (1).

Consider the switched system

$$\dot{x}(t) = f_{\sigma(t,x)}(x(t)), \quad x(t_0) = x_0, \tag{3}$$

where $x \in \mathbb{R}^n$ is the system state, $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \Sigma$ is a piecewise constant, right-continuous switching signal that can depend both upon state and time, $\Sigma \triangleq \{1, 2, \ldots, N\}$ with $N < \infty$, and $f_{\sigma(\cdot,\cdot)} : \mathbb{R}^n \to \mathbb{R}^n$ is the system vector field describing the active subsystem (called thereafter mode) under $\sigma(\cdot,\cdot)$. Let $\mathcal{I} = [0, \tau)$, with $\tau > 0$ be the domain of definition of the solutions of (3). A solution $x(\cdot)$ of (3) is an absolutely continuous function that satisfies $\dot{x}(t) = f_k(x(t))$ for almost all $t \in \mathcal{I}$ where $k = \sigma(t, x(t)) \in \Sigma$. The solution is maximal is $\mathcal{I}$ cannot be extended, and is *complete* if $\tau = \infty$. We state the following Lemma before we proceed to the main results.

Lemma 1. Let $a_i \geq b_i \geq 0$ for all $i \in \{1, 2, \ldots, K\}$ for some $K \in \mathbb{N}$. Then, for any $0 < r < 1$, we have

$$\sum_{i=1}^{K} (a_i^r - b_i^r) \leq \sum_{i=1}^{K} (a_i - b_i)^r. \tag{4}$$

Proof. Lemma 3.3 Zuo and Tie (2016) establish the following inequality for $z_i \geq 0$ and $0 < r \leq 1$,

$$\left(\sum_{i=1}^{M} z_i\right)^r \leq \sum_{i=1}^{M} z_i^r. \tag{5}$$

Hence, we have that for $a \geq b \geq 0$ and $0 < r \leq 1$, $a^r = (b + (a - b))^r \leq b^r + (a - b)^r$, or equivalently,

$$a^r - b^r \leq (a - b)^r. \tag{6}$$

Hence, we have that for any $0 < r \leq 1$,

$$\sum_{i=1}^{k} (a_i^r - b_i^r) \leq \sum_{i\in I_1} (a_i^r - b_i^r) \leq \sum_{i\in I_1} (a_i - b_i)^r.$$

3. MAIN RESULTS

3.1 Result1: FTS of Switched Systems

First, we define the notion of FTS for switched systems. Note that a mode $F \in \Sigma$ is called an FTS subsystem or FTS mode if the origin of $\dot{y} = f_F(y)$ is FTS. The standard notion of stability under arbitrary switching, as employed in Liberzon (2003); Branicky (1998); Zhao and Hill (2008); Lin and Antsaklis (2009); Fu et al. (2015), is restrictive in the following sense. The conditions therein require every single mode of the system (3) to be Lyapunov Stable (LS or simply, stable), AS, or FTS for the origin of the system (3) to be LS, AS, or FTS, respectively. We overcome this restriction by defining the corresponding notions of stability for hybrid system (inspired in part, from (Peleties and DeCarlo, 1991, Theorem 1)) as following. Let $\Pi \subset \mathrm{PWC}(\mathbb{R}_+ \times \mathbb{R}^n, \Sigma)$ denote the set of all possible pairs of switching signals, where PWC is the set of all piecewise constant functions mapping from $\mathbb{R}_+ \times \mathbb{R}^n$ to Σ.

Definition 3. The origin of the switched system (3) is called LS, AS or FTS if there exists an open neighborhood $X \subset \mathbb{R}^n$ such that for all $y \triangleq x(0) \in X$, there exists a subset of switching signals $\Pi_y \subset \Pi$ such that the origin of the system (3) is LS, AS or FTS, respectively, with respect to all $\sigma_f \in \Pi_y$. The origin is called globally AS or FTS if $X = \mathbb{R}^n$.

We make the following assumption for (3).

Assumption 1. The solution of (3) exists and are complete. In addition, there is a non-zero dwell-time for the FTS mode $F \in \Sigma$, i.e., $|T_{F_k}| = t_{F_k+1} - t_{F_k} \geq t_d$ for all $k \in \mathbb{N}$, where $t_d > 0$ is a positive constant.

Now, we present the conditions for FTS of the origin of (3) in terms of multiple Lyapunov functions. Let $\{i^0, i^1, \dots, i^p, \dots\}$ be the sequence of modes that are active during the intervals $[t_0, t_1), [t_1, t_2), \dots, [t_p, t_{p+1}), \dots$, respectively, for $i^p \in \Sigma, p \in \mathbb{Z}_+$.

Theorem 2. If there exist Lyapunov functions V_i for each $i \in \Sigma$, and a switching signal σ such that the following hold:

(i) There exists $\alpha_1 \in \mathcal{GK}$, such that
$$\sum_{k=0}^{p} \Big(V_{i^{k+1}}(x(t_{k+1})) - V_{i^k}(x(t_{k+1})) \Big) \leq \alpha_1(\|x_0\|), \quad (7)$$
holds for all $p \in \mathbb{Z}_+$;

(ii) There exists $\alpha_2 \in \mathcal{GK}$ such that
$$\sum_{k=0}^{p} \Big(V_{i^k}(x(t_{k+1})) - V_{i^k}(x(t_k)) \Big) \leq \alpha_2(\|x_0\|), \quad (8)$$
holds for all $p \in \mathbb{Z}_+$;

(iii) There exists $F \in \Sigma$ such that the origin of $\dot{y} = f_F(y)$ is FTS, and there exist a positive definite, continuously differentiable Lyapunov function V_F and constants $c > 0$, $0 < \beta < 1$ such that
$$\dot{V}_F(x(t)) \leq -c(V_F(x(t)))^{\beta}, \quad (9)$$
for all $t \in \bigcup_k [t_{F_k}, t_{F_k+1})$;

(v) The accumulated duration $|\bar{T}_F| \triangleq \sum_k |\bar{T}_{F_k}|$ corresponding to the period of time during which the mode F is active without any discrete jumps, satisfies
$$|\bar{T}_F| = \gamma(\|x_0\|) \triangleq \frac{(\alpha(\|x_0\|))^{1-\beta}}{c(1-\beta)} + \frac{M^{-\beta}(\bar{\alpha}(\|x_0\|))^{1-\beta}}{c(1-\beta)},$$
where $\alpha = \alpha_0 + \alpha_1 + \alpha_2$, $\bar{\alpha} = M(\alpha_1 + \alpha_2)$ and $\alpha_0 \in \mathcal{GK}$, and $M \in \mathbb{Z}_+$,

then, the origin of (3) is FTS with respect to the switching signal σ. Moreover, if all the conditions hold globally, the functions V_i are radially unbounded for all $i \in \Sigma$, and $\alpha_1, \alpha_2 \in \mathcal{GK}_\infty$, then the origin of (3) is globally FTS.

We first provide an intuitive explanation of the conditions of Theorem 2 (see Figure 1).

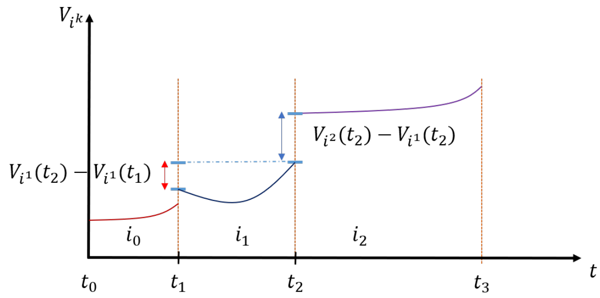

Fig. 1. Conditions (i) and (ii) Theorem 2 regarding the allowable changes in the values of the Lyapunov functions. The increments shown by blue and red double-arrows pertain to condition (i) and (ii), respectively.

- Condition (i) means that the cumulative value of the differences between the consecutive Lyapunov functions at the switching instants of the dynamics of continuous flows (i.e., at switches of the signal σ) is bounded by a class-$\mathcal{GK}$ function.
- Condition (ii) means that the cumulative increment in the values of the individual Lyapunov functions when the respective modes are active is bounded by a class-$\mathcal{GK}$ function.[2]
- Condition (iii) means that there exists an FTS mode $F \in \Sigma$ and a Lyapunov function V_F satisfying (9) for $\dot{x}(t) = f_F(x(t))$ on $[t_{F_k}, t_{F_k+1}) \setminus J_F$ for all $k \in \mathbb{Z}_+$.
- Condition (iv) means that the FTS mode F is active for a sufficiently long cumulative time $\gamma(\|x_0\|)$ without any discrete jump occurring in that cumulative period.

Now we provide the proof of Theorem 2.

Proof. First we prove the stability of the origin under conditions (i)-(ii). Let $x_0 \in D$, where D is some open neighborhood of the origin. For all $p \in \mathbb{Z}_+$, we have that
$$\begin{aligned} V_{i^p}(x(t_p)) =& V_{i^0}(x(t_0)) + \sum_{k=1}^{p} \Big(V_{i^k}(x(t_k)) - V_{i^{k-1}}(x(t_k)) \Big) \\ &+ \sum_{k=0}^{p-1} \Big(V_{i^k}(x(t_{k+1})) - V_{i^k}(x(t_k)) \Big) \\ \overset{(7),(8)}{\leq}& \alpha_0(\|x_0\|) + \alpha_1(\|x_0\|) + \alpha_2(\|x_0\|) \\ =& \alpha(\|x_0\|) \end{aligned}$$

[2] Note that some authors use the time derivative condition, i.e., $\dot{V}_i \leq \lambda V_i$ with $\lambda > 0$, in place of condition (ii), to allow growth of V_i, hence, requiring the function to be continuously differentiable (see, e.g., Wang et al. (2018)). Our condition allows the use of non-differentiable Lyapunov functions.

where $\alpha = \alpha_0 + \alpha_1 + \alpha_2$ with $\alpha_0(r) = \max_{i\in\Sigma_f,\ \|x\|\le r} V_i(x)$. Thus, we have:

$$V_{i^p}(x(t_p)) \le \alpha(\|x_0\|), \tag{10}$$

for all $p \in \mathbb{Z}_+$. Let $d_i(c) = \{x \mid V_i(x) \le c\}$ denote the c sub-level set of the Lyapunov function V_i, $i \in \Sigma_f$, and $B_\rho = \{x \mid \|x\| \le \rho\}$ denote a ball centered at the origin with radius $\rho \in \mathbb{R}_+$. Define $r(c) = \inf\{\rho \ge 0 \mid d_i(c) \subset B_\rho\}$ as the radius of the smallest ball centered at the origin that encloses the c sub-level sets $d_i(c)$, for all $i \in \Sigma_f$ (see Figure 2).

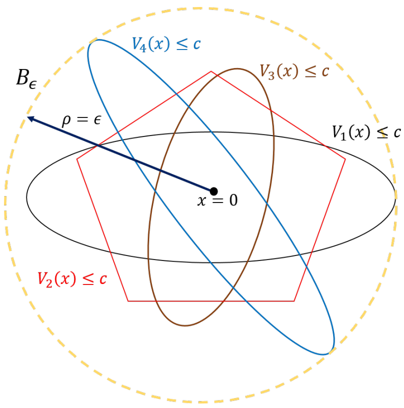

Fig. 2. The ball B_ρ, shown in dotted yellow, encloses c sublevel sets of the Lyapunov functions V_i, whose boundaries are shown in solid lines.

Since the functions V_i are positive definite, the sub-level sets $d_i(c)$ are bounded for small $c > 0$, and hence, the function r is invertible. The inverse function $c_\epsilon = r^{-1}(\epsilon)$ maps the radius $\epsilon > 0$ to the value c_ϵ such that the sub-level sets $d_i(c_\epsilon)$ are contained in B_ϵ for all $i \in \Sigma_f$. For any given $\epsilon > 0$, choose $\delta = \alpha^{-1}(r^{-1}(\epsilon)) > 0$ so that (10) implies that for $\|x_0\| \le \delta$, we have

$$V_{i^p}(x(t_p)) \le \alpha(\|x_0\|) \le \alpha(\alpha^{-1}(r^{-1}(\epsilon))) = r^{-1}(\epsilon)$$
$$\implies \|x(t_p)\| \le \epsilon,$$

for all $p \in \mathbb{Z}_+$, i.e., the origin is LS.

Next, we prove FTS of the origin when conditions (iii)-(iv) also hold. From (10), we have that

$$V_F(x(t_{F_i})) \le \alpha(\|x_0\|), \tag{11}$$

for all $i \in \mathbb{N}$. Let $M \in \mathbb{N}$ denote the total number of times the mode F is activated. From condition (iii), we have

$$\dot{V}_F(x(t)) \le -c(V_F(x(t)))^\beta. \tag{12}$$

for all $t \in \bigcup T_{F_k} = \bigcup [t_{F_k}, t_{F_k+1})$. We can integrate (12) to obtain

$$|T_{F_k}| \le \frac{V_{F_k}^{1-\beta}}{c(1-\beta)} - \frac{V_{F_k+1}^{1-\beta}}{c(1-\beta)}.$$

Thus, for any $M \in \mathbb{N}$, we have that

$$\sum_{k=1}^{M} |T_{F_k}| \le \sum_{k=1}^{M} \left(\frac{V_{F_k}^{1-\beta}}{c(1-\beta)} - \frac{V_{F_k+1}^{1-\beta}}{c(1-\beta)} \right)$$
$$= \frac{V_{F_1}^{1-\beta}}{c(1-\beta)} + \sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i+1}^{1-\beta}}{c(1-\beta)} - \frac{V_{F_M+1}^{1-\beta}}{c(1-\beta)}.$$

Using (11), we obtain that

$$\frac{V_{F_1}^{1-\beta}}{c(1-\beta)} \le \frac{(\alpha(\|x_0\|))^{1-\beta}}{c(1-\beta)}. \tag{13}$$

Define $\gamma_1(\|x_0\|) \triangleq \frac{(\alpha(\|x_0\|))^{1-\beta}}{c(1-\beta)}$ and note that $\gamma_1 \in \mathcal{GK}$. Now, let $F_s = \{q_1, q_2, \ldots, q_k\}, 0 \le q_l \le M$, be the set of indices such that $V_{F_{i+1}} \ge V_{F_i+1}$ for $i \in F_s$. We know that for $a \ge b \ge 0$, $a^r \ge b^r$ for any $r > 0$. Hence, we have that

$$\sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i+1}^{1-\beta}}{c(1-\beta)} \le \sum_{i\in F_s} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i+1}^{1-\beta}}{c(1-\beta)} \tag{14}$$

Using Lemma 1, we obtain that

$$\sum_{i\in F_s} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i+1}^{1-\beta}}{c(1-\beta)} \le \sum_{i\in F_s} \frac{(V_{F_{i+1}} - V_{F_i+1})^{1-\beta}}{c(1-\beta)}. \tag{15}$$

From the analysis in the first part of the proof, we know that

$$V_{F_{i+1}} - V_{F_i+1} = \sum_{k=l_1}^{l_2} \Big(V_{i^k}(x(t_k)) - V_{i^{k-1}}(x(t_k)) \Big)$$
$$+ \sum_{k=l_1}^{l_2-1} \Big(V_{i^k}(x(t_{k+1})) - V_{i^k}(x(t_k)) \Big)$$
$$\le \alpha_1 + \alpha_2,$$

where l_1, l_2 are such that t_{l_1} denotes the time when mode F becomes deactivated for the i-th time and t_{l_2} denotes the time when the mode F is activated for $(i+1)$-th time. Define $\bar{\alpha} = M(\alpha_1 + \alpha_2)$ so that we have

$$\sum_{i\in F_s} \big(V_{F_{i+1}} - V_{F_i+1} \big) \le \bar{\alpha}(\|x_0\|). \tag{16}$$

Hence, we have that

$$\sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i+1}^{1-\beta}}{c(1-\beta)} \overset{(15)}{\le} \frac{\sum_{i\in F_s} \big(V_{F_{i+1}} - V_{F_i+1} \big)^{1-\beta}}{c(1-\beta)}$$
$$\le \frac{M^{-\beta} \left(\sum_{i\in F_s} V_{F_{i+1}} - \bar{V}_{F_i+1} \right)^{1-\beta}}{c(1-\beta)}$$
$$\overset{(16)}{\le} \frac{M^{-\beta}(\bar{\alpha}(\|x_0\|))^{1-\beta}}{c(1-\beta)}, \tag{17}$$

where the second inequality follows from (Zuo and Tie, 2016, Lemma 3.4). Define $\gamma(\|x_0\|) \triangleq \gamma_1(\|x_0\|) + \frac{M^{-\beta}(\bar{\alpha}(\|x_0\|))^{1-\beta}}{c(1-\beta)}$ and $|T_F| = \sum_{k=1}^{M} |T_{F_k}|$ so that we obtain:

$$|T_F| + \frac{V_{F_M+1}^{1-\beta}}{c(1-\beta)} \le \frac{V_{F_1}^{1-\beta}}{c(1-\beta)} + \sum_{i=1}^{M-1} \frac{V_{F_{i+1}}^{1-\beta} - V_{F_i+1}^{1-\beta}}{c(1-\beta)} \le \gamma(\|x_0\|).$$

Clearly, $\gamma \in \mathcal{GK}$. Now, with $|T_F| = \gamma(\|x_0\|)$, we obtain

$$|T_F| + \frac{V_{F_M+1}^{1-\beta}}{c(1-\beta)} \le \gamma(\|x_0\|) = |T_F|,$$

which implies that $\frac{V_{F_M+1}^{1-\beta}}{c(1-\beta)} \le 0$. However, $V_F \ge 0$, which further implies that $V_{F_M+1} = 0$. Hence, if mode F is active for the accumulated time $|T_F| = \gamma(\|x_0\|)$ without any discrete jump in the system state, the value of the function V_F converges to 0 as $t \to t_{F_M+1}$, and thus, the origin of (3) is FTS.

Finally, if all the conditions (i)-(iv) hold globally and the functions V_i are radially unbounded, we have that α_0 is also radially unbounded and $\alpha_1, \alpha_2 \in \mathcal{GK}_\infty$. Thus, we have that $\alpha(\|x_0\|) < \infty$ and $\bar{\alpha}(\|x_0\|) < \infty$ for all $\|x_0\| < \infty$, and hence, $\gamma(\|x_0\|) < \infty$ for all $\|x_0\| < \infty$, which implies global FTS of the origin.

3.2 Result 2: Finite-Time Stabilizing Switching Signal

In this subsection, we present a method of designing a switching signal, based upon Theorem 2, so that the origin of the switched system is FTS. The approach is inspired from Zhao and Hill (2008) where a method of designing an asymptotically stabilizing switching signal is presented. Suppose there exist continuous functions $\mu_{ij} : \mathbb{R}^n \to \mathbb{R}$ satisfying:

$$\begin{aligned} \mu_{ij}(0) &= 0, \\ \mu_{ii}(x) &= 0 \quad \forall\, x, \\ \mu_{ij}(x) + \mu_{jk}(x) &\leq \min\{0, \mu_{ik}(x)\},\ \forall x \end{aligned} \tag{18}$$

for all $i, j, k \in \Sigma$. Define the following sets:

$$\begin{aligned} \Omega_i &= \{x \mid V_i(x) - V_j(x) + \mu_{ij}(x) \leq 0, j \in \Sigma\}, \\ \Omega_{ij} &= \{x \mid V_i(x) - V_j(x) + \mu_{ij}(x) = 0, i \neq j\}, \end{aligned} \tag{19}$$

where V_i is a Lyapunov function for each $i \in \Sigma$.

Now we are ready to define the switching signal. Let $\sigma(t_0, x(t_0)) = i$ and $i, j \in \Sigma$ be any arbitrary modes. For all times $t \geq t_0$, define the switching signal as:

$$\sigma(t,x) = \begin{cases} i, & \sigma(t^-, x(t^-)) = i,\ x(t^-) \in \mathrm{int}(\Omega_i); \\ j, & \sigma(t^-, x(t^-)) = F,\ x(t^-) \in \Omega_{Fj},\ \Delta_t \geq t_d; \\ j, & \sigma(t^-, x(t^-)) = i,\ i \neq F,\ x(t^-) \in \Omega_{ij}; \\ F, & \sigma(t^-, x(t^-)) = i,\ x(t^-) \in \Omega_{iF}; \end{cases} \tag{20}$$

where

- $\Delta_t = t - t_k$ is the time duration from the last switching instant t_k;
- $t_d > 0$ is some positive dwell-time;

Note that the condition for switching from mode F to mode j includes a dwell-time of t_d, so that Assumption 1 is satisfied. We now state the following result.

Theorem 3. Let the switching signal for (3) is given by (20). Let V_i are Lyapunov functions for $i = 1, 2, \ldots, N$, and μ_{ij} satisfy (18). Assume that the following hold:

(I) There exists continuous functions $\beta_{ij} : \mathbb{R}^n \to \mathbb{R}$ for $i, j \in \Sigma$ such that $\beta_{ij}(x) \leq 0$ for all $x \in \mathbb{R}^n$ and

$$\frac{\partial V_i}{\partial x} f_i(x) + \sum_{j=1}^{N_f} \beta_{ij}(x)(V_i(x) - V_j(x) + \mu_{ij}(x)) \leq 0, \tag{21}$$

holds for all $i \in \Sigma$, for all $x \in \mathbb{R}^n$;

(II) There exists a finite-time stable mode $F \in \Sigma$ satisfying condition (iii) and (iv) of Theorem 2;

(III) The functions μ_{ij} are continuously differentiable and satisfy

$$\frac{\partial \mu_{ij}}{\partial x} f_i \leq 0,\ i, j = 1, 2, \ldots, N. \tag{22}$$

(IV) No sliding mode occurs at any switching surface.

Then, the origin of (3) is FTS.

Proof. We show that all the conditions of Theorem 2 and Assumption 1 are satisfied to establish FTS of the origin for (3), when the switching signal is defined as per (20). As per the analysis in (Zhao and Hill, 2008, Theorem 3.18), we obtain that the conditions (i)-(ii) of Theorem 2 are satisfied with

$$\alpha_1(r) = \max_{\|x\| \leq r,\ i,j \in \Sigma_f} |\mu_{ij}(x)|, \tag{23}$$

$$\alpha_2(r) = 0, \tag{24}$$

for any $r \geq 0$. From (II), we obtain that conditions (iii) and (iv) of Theorem 2 hold as well. Per (20), Assumption 1 is also satisfied. Thus, all the conditions of the Theorem 2 and Assumption 1 are satisfied. Hence, we obtain that the origin of (3) with switching signal defined as per (20) is FTS.

Remark 1. Note that an arbitrary switching signal σ may not satisfy the conditions of Theorem 2, particularly condition (v), where the mode F is required to be active for $T_F(x_0)$ time duration. For any given initial condition x_0, the switching signal can be defined as per (20) to render the origin of (3) FTS. Definition 3 allows us to choose the switching signal σ as per (20) so that the switched system (3) satisfies the conditions of Theorem 2. Moreover, one can verify that the only difference between the switching signal defined in Zhao and Hill (2008) and (20) is the introduction of dwell-time t_d when switching from mode F. This observation re-emphasizes on the fact a system whose origin is uniformly stable can be made FTS by ensuring that the dwell-time condition and the cumulative activation time requirements are satisfied for an FTS mode.

A note on construction of functions μ_{ij}, V_i: For a class of switched systems consisting of $N-1$ linear modes and one FTS mode F, one can follow a design procedure similar to (Zhao and Hill, 2008, Remark 3.21) to construct the functions μ_{ij}, as well as the Lyapunov functions V_i, for all $i \neq F$. The design procedure includes choosing quadratic functions $\mu_{ij} = x^T P_{ij} x$ and $V_i = x^T R_i x$ with R_i as positive definite matrices, and using the conditions (18) and (22) along with the conditions of Theorem 2, to formulate a linear matrix inequality (LMI) based optimization problem. For system consisting of polynomial dynamics f_i, one can formulate a sum-of-square (SOS) problem to find polynomial functions V_i, μ_{ij} and β_{ij} by posing (18), (21) and (22) inequalities as SOS constraints (see e.g., Prajna et al. (2002) for methods of solving SOS problems). The "min-switching" law as described in Liberzon and Morse (1999), can be defined by setting the functions $\mu_{ij} = 0$, which would imply that the Lyapunov functions should be non-increasing at the switching instants. Our conditions on the lines of the generalization of min-switching law, as presented in Zhao and Hill (2008), overcome this limitation and allow the Lyapunov functions to increase at the switching instants.

3.3 FTS output-feedback for Switched Linear Systems

In this subsection, we consider a switched linear system with N modes such that only one mode is observable and controllable, and design an output-feedback to stabilize the system trajectories at the origin in a finite time. Consider the system:

$$\begin{aligned} \dot{x} &= A_{\sigma(t,x)} x + B_{\sigma(t,x)} u, \\ y &= C_{\sigma(t,x)} x, \end{aligned} \tag{25}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$ are the system states, and input and output of the system, respectively, with $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times 1}$ and $C_i \in \mathbb{R}^{1\times n}$. The switching signal $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \Sigma \triangleq \{1, 2, \dots, N\}$ is a piecewise constant, right-continuous function. We make the following assumption:

Assumption 2. There exists a mode $\sigma_0 \in \Sigma$ such that $(A_{\sigma_0}, B_{\sigma_0})$ is controllable and $(A_{\sigma_0}, C_{\sigma_0})$ is observable.

Without loss of generality, one can assume that the pair $(A_{\sigma_0}, C_{\sigma_0})$ is in the controllable canonical form and $(A_{\sigma_0}, C_{\sigma_0})$ is in the observable canonical form, i.e., $A_{\sigma_0} = \begin{bmatrix} 0_n & \begin{bmatrix} I_{n-1} \\ 0_{n-1}^T \end{bmatrix} \end{bmatrix}$, $B_{\sigma_0} = [0\ 0\ 0\ \cdots\ 0\ 1]^T$ and $C_{\sigma_0} = [1\ 0\ 0\ \cdots\ 0\ 0]$, where $I_{n-1} \in \mathbb{R}^{n-1\times n-1}$ is an identity matrix and $0_k = [0\ 0\ \cdots\ 0\ 0]^T \in \mathbb{R}^{k\times 1}$.

The objective is to design an output feedback for (25) so that the closed loop trajectories $x(\cdot)$ reach the origin in a finite time. To this end, we first design an FTS observer, and use the estimated states $\hat{x}$ to design the control input u. The form of the observer is:

$$\dot{\hat{x}} = A_\sigma \hat{x} + g_\sigma(C_\sigma x - C_\sigma \hat{x}) + B_\sigma u. \tag{26}$$

Following (Perruquetti et al., 2008, Theorem 10), we define the function $g : \mathbb{R} \to \mathbb{R}^n$ as:

$$g_i(y) = l_i\, \mathrm{sign}(y)|y|^{\alpha_i}, i = 1, 2, \dots, n, \tag{27}$$

where l_i are such that the matrix $\bar{A}$ defined as $\bar{A} = \begin{bmatrix} -\bar{l} & \begin{bmatrix} I_{n-1} \\ 0_{n-1} \end{bmatrix} \end{bmatrix}$ where $\bar{l} = [l_1\ l_2\ \cdots\ l_n]^T$ is Hurwitz, and the exponents α_i are chosen as $\alpha_i = i\alpha - (i-1)$ for $1 < i \le n$, where $1 - \frac{n-1}{n} < \alpha < 1$. Define the function g_σ as:

$$g_\sigma(y) = \begin{cases} g(y), & \sigma(t) = \sigma_0; \\ 0, & \sigma(t) \neq \sigma_0; \end{cases} \tag{28}$$

Let the observation error be $e = x - \hat{x}$, with $e_i = x_i - \hat{x}_i$ for $i = 1, 2, \dots, N$. Its time derivative reads:

$$\dot{e} = A_\sigma e - g_\sigma(C_\sigma e). \tag{29}$$

Next, we design a feedback $u = u(\hat{x})$ so that the origin is FTS for the closed-loop trajectories of (25). Inspired from control input defined in (Bhat and Bernstein, 2005, Proposition 8.1), we define the control input as

$$u(\hat{x}) = \begin{cases} -\sum_{i=1}^{n} k_i\, \mathrm{sign}(\hat{x}_i)|\hat{x}_i|^{\beta_i}, & \sigma(t) = \sigma_0; \\ 0, & \sigma(t) \neq \sigma_c; \end{cases}, \tag{30}$$

where $\beta_{j-1} = \frac{\beta_j \beta_{j+1}}{2\beta_{j+1} - \beta_j}$ with $\beta_{n+1} = 1$ and $0 < \beta_n = \beta < 1$, and k_i are such that the polynomial $s^n + k_n s^{n-1} + \cdots + k_2 s + k_1$ is Hurwitz. We now state the following result.

Theorem 4. Let the switching signal σ for (25) be given by (20) with $F = \sigma_0$. Assume that there exist functions μ_{ij} as defined in (18), and that the conditions (i)-(iii) of Theorem 3 are satisfied. Then, the origin of the closed-loop system (25) under the effect of control input (30) is an FTS equilibrium.

Proof. We first show that there exists $T_1 < \infty$ such that for all $t \ge T_1$, $\hat{x}(t) = x(t)$. Note that the origin is the only equilibrium of (29). From the analysis in Theorem 3, we know that the conditions (i) and (ii) of Theorem 2 are satisfied. The observation-error dynamics for mode σ_0 reads:

$$\dot{e} = \begin{bmatrix} e_2 - l_1 \mathrm{sign}(e_1)|e_1|^{\alpha_1} \\ e_3 - l_2 \mathrm{sign}(e_1)|e_1|^{\alpha_2} \\ \vdots \\ e_n - l_{n-1} \mathrm{sign}(e_1)|e_1|^{\alpha_{n-1}} \\ -l_n \mathrm{sign}(e_1)|e_1|^{\alpha_n} \end{bmatrix}. \tag{31}$$

Now, using (Perruquetti et al., 2008, Theorem 10), we obtain that the origin is an FTS equilibrium for (31), i.e., for mode σ_0 of (29). From (Perruquetti et al., 2008, Lemma 8), we also know that (31) is homogeneous with degree of homogeneity $d = \alpha - 1 < 0$. Hence, using (Bhat and Bernstein, 2005, Theorem 7.2), we obtain that there exists a Lyapunov function V_o satisfying $\dot{V}_o \le -cV_o^\beta$ where $c > 0$ and $0 < \beta < 1$. Hence, condition (iii) of Theorem 2 is also satisfied. From the proof of Theorem 3, we obtain that the condition (iv) of Theorem 2 and Assumption 1 are also satisfied. Hence, we obtain that the origin of (29) is an FTS equilibrium. Thus, there exists $T_1 < \infty$ such that for all $t \ge T$, $\hat{x}(t) = x(t)$. So, for $t \ge T_1$, the control input satisfies $u = u(\hat{x}) = u(x)$. Again, it is easy to verify that the origin is the only equilibrium for (25) under the effect of control input (30). The closed-loop trajectories take the following form for the mode $\sigma = \sigma_0$

$$\dot{x} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n - \sum_{i=1}^{n} k_i \mathrm{sign}(x_i)|x_i|^{\beta_i} \end{bmatrix}. \tag{32}$$

From (Bhat and Bernstein, 2005, Proposition 8.1), we know that the origin of the closed-loop trajectories for mode $\sigma = \sigma_0$ is FTS. Hence, repeating same set of arguments as above, we obtain that there exists $T_2 < \infty$ such that for all $t \ge T_1 + T_2$, the closed-loop trajectories of (25) satisfy $x(t) = 0$.

We presented a way of designing switching signal σ and control input u for a class of switched linear system where only of the modes is controllable and observable.

4. SIMULATIONS

We present a numerical example to demonstrate the efficacy of the proposed method. The example considers a switched linear control system with five modes such that only one mode is both controllable and observable. We design an FTS output controller for the considered switched system, and demonstrate that the closed-loop trajectories reach the origin despite presence of unobservable modes, and that some of the uncontrollable modes are unstable.

We consider linear switched system of the form (25) and design an output feedback that stabilizes the origin for the closed-loop system in a finite time. For illustration purposes, we consider a system of order $n = 2$, $\sigma \in \{1, 2, 3, 4, 5\}$, and assume that mode $\sigma = 5 \triangleq \sigma_0$ is controllable and observable, i.e., that the pair $(A_{\sigma_0}, B_{\sigma_0})$ is controllable and $(A_{\sigma_0}, C_{\sigma_0})$ is observable, while other modes are either uncontrollable or unobservable, or both. The simulation parameters are:

- Number of modes $N = 5$, FTS mode $F = 5$, $t_d = 0.1$, $\alpha = 0.9$, $a_1 = -10$, $a_2 = 10$, $\beta = .9$ $k_1 = 20$ and $k_2 = 10$;
- The matrices A_i, B_i, C_i are chosen as $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$, $A_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1.2 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 0.1 \\ 0.1 & 2 \end{bmatrix}$, $A_5 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B_1 = B_2 = B_3 = B_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $B_5 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C_1 = C_2 = C_3 = C_4 = [0\ 0]$, $C_5 = [1\ 0]$.
- Generalized Lyapunov functions are chosen as $V_i(x) = x^T P_i x$ where matrices P_i are chosen as $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$, $P_3 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $P_4 = \begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix}$, and $V_5(x) = \frac{k_2}{2\alpha}|x_1|^{2\alpha} + \frac{1}{2}|x_2|^2$;
- Functions μ_{ij} as
$$\mu_{ij}(x) = \begin{cases} -\|x\|^2, & i \in \{1, 2, 4\}; \\ 0, & i \in \{3, 5\}; \end{cases}$$
for all $j \in \sigma$.

Note that open-loop mode 1 is Lyapunov stable, mode 3 is asymptotically stable, and modes 2, 4 and 5 are unstable. The generalized Lyapunov candidates V_i, being quadratic, satisfy condition (i) of Theorem 2. Modes 1, 3 and 5, being stable, satisfy condition (ii) with $\alpha_2 = 0$, and modes 2 and 4, being active only for a finite time, satisfy condition (ii) with $\alpha_2 = k\|x_0\|^2$ for some $k > 0$. Conditions (iv) and (v) are satisfied by carefully designing the switching signal, as discussed in Section 3.2.

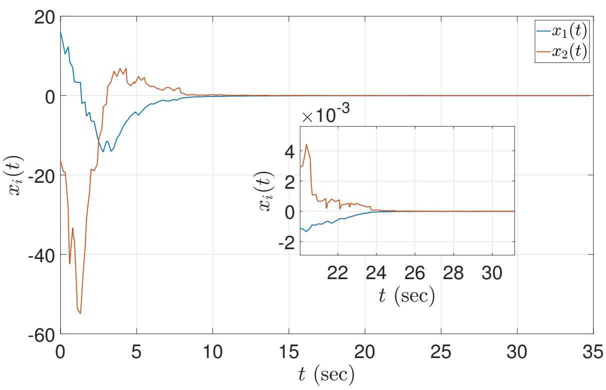

Fig. 3. Closed-loop system states $x_1(t), x_2(t)$ with time for linear switched system.

Figure 3 illustrates the state trajectories $x_1(t), x_2(t)$ of the closed-loop system over time for randomly chosen initial conditions, and Figure 4 depicts the norm of the states $\|x(t)\|$. Figure 5 plots the norm of the state-estimation error, $\|x - \hat{x}\|$ with time. It can be seen from the these figures that both the norms $\|x\|$ and $\|x - \hat{x}\|$ go to zero in finite time.

Figure 6 shows the evolution of Lyapunov functions $V_i(x - \hat{x})$ for the FTS observer of the linear switched system. It can be seen that there are unstable modes in the observer, where the value of the functions increase when the respective modes are active (e.g., mode 2 and 4). Finally, Figure 7 plots the switching signal σ with time. The switching signal is designed as per the design procedure listed in Section 3.2. It can be seen that all the five modes (including

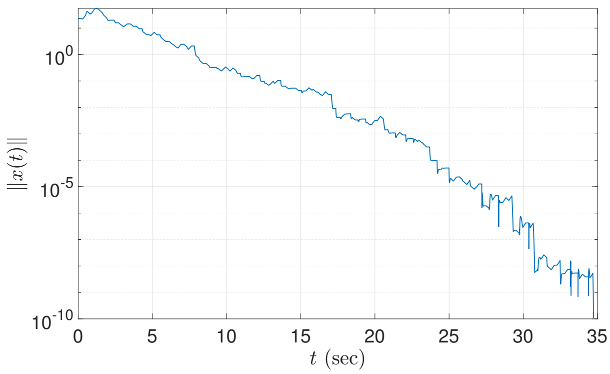

Fig. 4. The norm of the state vector $x(t)$ for the closed-loop trajectories of linear switched system with time.

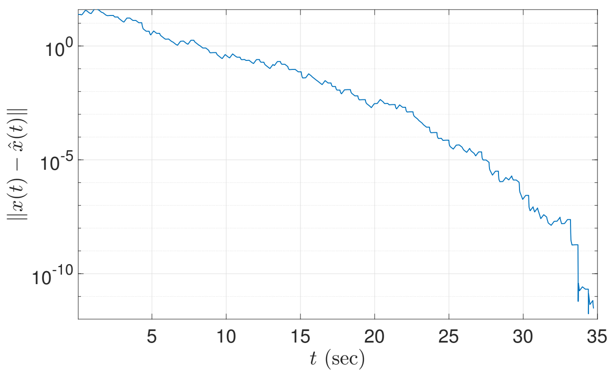

Fig. 5. The norm of the state-estimation error $x(t) - \hat{x}(t)$ for the linear switched system with time.

the unstable modes) get activated for the switched linear system, while FTS of the origin is still ensured.

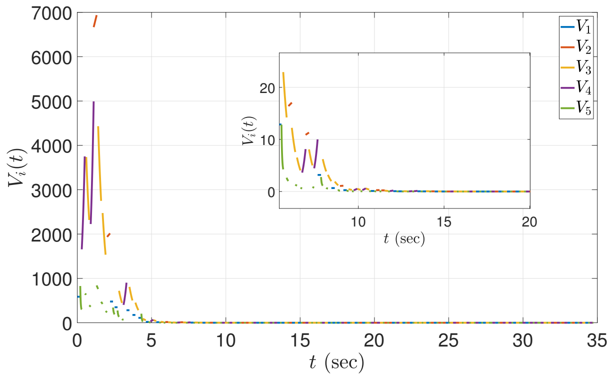

Fig. 6. The evolution of the Lyapunov functions $V_i(t)$ for the FTS observer of the linear switched system.

The provided examples validate that the system can achieve FTS even when one or more modes are unstable, if the FTS mode is active for long enough.

5. CONCLUSIONS

In this paper, we studied FTS of a class of switched systems. We showed that under some mild conditions on the bounds on the difference of the values of Lyapunov functions, if the FTS mode is active for a sufficient cumulative time, then the origin of the switched system is FTS. As an application of the theoretical results, we

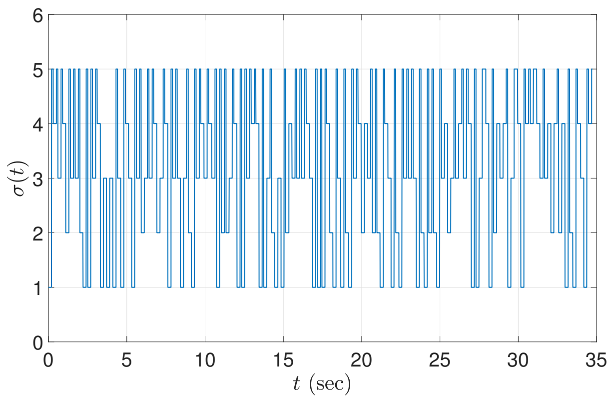

Fig. 7. Switching signal for the linear switched system.

designed an FTS output feedback for a class of linear switched systems in which only one of the modes is both controllable and observable.

As future research, we want to investigate how the results presented in this paper can be used for the systematic control synthesis of switched systems under spatiotemporal constraints, requiring closed-loop trajectories to reach a given goal set within a finite time.

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