Pricing of Electricity Swing Options[*]

Jussi Keppo

Department of Industrial and Operations Engineering
University of Michigan, 1205 Beal Avenue
Ann Arbor, Michigan 48109-2117, USA
E-mail: keppo@umich.edu

This version: January 21, 2004

Abstract

We consider the pricing of electricity swing options that hedge the electricity price risk and also partly hedge the risks in the option owner's electricity consumption process. The swing derivative sets instantaneous and cumulative boundaries for the electricity consumption and it specifies the price at which the option owner can buy energy. The name swing option comes from these consumption boundaries since the consumption swings between the lower and upper boundaries. We show that the swing options can be priced and hedged with regular electricity forwards and call options. Then we illustrate in a numerical example how swing options can be used to model and hedge power plants.

Keywords: Derivative pricing, electricity market, consumption uncertainty.

---

[*] I am grateful for Jukka Ruusunen, Iivo Vehviläinen, and Benjamin Kanoui who offered insights into this work. I also thank anonymous referees, Stephen Figlewski, Sophie Shive, and conference participants at the INFORMS Annual Meeting 2001 for helpful comments. All errors are mine.

1. Introduction

One major difference between the stock market and many commodity markets, especially the electricity market, is that commodity usage is usually given by an exogenous consumption process while a stock investor can choose the asset holding. Therefore, in commodity derivative markets there is a need to produce derivative instruments that take into account the uncertain commodity usage. One example of this kind of electricity instrument is a swing option. The aim of this paper is to model electricity swing options in terms of regular electricity derivatives. Our pricing model helps in understanding how different variables affect swing option prices and also gives a practical hedging strategy for the options. Further, the proposed model can be used in the optimization and hedging of power plants.

Swing options have been traded in electricity over-the-counter (OTC) markets for many years but their embedded optionality is sometimes mispriced. A swing option, also called a variable base-load factor, is an agreement to purchase energy at a certain fixed price during a fixed time interval [see e.g. Barbieri and Garman (1996, 1997) and Pilipovic and Wengler (1998)]. The option gives some flexibility on the purchased energy amount and on the purchase timing. In practice, swing option owners do not always use this flexibility to maximize the option value because often they are unable to control their electricity usage. Therefore, the purchase constraints are selected such that the consumption process is inside the constraints with high probability. There exist two types of swing options. The first type is called One-Swing option and the other is Full-Swing option. With the One-Swing option a single consumption swing right is purchased and with Full-Swing option the number of swing rights equal to the number of delivery dates within the delivery period. Both these option types have similar instantaneous and cumulative consumption constraints (power and energy boundaries) and, therefore, they can be analyzed in our framework. Further, by using this model we can price and hedge more general swing options that might have continuously changing instantaneous consumption constraints. In this paper we consider an option seller who, in order to avoid losses, assumes that the option owner maximizes the value of the swing option by optimizing the consumption process.

As mentioned earlier, we show that in the presence of electricity forwards and options a swing option is just a portfolio of these derivatives. Electricity forwards and options are traded in OTC markets and electricity exchanges [e.g., Nord Pool (Scandinavia), APX (USA and Netherlands), NYMEX (New York), and VicPool (Victoria, Australia)]. The results obtained from our model show, for example, that the higher the instantaneous and cumulative consumption upper boundaries the higher the swing option price, because then there is more optionality in the swing option and, therefore, the electricity consumer can better hedge the consumption uncertainties. We show that in this high optionality case the swing option price depends on the electricity forward curve uncertainties since the value of the flexibility is an increasing function of the forward volatility. In the numerical example we illustrate how the model can be applied in a power plant's production optimization and hedging. For instance, hydropower plants can be modeled with our framework because water reservoir constraints are similar to the swing options' consumption constraints [the optimization of hydropower plants is considered, e.g., in Gjelsvik, Røtting and Røynstrand (1992), Keppo (2002), and Pereira (1989)].

Many other papers have also considered commodity and electricity derivative pricing. The basic financial asset pricing methods can be found e.g. in Harrison and Kreps (1979), Harrison and Pliska (1981), Heath, Jarrow, and Morton (1992), and Duffie (1992). Black (1976) and Wolf (1982) derive pricing models for commodity contracts. Deng, Johnson, and Sogomonian (2001) study electricity spark and locational spread options. Keppo and Räsänen (1999) derive a pricing model for an electricity contract that does not have explicit constraints for electricity consumption. An extension to that paper is illustrated in Keppo and Räsänen (2000) where fixed budget instruments are introduced. These instruments allow the customers to fix the amount they spend on electricity consumption. The difference between fixed budget instruments and swing options is that with swing options the option owner is assumed to optimize the electricity consumption while in the models of Keppo and Räsänen the electricity consumption process is an exogenous stochastic variable.

Swing option pricing is considered, e.g., in Jaillet, Ronn, and Tompaidis (2001) and Thompson (1995). Both these papers have solved swing option prices by using numerical techniques, e.g., a lattice-based method. In contrast to these papers we first characterize the swing option prices in terms of electricity forwards and options that are traded in electricity financial markets. Due to the swing option's path dependency and Bermudan option type nature we are not able to obtain analytical pricing function for swing options [for the pricing of Bermudan options see Schweizer (2002), Carr and Yang (2002), Carrière (2001), and

Broadie and Yamamoto (2002)]. Because the swing option's optimal consumption process does not depend only on the current electricity spot price (and the cumulative energy usage) but also on the whole electricity forward curve that depends on several risk factors [for the modeling of electricity forward curves see e.g. Audet, Heiskanen, Keppo, and Vehviläinen (2003) and Koekebakker and Ollmar (2001)], the exercise times of the Bermudan option characteristics depend on multidimensional uncertainty. This means that the numerical pricing algorithms are hard to implement and the swing option price depends highly on the forward curve model. By using Markov electricity consumption processes, however, we solve for the swing option's analytical lower boundary as a result of a linear optimization model. This lower boundary and the corresponding hedging strategy are easily implemented to every day industry practice.

The rest of the paper is divided as follows. Section 2 introduces the financial instruments used in the paper. Section 3 derives the swing option pricing model and Section 4 illustrates the model with two examples. Finally Section 5 concludes.

2. Model

We consider an electricity market where financial instruments are traded continuously within a time horizon $[0,\tau]$. We denote the swing option's delivery period by $[T_0,T_1]$, which is a subset of $[0,\tau]$. The swing option's owner has to purchase at least the minimum amount of energy at a certain price during T_0 - T_1. In addition to that, the owner has an option to purchase an extra amount of energy at the same fixed price on the same delivery period. In this paper we consider an option seller and in order to avoid losses the seller assumes that the option's owner selects such a consumption process, p, that maximizes the option's value. In describing the probabilistic structure of the market, we will refer to a filtered complete probability space $\left(\Omega, F, P, \left(F_t\right)_{0\le t\le \tau}\right)$ that satisfies the usual hypotheses [see e.g. Protter (1990)]. Here Ω is a set, F is a σ-algebra, P is a probability measure on F, and $\left(F_t\right)_{0\le t\le \tau}$ is an increasing family of σ-algebras. The following assumptions characterize our electricity market.

Assumption 1. *There exist European call options and forward contracts on the electricity price. The electricity derivative market is complete and there is no arbitrage.*

In a competitive electricity markets, there are options and forward contracts continuously traded in exchange places (e.g. Nord Pool) and in the OTC market. This implies that the electricity derivative market has priced the risks associated with the electricity price process. In deriving our swing option pricing model we will use different maturity forwards and European call options that have the same strike price as the swing option under consideration. Given the no arbitrage condition all the electricity derivative portfolios with the same future payoffs have the same current value, i.e., the law of one price holds in our electricity market. If this were not the case, traders would sell the overpriced portfolio and buy the lower priced portfolio and collect these opportunities out from the market. Market completeness means that the tradable instruments are priced according to the same unique linear pricing function [see e.g. Duffie (1992)]. Therefore, in the electricity market there exist at least as many derivative contracts as there are sources of uncertainties. For instance, if the market is driven by n independent Brownian motions then there are at least n electricity derivative instruments traded in the market. Note that without the derivative instruments the electricity market is incomplete, because electricity is a nontradable instrument.

We denote by $f(t,T)$ the T-maturity electricity forward price on electricity spot price at time t. By allowing T to vary from t to τ we get the whole forward curve $f(t,\cdot):[t,\tau]\rightarrow\mathbf{R}_+$. There are huge cycles in the forward curve because of the cycles in electricity demand. The maturity $T = t$ corresponds to the starting point of the curve $f(t,t) = S(t)$ which is the electricity spot price at time t. We assume that at each time $t \in [0,\tau]$ there exists a right continuous electricity forward curve. For the modeling of electricity forward curves see, e.g., Audet, Heiskanen, Keppo, and Vehviläinen (2003) and Koekebakker and Ollmar (2001).

The T-maturity European call option price at time t is denoted by $C(t,T,K)$, where K is the strike price of the option and K is equal to the swing option's strike price that we define later. We assume that there exist call options for all maturities, and for all t the curve $C(t,\cdot,K):[t,\tau]\rightarrow\mathbf{R}_+$ is right continuous. The next assumption characterizes the derivative price processes and other stochastic processes used in this paper.

ASSUMPTION 2. *The stochastic variables follow right continuous semimartingale processes that are driven by continuous and jump uncertainties on the probability space* (Ω, F, P) *along with the standard filtration* $\{F_t\!: t \in [0,\tau]\}$. *These variables belong to the class of functions* $g(t,\omega)$: $[0,\tau]\times\Omega\rightarrow\mathbf{R}$ *that satisfy*

i) $(t,\omega)\rightarrow g(t,\omega)$ *is* $\boldsymbol{B}\times F$*-measurable, where* $\boldsymbol{B}$ *denotes the Borel* σ*-algebra on* $[0,\tau]$.

ii) $g(t,\omega)$ *is* F_t*-adapted.*

iii) $E\left[g^2(t,\omega)\right] < \infty$ *for all* $t \in [0,\tau]$.

Assumption 2 implies that we have both continuous and jump uncertainties in the electricity market. For simplicity, we will work in Hilbert space $L^2(\Omega,F,P)$. Semimartingale processes in financial markets are studied, e.g., in Goll and Kallsen (2000), Delbaen and Schachermayer (1994, 1996), and Carr and Wu (2003).

Under assumptions 1 and 2 the price of a T-maturity electricity derivative instrument at time t is given by the risk-neutral pricing formula [see e.g. Duffie (1992)]

$$\text{(1)} \qquad \pi(t,T) = e^{-r(T-t)} E^Q\left[\Phi\left(S(T)\right) \middle| F_t\right] \text{ for all } \; t \in [0,T], \; T \in [0,\tau],$$

where $\Phi(\cdot)$ is the payoff function, r is the risk-free interest rate that is for simplicity assumed to be constant, and E^Q is the expectation operator under the risk-neutral pricing measure Q.

If π is the value of T-maturity forward contract we get that the forward price at time t

$$\text{(2)} \qquad f(t,T) = E^Q\left[S(T) \middle| F_t\right]$$

because now $\Phi(S(T)) = S(T) - f(t,T)$ and the value of the forward contract when initiated is by definition zero, i.e., $\pi(t,T) = 0$. The pricing measure Q can be estimated from the electricity forward prices because given assumptions 1 and 2 the only unknown term in equation (2) is the Radon-Nikodym derivative dQ/dP.

By the definition of swing option, there exist constraints for the swing option's consumption process. The objective of the swing option holder is to maximize the value of the contract by selecting the optimal consumption process among the processes that satisfy almost surely the following cumulative and instantaneous constraints (energy and power constraints)

$$
\begin{aligned}
&e_{low} \le \int_{T_0}^{T_1} p(y)dy \le e_{up} \\
&p_{low}(t) \le p(t) \le p_{up}(t) \quad \text{for all} \quad t \in [T_0, T_1],
\end{aligned}
\tag{3}
$$

where $p(\cdot)$: $[T_0, T_1] \to \mathbf{R}_+$ is the swing option's consumption process and it is stochastic, e_{low} and e_{up} are constant lower and upper boundaries for the purchased energy, $p_{low}(\cdot):[T_0,T_1] \to \mathbf{R}_+$ and $p_{up}(\cdot):[T_0,T_1] \to \mathbf{R}_+$ are deterministic right continuous lower and upper boundary functions for power. Hence, p_{low} and p_{up} are the instantaneous consumption constraints and e_{low} and e_{up} are the cumulative constraints. The instantaneous constraints imply that the option owner must buy at least $p_{low}(t)$ and he/she cannot buy more than $p_{up}(t)$ at each time $t \in [T_0, T_1]$. Due to the cumulative constraints total purchases during T_0 – T_1 must be at least e_{low} and cannot be more than e_{up}. Because of the instantaneous constraints we assume that the cumulative constraints satisfy

$$
\int_{T_0}^{T_1} p_{low}(y)dy \le e_{low}, \quad e_{up} \le \int_{T_0}^{T_1} p_{up}(y)dy. \tag{4}
$$

We assume that the swing option seller hedges the option by trading electricity derivative instruments. Because the value of the hedging portfolio equals the swing option price, we have

$$
\int_{t}^{T_1} e^{-ry} p^*(y)[S(y)-K]dy = e^{-rt} z(t) + \int_{t}^{T_1} dM^Q(y), \tag{5}
$$

where $z(t)$ is the value of the hedging portfolio at time t, $p^*(\cdot)$ is the optimal consumption process of the swing option and it satisfies Assumption 2 and equation (3), K is the electricity price of the swing option (strike price), and $M^Q(\cdot)$ is a martingale under Q and it corresponds to the discounted gains and losses from the swing option's hedging strategy. Thus, the left-hand-side of (5) is the discounted cash flows from the swing option and the-right-hand side is the discounted gains and losses from the hedging strategy. Each time $t \in [T_0, T_1]$ the swing option's cash flow equals $p^*(t)[S(t) - K]$. If $p^*(t) > 0$, which is the case for all states of the world if $p_{low}(t) > 0$, and if $S(t) > K$ then the cash flow is positive. If $p^*(t) > 0$ and $S(t) < K$ the cash flow is negative, i.e., if the strike price K is high enough the swing option holder might make losses, because the consumption process has to satisfy the instantaneous and cumulative lower boundaries in (3). As mentioned earlier, the objective of the swing option's owner is to maximize the value of the contract by controlling the electricity consumption and, therefore, by taking the conditional expectation under Q equation (5) can be represented as follows

$$(6) \qquad z(t, T_0, T_1) = \sup_{p(\cdot) \in A} E^Q \left[\int_{t \vee T_0}^{T_1} e^{-r(y-t)} p(y)[S(y) - K]\, dy \,\middle|\, F_t \right] \quad \text{for all} \quad t \in [0, T_1],$$

where $t \vee T_0 = \max[t, T_0]$, $z(t, T_0, T_1)$ $(= z(t))$ is the arbitrage-free swing option price at time t for delivery period $[t \vee T_0, T_1]$, and A is a nonempty set of consumption processes that satisfy the conditions of Assumption 2 and the constraints in equations (3) and (4). Thus, according to (6) swing option is defined by the set of parameters $\left(p_{low}(\cdot), p_{up}(\cdot), e_{low}, e_{up}, T_0, T_1, K\right)$ and the price is given by the usual risk-neutral pricing equation.

As indicated in (5) and (6), we will see that swing options can be replicated by a basket of regular electricity derivatives and the amount of these derivatives depends on the instantaneous and cumulative consumption constraints. In practice, the constraints are set so that the realized consumption process of the option's owner is between the boundaries with high probability. However, because the realized consumption is unknown when the swing option is sold and because in many practical situations the option owner is not able to control the consumption process, there is always a risk that the realized consumption is outside the given region. Therefore, the option owner always carries part of the consumption risk. The swing option owner buys or sells the excess or the shortage of energy on the spot market. Thus, with swing options the planning of the power and energy constraints is a very important task in serving consumer needs. An example of a contract that also hedges the stochastic consumption process is given in Keppo and Räsänen (2000). As described in (6), the seller assumes in order to avoid losses that the option owner maximizes the value of the swing option by optimizing the consumption process.

3. Pricing

This section considers the pricing of swing options using different electricity derivative instruments. First, we divide the swing option's consumption into three terms as described in the following lemma.

LEMMA 1. *The swing option's electricity consumption can be represented as*

$$p(t) = p_{low}(t) + p_S(t) + p_C(t)\mathbf{1}\{S(t) \geq K\} \quad \text{for all} \quad t \in [T_0, T_1], \tag{7}$$

where $p_S(\cdot) \in A_S$, $p_C(\cdot) \in A_C(p_S)$, A_S *is the class of stochastic processes that satisfy the conditions of Assumption* 2 *and*

$$\int_{T_0}^{T_1} p_S(y)dy = e_{low} - \int_{T_0}^{T_1} p_{low}(y)dy$$
$$0 \leq p_S(t) \leq p_{up}(t) - p_{low}(t) \ \text{ for all } \ t \in [T_0, T_1],$$

$A_C(p_S)$ *is the class of stochastic processes that satisfy the conditions of Assumption* 2 *and*

$$0 \leq \int_{T_0}^{T_1} p_C(y)\mathbf{1}\{S(y) \geq K\}dy \leq e_{up} - e_{low}$$
$$0 \leq p_C(t)\mathbf{1}\{S(t) \geq K\} \leq p_{up}(t) - p_{low}(t) - p_S(t) \ \text{ for all } \ t \in [T_0, T_1],$$

and the indicator $\mathbf{1}\{S(t) \geq K\} = \begin{cases} 1, & \text{if } S(t) \geq K \\ 0, & \text{otherwise} \end{cases}$.

PROOF. Because with p_C the lower instantaneous and cumulative constraints (power and energy constraint) are zero and because the owner of the swing option maximizes the value of the swing option, he/she selects $p_C(t) = 0$ if $S(t) < K$. Therefore, we can represent the third term of (7) as $p_C(t)\mathbf{1}\{S(t) \geq K\}$. From equation (7) we get

$$e_{low} \leq \int_{T_0}^{T_1} \left(p_{low}(y) + p_S(y) + p_C(y)\mathbf{1}\{S(y) \geq K\}\right)dy \leq e_{up}$$

and

$$p_{low}(t) \leq p_{low}(t) + p_S(t) + p_C(t)\mathbf{1}\{S(t) \geq K\} \leq p_{up}(t) \quad \text{for all} \quad t \in [T_0, T_1].$$

These are the same as equations (3) and (4). *Q.E.D.*

All the consumption terms in equation (7) are nonnegative and the two last terms are stochastic. The first term, p_{low}, is the lower power boundary in (3). Thus, each time $t \in [T_0, T_1]$ the swing option owner consumes at least this minimum power. The set A_S for the second term, p_S, corresponds to the additional purchases, of which the cumulative value equals the difference between the lower energy boundary, e_{low}, and the cumulative lower power boundary, $\int_{T_0}^{T_1} p_{low}(y)dy$. Therefore, $p_{low} + p_S$ is the minimum consumption that satisfies both of the swing option constraints in equation (3). Note that there is some optionality in the timing of p_S but its cumulative value has to be equal to $e_{low} - \int_{T_0}^{T_1} p_{low}(y)dy$. That is, the option owner can select the times where $p_S(t) > 0$ but the cumulative value satisfies $\int_{T_0}^{T_1} p_S(y)dy = e_{low} - \int_{T_0}^{T_1} p_{low}(y)dy$. If the swing option's owner selects $p_S(t) > 0$ then according to (6) the cash flow at time t is $[p_{low}(t) + p_S(t)][S(t) - K]$ (assuming $p_C(t) = 0$, this is due to the constraints of p_C and is explained after Proposition 1) and then the owner losses the opportunity to use this consumption option in the future and the possibility to collect even higher cash flows. In some cases, however, the cumulative constraint of p_S forces $p_S(t)$ to be strictly positive and then we might have negative cash flow at time t. The set A_C for the third term, p_C, is the purchase set that has timing and quantity optionality. Thus, if necessary, the owner of the swing option can set $p_C(t) = 0$ for all $t \in [T_0, T_1]$ and, because the owner maximizes the swing option value, the third term is modeled as $p_C(t)\mathbf{1}\{S(t) \geq K\}$, i.e., the third term is nonzero only if $S(t) \geq K$. The maximum cumulative value of p_C is $e_{up} - e_{low}$. If the swing option's owner selects $p_C(t) > 0$ then the cash flow at time t is $\left[p_{low}(t) + p_C(t)\right]\left[S(t) - K\right] \geq 0$ (assuming now $p_S(t) = 0$), i.e., due to the quantity optionality if $p_C(t) > 0$ then the cash flow at time t is positive. However, by using the consumption option at time t the option owner losses the opportunity to use this option in the future, i.e., he/she losses the possibility to collect even higher profits in the future. In summary, the first term in equation (7) is the deterministic lower instantaneous consumption constraint, the second term corresponds to the lower cumulative consumption constraint, and the last term corresponds to the upper cumulative constraints.

Now by using Lemma 1 and equation (6) we get the following proposition that characterizes the swing option price in terms of electricity forwards.

PROPOSITION 1. *The swing option price is given as follows*

$$\begin{aligned} z(t,T_0,T_1) = {} & \int_{t\vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y)\left(f(t,y) - K\right)dy + \int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] \\ & \cdot e^{-r(y-t)} \left|f(t,y) - K\right| \left[Q_{f(t,y)}\left(p_S^*(y) > 0, S(y) \geq K\right) - Q_{f(t,y)}\left(p_S^*(y) > 0, S(y) < K\right)\right. \\ & \left. + Q_{f(t,y)}\left(p_C^*(y) > 0, p_S^*(y) = 0\right)\right] dy \quad \text{for all} \quad t \in [0,T_1], \quad T_0 \in [0,T_1), \quad T_1 \in (0,\tau], \end{aligned} \tag{8}$$

where the optimal consumption terms $p_S^*(\cdot)$ *and* $p_C^*(\cdot)$ *satisfy*

$$\begin{aligned} & \int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] \mathbf{1}\left\{p_S^*(y) > 0\right\} dy = e_S(t) \\ & \int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] \mathbf{1}\left\{p_C^*(y) > 0\right\} \mathbf{1}\left\{p_S^*(y) = 0\right\} dy \leq e_C(t), \end{aligned}$$

$f(t,T) = E^Q\left[S(T) \middle| F_t\right]$ *is the T-maturity forward price at time t,* $Q_{f(t,T)}$ *is the electricity forward martingale measure corresponding to* $f(t,T)$ [*see e.g. Geman* (1989) *and Björk* (1998)],

$Q_{f(t,T)}(0<x,0<y)=E^{Q_{f(t,T)}}\left[\mathbf{1}\{0<x\}\mathbf{1}\{0<y\}\middle|F_t\right]$ *is the probability under* $Q_{f(t,T)}$ *that* x *and* y *are strictly positive,* $e_S(t)=e_{low}-\int_{T_0}^{T_1}p_{low}(y)dy-\int_{T_0}^{t\vee T_0}p_S^*(y)dy$ *is the cumulative constraint of* p_S *at time* t*, and* $e_C(t)=e_{up}-e_{low}-\int_{T_0}^{t\vee T_0}p_C^*(y)dy$ *is the cumulative upper constraint of* p_C *at time* t*.*

PROOF. See appendix.

Proposition 1 implies that the swing options can be priced and hedged in terms of regular forwards. These different maturity forward contracts have forward price equal to the swing option's strike price, and these forwards can be replicated by using the current forward contracts and risk-free instruments. For instance, in Proposition 1 $e^{-r(T-t)}\left(f(t,T)-K\right)$ is equal to the value of one T-maturity forward contract plus $e^{-r(T-t)}\left(f(t,T)-K\right)$ in the risk-free asset at time t. Therefore, the existence of the risk-free asset and forward curve implies the existence of the forwards used in Proposition 1. From now on when we refer to forward contracts we mean forwards with forward price equal to the swing option's strike price. We use the absolute values of the forward contracts in equation (8) because then the numeraires are always positive [see e.g. Björk (1998)]. This is done in order to get the representation of Proposition 1. Note that if $f(t,T)\geq K$ then $e^{-r(T-t)}\left|f(t,T)-K\right|=e^{-r(T-t)}\left(f(t,T)-K\right)$ which is the value of one long forward contract at time t, and if $f(t,T)<K$ then $e^{-r(T-t)}\left|f(t,T)-K\right|=e^{-r(T-t)}\left(K-f(t,T)\right)$ and this is the value of the short position.

Proposition 1 is quite obvious since it says that given the optimal exercise times of swing rights that satisfy the consumption constraints, i.e., given the events where $p_S^*(t)>0$ and where $p_C^*(t) > 0$ for all $t \in [T_0, T_1]$ the swing option price is equal to the portfolio of forward contracts and the forward holdings are given by the probabilities $Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \geq K\right)$, $Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right)$, and $Q_{f(t,T)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right)$, where $T \in [T_0, T_1]$. Note that these probabilities are not under the objective measure P but under the forward measures. The exercise events where $p_S^*(t) > 0$ or $p_C^*(t) > 0$ depend on the energy used on $[T_0, t]$ as well as the forward curve and its uncertainties. Therefore, the solving of these exercise times is difficult even with numerical methods.

The first integral in equation (8) is the value of the lower instantaneous consumption boundary. The term $Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \geq K\right)$ is the probability under the forward measure $Q_{f(t,T)}$ that $p_S^*(T) > 0$ and $S(T) \geq K$, and $Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right)$ is the probability that $p_S^*(T) > 0$ and $S(T) < K$. According to the appendix, we have that the expectation under the risk-neutral probability measure

$$\begin{aligned} &E^Q\left[p_S^*(T)\big(S(T) - K\big)\middle|F_t\right] = \left|f(t,T) - K\right|\left[p_{up}(T) - p_{low}(T)\right] \\ &\quad \cdot \left[Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \geq K\right) - Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right)\right], \end{aligned}$$

where $t \in [0, T]$ and $T \in \left[T_0, T_1\right]$. Therefore, in (8) the term

$$\left[p_{up}(T) - p_{low}(T)\right]e^{-r(T-t)}\left|f(t,T) - K\right|\left[Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \geq K\right) - Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right)\right]$$

is the present value of $p_S^*(T)$ units of T-maturity forwards. Note that $p_S^*(T) \in \left\{p_{up}(T) - p_{low}(T), 0\right\}$ and it is an F_T-measurable random variable, i.e., its value is unknown before time T. Similarly, the expectation

$$E^Q\left[p_C^*(T)\big(S(T) - K\big)\middle|F_t\right] = \left|f(t,T) - K\right|\left[p_{up}(T) - p_{low}(T)\right]Q_{f(t,T)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right)$$

and this gives that in (8) the term

$$\left[p_{up}(T)-p_{low}(T)\right]e^{-r(T-t)}\left|f(t,T)-K\right|Q_{f(t,T)}\left(p_C^*(T)>0,p_S^*(T)=0\right)$$

is the present value of $p_C^*(T)$ units of T-maturity forwards. Combining we see that equation (8) represents the net present value corresponding to the consumption terms of Lemma 1 under optimal policy.

The above equations imply several facts about the swing option's optimal consumption strategy. First, the second and third consumption terms satisfy $p_S^*(T)p_C^*(T)=0$ for all $T\in\left[T_0,T_1\right]$ and this is due to the representation of Lemma 1. Second, the optimal consumption process is either p_{low} or p_{up}, i.e., the consumption process swings between the instantaneous lower and upper boundaries. As illustrated in the appendix, this is due to the facts that the swing option value is linear with respect to p_S and p_C, and the instantaneous consumption at time t never violates the cumulative upper constraint if $\int_0^{t-}p(y)dy<e_{up}$.

According to equation (8), if the swing option seller holds T-maturity forward contracts at time t as follows

$$\begin{aligned}
&p_{low}(T)+\left[p_{up}(T)-p_{low}(T)\right]\Big[Q_{f(t,T)}\left(p_S^*(T)>0,S(T)\geq K\right)-Q_{f(t,T)}\left(p_S^*(T)>0,S(T)<K\right)\\
&\quad+Q_{f(t,T)}\left(p_C^*(T)>0,p_S^*(T)=0\right)\Big]\quad\text{if}\quad f(t,T)\geq K\\
&p_{low}(T)-\left[p_{up}(T)-p_{low}(T)\right]\Big[Q_{f(t,T)}\left(p_S^*(T)>0,S(T)\geq K\right)-Q_{f(t,T)}\left(p_S^*(T)>0,S(T)<K\right)\\
&\quad+Q_{f(t,T)}\left(p_C^*(T)>0,p_S^*(T)=0\right)\Big]\quad\text{if}\quad f(t,T)<K
\end{aligned}$$

for all $T\in\left[T_0\vee t,T_1\right]$ then the swing option is hedged. Note that the seller needs to hold forwards with maturities between T_0 and T_1 and if the above holding of T-maturity forward is negative then the seller has a short position. Because the forward holdings depend on the probabilities that are calculated with respect to information at time t, these probabilities change over time and therefore the hedging strategy of a swing option is dynamic. In order to illustrate this let us assume that $f(0,T) > K$ for all $T \in [T_0, T_1]$. In this case if the forward prices on $\left[T_0, \tilde{T}\right]$ increases and the forward prices on $\left(\tilde{T}, T_1\right]$ decreases then the forward holdings corresponding to maturities between T_0 and $\tilde{T}$ increases and the forward holdings corresponding to maturities between $\tilde{T}$ and T_1 decreases, where $\tilde{T} \in [T_0, T_1]$. This is because in the above equation the term

$$Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \geq K\right) - Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right) + Q_{f(t,T)}\left(p_C^*(T) > 0, p_S^*(T) = 0\right)$$

increases for all $T \in \left[T_0, \tilde{T}\right]$ and decreases for all $T \in \left(\tilde{T}, T_1\right]$. Note that when the forward prices increases then there is a higher probability that the corresponding spot price $S(T) \geq K$ and, therefore, it is more likely that the optimal consumption at time T is $p_{up}(T)$.

In Proposition 1 we only used forward contracts to hedge the swing options. We can also solve for the corresponding swing option's replication strategy by using the forwards and call options. This is given in the following corollary.

COROLLARY 1. *The swing option price is given by*

$$
\begin{aligned}
z(t,T_0,T_1) = &\int_{t\vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y)\big(f(t,y)-K\big)\,dy \\
&+\int_{t\vee T_0}^{T_1}\big[p_{up}(y)-p_{low}(y)\big]e^{-r(y-t)}\left|f(t,y)-K\right|\Big[Q_{f(t,y)}\big(p_S^*(y)>0,S(y)\geq K\big) \\
&-Q_{f(t,y)}\big(p_S^*(y)>0,S(y)<K\big)\Big]dy \qquad (9)\\
&+\int_{t\vee T_0}^{T_1}\big[p_{up}(y)-p_{low}(y)\big]C(t,y,K)Q_{C(t,y,K)}\big(p_C^*(y)>0,p_S^*(y)=0\big)\,dy \\
&\qquad\text{for all}\quad t\in[0,T_1],\quad T_0\in[0,T_1),\quad T_1\in(0,\tau],
\end{aligned}
$$

where

$$
\begin{aligned}
&\int_{t\vee T_0}^{T_1}\big[p_{up}(y)-p_{low}(y)\big]\mathbf{1}\big\{p_S^*(y)>0\big\}\,dy = e_S(t) \\
&\int_{t\vee T_0}^{T_1}\big[p_{up}(y)-p_{low}(y)\big]\mathbf{1}\big\{p_C^*(y)>0\big\}\mathbf{1}\big\{p_S^*(y)=0\big\}\,dy = e_C(t),
\end{aligned}
$$

the T-maturity European call option price with strike price K at time t is $C(t,T,K)=e^{-r(T-t)}E^Q\big[\mathbf{1}\{S(T)\geq K\}(S(T)-K)\big|F_t\big]$, $Q_{C(t,T,K)}$ *is the martingale measure corresponding to* $C(t,T,K)$, *and* $Q_{C(t,T,K)}(x>0,y>0)=E^{Q_{C(t,T,K)}}\big[\mathbf{1}\{x>0\}\mathbf{1}\{y>0\}\big|F_t\big]$ *is the probability under* $Q_{C(t,T,K)}$ *that x and y are strictly positive.*

PROOF. See appendix.

In Corollary 1 we have written Proposition 1 in a different form by using the call option measures, $Q_{C(t,\cdot,K)}$. Thus, we get from the last integrals of (8) and (9)

$$
\frac{Q_{C(t,T,K)}\big(p_C^*(T)>0,p_S^*(T)=0\big)}{Q_{f(t,T)}\big(p_C^*(T)>0,p_S^*(T)=0\big)}=\frac{e^{-r(T-t)}\left|f(t,T)-K\right|}{C(t,T,K)}\quad\text{for all}\quad t\in[0,T],\quad T\in[T_0,T_1].
$$

That is, the ratio of the probabilities is given by the ratio of the absolute value of the forward and the call option price. The representation of Corollary 1 is more convenient than Proposition 1 in the sense that the indicator in p_C is modeled with call options and, therefore, it is easier to understand the consumption terms of swing option. $p_{low} + p_S^*$ is the minimum consumption that satisfies both the swing option constraints in equation (3) and, therefore, it is modeled with forward holdings. Because p_C^* has quantity optionality, this consumption term is modeled with call options.

From Corollary 1 we get that if $p_{low} = p_{up}$ or if $e_{low} = e_{up}$ then the portfolio consists only of forwards. Further, if e_{low}, $p_{low} = 0$, and e_{up}, $p_{up} > 0$ then there are only options of which quantities have to be optimized. In this case the swing option price depends more on the forward curve volatility because the forward curve uncertainty affects the call option prices. Since option prices are always positive the increase in e_{up} and p_{up} never decreases the swing option value.

Proposition 1 and Corollary 1 imply that the swing option is equivalent to a basket of regular electricity derivatives, i.e., swing options can be represented in terms of the existing electricity derivatives. Note that because the consumption process depends on the market forward and call option prices and because these prices are stochastic, the swing option's replication strategy is dynamic.

The purchase times, where $p_S^*(t) > 0$ or $p_C^*(t) > 0$, are stopping times of stochastic analysis and they are solved similarly as exercise strategies of Bermudan options. Bermudan derivatives are between American and European derivatives because they have a set of permitted exercise dates. Bermudan options are studied in Broadie and Yamamoto (2002), Schweizer (2002), Carr and Yang (2002), and Carrière (2001). The optimal purchase times can be solved by using stochastic dynamic programming. This implies that since there are at most four consumption alternatives we get the optimal decision at time t by assuming an optimal consumption policy on $\left(t \vee T_0, T_1\right]$ and by comparing the swing option value under the following alternatives

$$p_S(t) = \begin{cases} p_{up}(t) - p_{low}(t) \\ 0 \end{cases} \text{ and } p_C(t) = \begin{cases} p_{up}(t) - p_{low}(t) \\ 0 \end{cases},$$

where we assume that $p_S(t) = 0$, $p_S(t) = p_{up}(t) - p_{low}(t)$, and $p_C(t) = p_{up}(t) - p_{low}(t)$ are admissible, i.e., they satisfy the corresponding conditions of Lemma 1 and if some of them is not admissible then we just have fewer alternatives to compare. However, this dynamic programming problem is hard to solve because the swing option's optimal consumption strategy depends on the whole forward curve and its uncertainties as well as on the cumulative consumption process, i.e., the problem is a multidimensional stochastic control problem. For instance, according to Audet, Heiskanen, Keppo, and Vehviläinen (2003) and Koekebakker and Ollmar (2001), there are more uncertainties in the electricity forward curve than there are usually in the yield curve. Further, the stochastic processes of the electricity forward curve's risk factors are more complicated than risk factor processes in regular financial markets.

Due to the above mentioned problems in applying stochastic control techniques to modeling swing options, we introduce a practical pricing and hedging approach that gives the swing option's lower boundary. We model $p_S^*(\cdot)$ and $p_C^*(\cdot)$ with Markov processes that depend on the electricity spot price. This implies that the probabilities of equation (9) are indicator functions of time because otherwise there is no guarantee that the power and energy constraints are satisfied. Because the class of Markov consumption processes is smaller than the initial space A, we get the following result.

COROLLARY 2. *The swing option's lower boundary is given by*

$$\begin{aligned} z_{low}(t,T_0,T_1) &= \int_{T_0 \vee t}^{T_1} e^{-r(y-t)}\left(f(t,y)-K\right)p_{low}(y)dy + \int_{T_0 \vee t}^{T_1} e^{-r(y-t)}\left(f(t,y)-K\right)\left[p_{up}(y)-p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_S^*(t)\right\}dy \\ &+ \int_{T_0 \vee t}^{T_1} C(t,y,K)\left[p_{up}(y)-p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_C^*(t)\right\}dy \quad \text{for all} \quad t \in [0,T_1], \quad T_0 \in [0,T_1), \quad T_1 \in (0,\tau] \end{aligned} \tag{10}$$

where $p_{low}(\cdot) + \left[p_{up}(\cdot) - p_{low}(\cdot)\right]\left[\mathbf{1}\left\{\cdot \in \Gamma_S^*(t)\right\} + \mathbf{1}\left\{\cdot \in \Gamma_C^*(t)\right\}\right] : \left[t \vee T_0, T_1\right] \to \mathbf{R}_+$ *is the optimal Markov consumption process at time* t *for the future time period* $\left[t \vee T_0, T_1\right]$, $\Gamma_S^*(t)$ *and* $\Gamma_C^*(t)$ *are disjoint sets. These sets are obtained from the following linear optimization problem at time* t

$$\max_{\Gamma_S(t)\subset[t,\tau]}\left\{\int_{T_0 \vee t}^{T_1} e^{-r(y-t)}\left(f(t,y)-K\right)\left[p_{up}(y)-p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_S(t)\right\}dy \right.$$
$$\left. + \max_{\Gamma_C(t)\subset[t,\tau]-\Gamma_S(t)} \int_{T_0 \vee t}^{T_1} C(t,y,K)\left[p_{up}(y)-p_{low}(y)\right]\mathbf{1}\left\{y \in \Gamma_C(t)\right\}dy\right\}$$

subject to

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] \mathbf{1}\left\{y \in \Gamma_S(t)\right\} dy = e_S(t)$$

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] \mathbf{1}\left\{y \in \Gamma_C(t)\right\} \mathbf{1}\left\{y \notin \Gamma_S(t)\right\} dy = e_C(t)$$

where

$$e_S(t) = e_{low} - \int_{T_0}^{T_1} p_{low}(y) dy - \int_{T_0}^{t\vee T_0} \left[p_{up}(y) - p_{low}(y)\right] \mathbf{1}\left\{y \in \Gamma_S^*(y)\right\} dy \quad \textit{and}$$

$$e_C(t) = e_{up} - e_{low} - \int_{T_0}^{t\vee T_0} \left[p_{up}(y) - p_{low}(y)\right] \mathbf{1}\left\{y \in \Gamma_C^*(y)\right\} \mathbf{1}\left\{y \notin \Gamma_S^*(y)\right\} dy \, .$$

Proof. Under the Markov consumption strategies we get from Corollary 1

$$\begin{aligned} z(t,T_0,T_1) &= \int_{t\vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y) \left(f(t,y) - K\right) dy \\ &+ \int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] e^{-r(y-t)} \left|f(t,y) - K\right| \left[Q_{f(t,y)}\left(S(y) \ge K\right) - Q_{f(t,y)}\left(S(y) < K\right)\right] \mathbf{1}\left\{y \in \Gamma_S^*(t)\right\} dy \\ &+ \int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] C(t,y,K) \mathbf{1}\left\{y \in \Gamma_C^*(t)\right\} dy, \end{aligned}$$

where events $p_S^*(t) > 0$ and $p_C^*(t) > 0$ are modeled with the indicator functions. Indicator functions are used because otherwise there is no guarantee with spot-price dependent Markov strategies that the consumption constraints are satisfied. According to (A2) and (A5) in the appendix, we have

$$\begin{aligned} E^Q\left[p_S^*(T)\left(S(T) - K\right) \middle| F_t\right] &= \left|f(t,T) - K\right| \left[p_{up}(T) - p_{low}(T)\right] \\ &\cdot \left[Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) \ge K\right) - Q_{f(t,T)}\left(p_S^*(T) > 0, S(T) < K\right)\right]. \end{aligned}$$

By using (A5) again and the Markov assumption, we get

$$E^Q\left[\left|S(T)-K\right| \middle| F_t\right] = \left|f(t,T)-K\right|\left[Q_{f(t,T)}\left(S(T)\geq K\right) - Q_{f(t,T)}\left(S(T)<K\right)\right].$$

Using this with (2) and the second integral above gives equation (10). *Q.E.D.*

Due to the Markov assumption Corollary 2 solves the swing option's lower boundary. We select the sets Γ_S and Γ_C (times where $p_S(t)>0$ and $p_C(t)>0$) based on the current forward curve and European call option values. Since the forward curve and option prices are continuously changing, these sets and, therefore also the asset holdings, are dynamic. By definition of swing option there is some freedom on the purchase timing of p_S and p_C, however, the Markov assumption is implicitly partly ignoring the swing option's Bermudan option type characteristics. Because a Bermudan derivative is always more valuable than the corresponding European one, Corollary 2 gives the lower boundary of the swing option. This is similar to approximating a Bermudan option with the highest European option as follows

$$\tilde{X}(t) = \max_{\tau\in D} X(t,\tau),$$

where D is the set of possible exercise dates of the Bermudan option, $\tilde{X}(t)$ is the price of the Bermudan option at time t, and $X(t,T)$ is the prices of T-maturity European option at time t. Because the Bermudan option is more valuable than any European option on the right-hand-side, the above equation gives the lower boundary for the Bermudan option price. The difference between the lower boundary and the true value depends on the used stochastic process for the underlying asset as well as the option parameters. For instance, Broadie and Yamamoto (2002) report in their numerical example about 3.5% difference between a European option and the corresponding Bermudan option.

The optimization problem of the additional electricity purchase $\left[p_{up}(\cdot)-p_{low}(\cdot)\right]\left[\mathbf{1}\left\{\cdot\in\Gamma_S(t)\right\}+\mathbf{1}\left\{\cdot\in\Gamma_C(t)\right\}\right]$ in Corollary 2 is a linear optimization problem and can be solved by using standard techniques [for linear optimization problems see e.g. Taha (1997)]. Note that if $f(t,T)\ \leq\ K$ then $e^{-r(T-t)}\left(f(t,T)-K\right)\leq 0$ and $C(t,T,K)\geq 0$ and, therefore, in the optimization we try to select $\Gamma_S(t)$ and $\Gamma_C(t)$ such that $\mathbf{1}\left\{T\in\Gamma_S(t)\right\}=0$ and $\mathbf{1}\left\{T\in\Gamma_C(t)\right\}=1$. That is, at the points where the forward curve is greater than the swing option's strike price K we prefer to use forwards, and at the points where the curve is lower than the strike price we prefer options. This is illustrated in our numerical example.

From Corollary 2 and equation (A1) in the appendix we get the lower and upper boundaries of a swing option. That is, the swing option price z satisfies

$$z_{low}(t,T_0,T_1)\leq z(t,T_0,T_1)\leq z_{up}(t,T_0,T_1)<\infty \tag{11}$$

where

$$z_{up}(t,T_0,T_1)=\int_{T_0\vee t}^{T_1}\left\{e^{-r(y-t)}\left[f(t,y)-K\right]p_{low}(y)+C(t,y,K)\left[p_{up}(y)-p_{low}(y)\right]\right\}dy\,.$$

Thus, Corollary 2 gives the correct swing option price if $e_{up}\geq\int_{T_0}^{T_1}p_{up}(y)dy$ and $e_{low}\leq\int_{T_0}^{T_1}p_{low}(y)dy$ (also if $e_{up}=e_{low}$). However, this is a special case and this is ruled out in (4). In general $z_{low}(t,T_0,T_1)$ should be closer to $z(t,T_0,T_1)$ the closer e_{up} and e_{low} are to $\int_{T_0}^{T_1}p_{up}(y)dy$

and $\int_{T_0}^{T_1} p_{low}(y)dy$. The difference between the true value and the lower boundary depends on the forward curve and its uncertainties.

Because of the additional consumption jumps between the power constraints, i.e., $p_S^*(t) + p_C^*(t) = 0$ or $p_S^*(t) + p_C^*(t) = p_{up}(t) - p_{low}(t)$, we can model different swing options in our setup. If we use discrete time intervals $T_0 = \tau_1 < \ldots < \tau_{n+1} = T_1$, where $\Delta\tau = \tau_{i+1} - \tau_i$ for all $i \in \{1,\ldots,n\}$, then the following corollary models the One-Swing option of which swing period is equal to $\Delta\tau$.

COROLLARY 3. *If* A *in equation* (6) *is a set of consumption processes satisfying*

$$\begin{aligned} &(T_1 - T_0)\, p_{low} \le \Delta\tau \sum_{i\in\{1,\ldots,n\}} p(\tau_i) \le (T_1 - T_0)\, p_{low} + \left(p_{up} - p_{low}\right)\Delta\tau \\ &p(\tau_1) = p_{low};\ p_{low} \le p(\tau_i) \le p_{up} \quad \text{for all} \quad i \in \{2,\ldots,n\} \end{aligned} \tag{12}$$

where the electricity consumption is constant over the discrete time interval and the lower and upper boundaries are constant, then equation (12), *Lemma* 1, *Proposition* 1*, Corollary* 1, *and Corollary* 2 *solve the One-Swing option price at time t.*

PROOF. From Proposition 1 we get $p(\tau_i) = p_{low}$ or $p(\tau_i) = p_{up}$ and at the beginning of the swing option delivery period we have $p(T_0) = p_{low}$ because $p(\tau_1) = p_{low}$. If the option holder chooses to select $p(\tau_i) = p_{up}$ at some point $\tau_i > T_0$ then due to the cumulative constraints we have to have again $p(\tau_j) = p_{low}$ for all $j > i$. Thus, there is only one swing from p_{low} to p_{up} and the time spent in p_{up} is $\Delta\tau$. *Q.E.D.*

By using Proposition 1 and corollaries 1-3 we get the value of the One-Swing option at time t. Note that Proposition 1 and corollaries 1 and 2 give directly the Full-Swing option price. By utilizing the cumulative boundaries e_{low} and e_{up} we can give constraints for the time that the consumption process p equals p_{low} and p_{up}. That is, if we use the same discrete intervals as in Corollary 3 and if the consumption process satisfies

$$\begin{aligned} &(T_1 - T_0)\, p_{low} + \left(p_{up} - p_{low}\right) k_{low} \Delta\tau \leq \Delta\tau \sum_{i \in \{1,\ldots,n\}} p(\tau_i) \leq (T_1 - T_0)\, p_{low} + \left(p_{up} - p_{low}\right) k_{up} \Delta\tau \\ &p_{low} \leq p(\tau_i) \leq p_{up} \quad \text{for all} \quad i \in \{1,\ldots,n\} \end{aligned} \tag{13}$$

where $0 \leq k_{low} \leq k_{up} \leq n$, then the consumption is equal to p_{up} at least $\frac{k_{low}\Delta\tau}{T_1 - T_0} \cdot 100\%$ of the time and at most $\frac{k_{up}\Delta\tau}{T_1 - T_0} \cdot 100\%$ of the time.

4. Example

We illustrate our swing option pricing model with two examples and, for simplicity, we consider the option's lower boundary, i.e., we use Corollary 2. The first example solves a swing option price's lower boundary with different energy constraints and the second example illustrates how the pricing model can be used in the optimization and hedging of the electricity production process. We use forward price data from the Scandinavian exchange Nord Pool (The Nordic Power Exchange) between 1/1/01 and 12/31/01. The Scandinavian electricity market is hydro power dominated with roughly 50% of the electricity supply being hydro based. The winters are cold and much of the precipitation comes as snow. In the spring the snow melts, causing floods whose timing varies a lot from year to year due to the temperature. There is a significant electricity heating load but the mild summers do not require a lot of air conditioning, so that electricity demand is concentrated on the winter season. The time dependent variation present in the demand results in a seasonal, weekly, and daily profile in the electricity spot price and electricity forward curve. However, these price variations are smoothed to some extent in the Scandinavian market because of hydropower production.

Figure 1 illustrates the one year forward curve during 2001 by using the Block products that are electricity forwards of which underlying asset is the average electricity price over a four weeks period. Because in the pricing of swing options we need forward curve on the electricity spot price that has instantaneous duration we have interpolated the Block data points to model better the forward curve on the spot price. As expected, according to Figure 1 the forward price is higher during the winter and lower during the summer. However, usually this price variation is much higher in Scandinavia, and 2001 was uncommon since the price was always quite close to the yearly average price. The hydrological situation changed from relatively wet conditions to dry in the beginning of year 2001. The change was due to cold and relatively dry weather in the first months of 2001 and was reflected as a sharp rise in the spot price. The hydrological situation improved during the year thus causing the spot price to fall towards autumn.

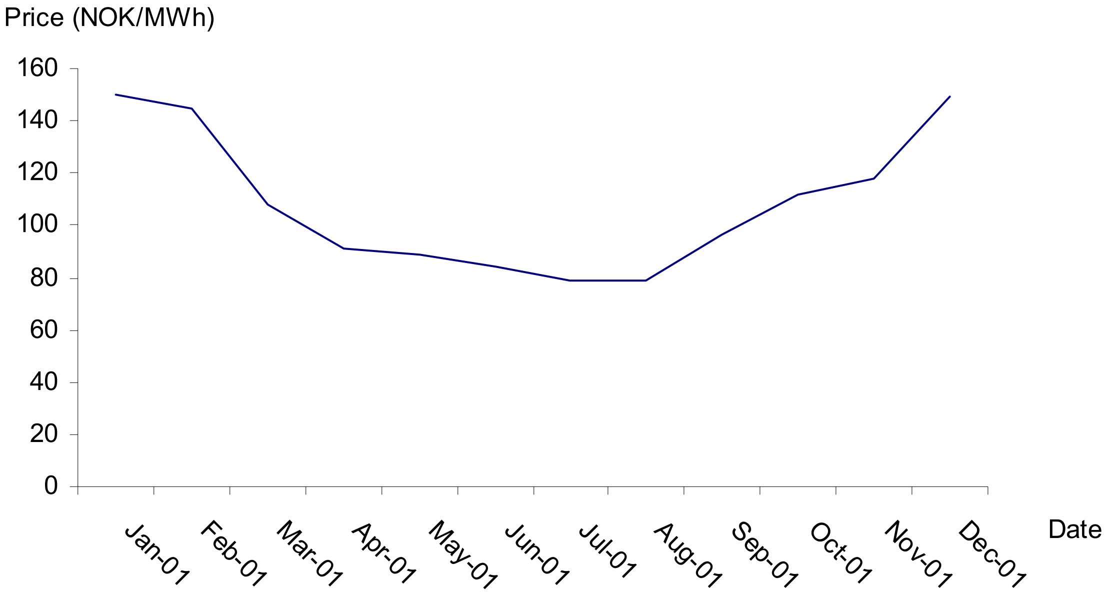

Figure 1. Forward curve based on Nord Pool's Block products in 2001.

Figure 2 illustrates the corresponding historical volatility curve. Usually in Scandinavia the volatility is high during the spring because the melting snow causes floods, but their timing varies from year to year due to the temperature. As mentioned earlier, in the winter 2000-2001 there was less snow than normal. Therefore, there was no uncertainty on the spring flood and the volatility in Figure 2 is significantly lower than usual.

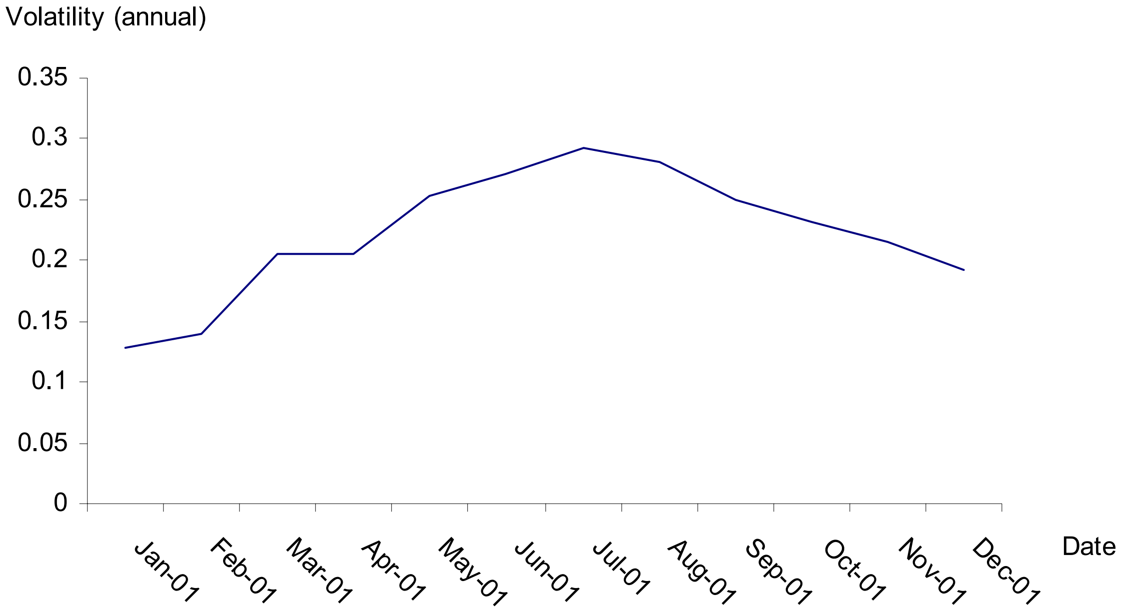

Figure 2. Historical volatility curve based on Nord Pool's Block products in 2001.

We assume that current date is 1st of December 2000 and the swing option's delivery period is one year, from 1/1/01 to 12/31/01. The strike price is 100 NOK/MWh (average forward price is 112 NOK/MWh) and the annual continuous compounding risk free rate is 5%. Now from figures 1 and 2 we can solve the European call option prices of Corollary 2 by using the Black-76 model. This is illustrated in Figure 3.

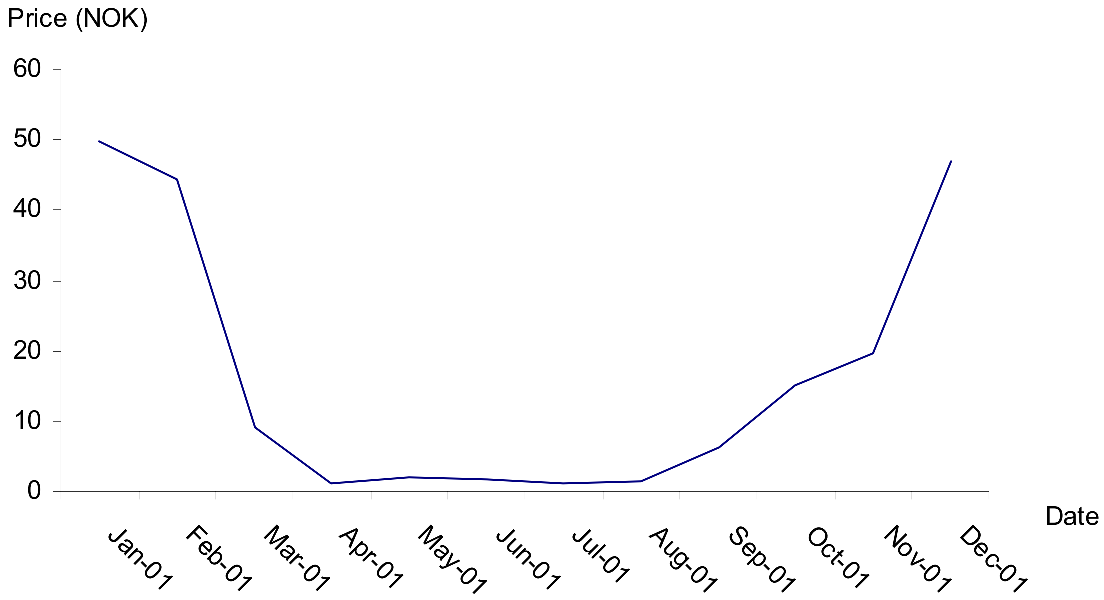

Figure 3. Call option prices from Black-76 model and with strike price equal to 100 NOK/MWh, risk-free rate 5%, and volatility given by Figure 2.

The option prices for the summer months are low and for the winter period high because of the forward curve in Figure 1.

4.1 Swing option example

We analyze the swing option price's lower boundary with different energy (cumulative) constraints. The results are illustrated in Table 1.

Table 1. Swing option price's lower boundary with different consumption constraints.

e_{low} (MWh)	e_{up} (MWh)	p_{low} (MW)	p_{up} (MW)	**Swing option price (NOK)**
400	800	0.05	0.1	10456
600	800	0.05	0.1	10196
600	600	0.05	0.1	9693

Comparing the first row and the second row we see that if the energy lower boundary is increased then the price's lower boundary decreases because some call options are changed to forwards. The third row implies that if the upper energy boundary is decreased then the swing option's lower boundary decreases even more because, according to Corollary 2, the call option holding is decreased.

4.2 Production example

Next we illustrate how the swing option model can be used in the electricity production optimization and hedging. We assume that the production costs are constant and equal to 100 NOK/MWh. Due to the production constraints the minimum production power $p_{low} =$ 500 MW and the maximum production power $p_{up} = 1000$ MW. There exist also boundaries on cumulative production and, therefore, we have the following energy constraints for the production between 1/1/01 and 12/31/01: $e_{low} = 4 \cdot 10^6$ MWh and $e_{up} = 8 \cdot 10^6$ MWh. For instance, in the case of a hydropower plant these upper and lower energy constraints correspond to the size of the water reservoir [see e.g. Gjelsvik, Røtting and Røynstrand (1992), Keppo (2002), and Pereira (1989)]. From the swing option's instantaneous and cumulative constraints we see that owning a power plant is similar to holding a swing option with strike price equal to the production cost and, therefore, we model the production strategy by using Corollary 2. Figure 4 illustrates the production strategy that corresponds to the forward holding p_S and that can be hedged by selling the electricity forwards.

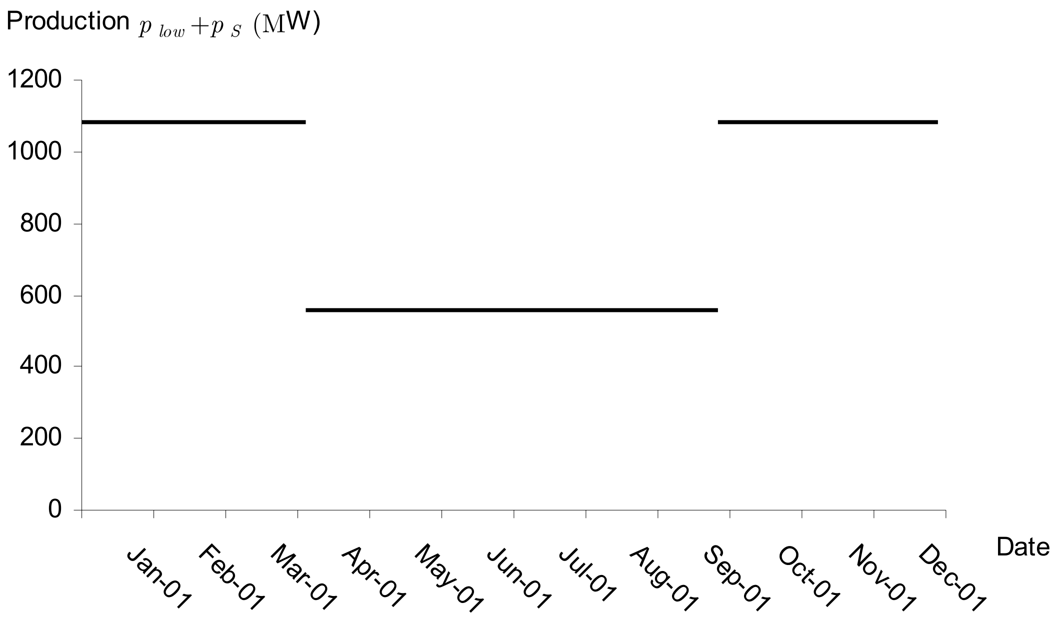

Figure 4. Production strategy corresponding to forwards with production cost equal to 100 NOK/MWh.

As can be seen from Figure 4, this production is high during the winter and low during the summer. This is due to the shape of the forward curve in Figure 1. Figure 5 illustrates the optional production strategy p_C. According to this figure the optional electricity is produced during the summer if the electricity spot price is higher than the production cost. Because this optional production strategy corresponds to the call option holding, this part of the production can be hedged by selling call options.

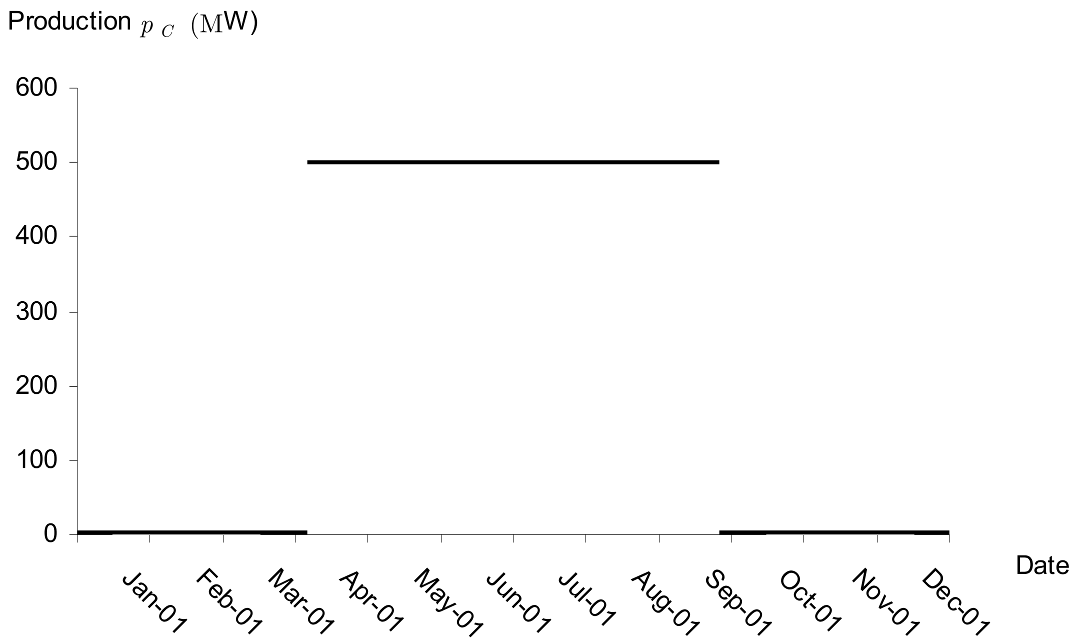

Figure 5. Optional production strategy with production cost equal to 100 NOK/MWh.

According to figures 4 and 5 the energy production strategy (of e.g. a hydropower plant) can be modeled as $p_{low} + p_S + p_C$, where $p_{low} + p_S$ is a production strategy that depends only on time and p_C is an optional production strategy that depends on time and electricity spot price. Therefore, production part $p_{low} + p_S$ can be hedged by selling forwards and p_C by selling call options.

5. Conclusions

In this paper we have studied the pricing and hedging of electricity swing options. We have shown that the swing options can be replicated with regular electricity forwards and call options. Therefore, the pricing of swing options is based on electricity market data that is obtained from electricity exchanges. The hedging strategy of a swing option is equal to the solution of a constrained optimization problem. Because this optimization problem requires numerical methods, we introduced a practical hedging strategy that replicates the swing option's lower boundary. In this hedging strategy the minimum energy amount was modeled with electricity forward holdings and the optional energy amount with call option holdings. Because electricity options are functions of the forward volatility, the electricity forward curve uncertainty affects the swing option price more if there is flexibility in the purchased energy amount.

The developed pricing model eases the understanding how different variables affect the swing option prices. Due to the introduced hedging strategy the model also makes the production of these instruments easier. Further, as illustrated in the numerical example the swing option pricing framework helps in the optimization and hedging of power plants, because many power plants, for instance hydropower plants, have swing option type production (consumption) constraints.

References

Audet, N., P. Heiskanen, J. Keppo, and I. Vehviläinen, 2003, "Modeling Electricity Forward Curve Dynamics in the Nordic Market," forthcoming in the *Modeling Prices in Competitive Electricity Markets* (edited by D. W. Bunn), Wiley Series in Financial Economics.

Barbieri, A. and M.B. Garman, 1996, “Understanding the Valuation of Swing Contracts,” *Energy and Power Risk Management.*

Björk, T., 1998, *Arbitrage theory in continuous time*, Oxford University Press, New York.

Black, F., 1976, “The Pricing of Commodity Contracts,” *Journal of Financial Economics*, 3: 167 - 179.

Broadie, M. and Y. Yamamoto, 2002, “Application of the Fast Gauss Transform to Option Pricing”, Preprint, Columbia University.

Carr, P and L. Wu, 2003, “Time-Changed Levy Processes and Option pricing,” forthcoming in *Journal of Financial Economics.*

Carr, P. and G. Yang, 2001, “Simulating Bermudan Interest Rate Derivatives,” in: M. Avellaneda (ed), *Quantitative Analysis in Financial Markets*, Vol II, World Scientific Publishing, 295-316.

Carrière, J. F., 2001, “Linear Regression and Standardized Quasi-Monte Carlo for Approximating Bermudan Options,” Working Paper, University of Alberta.

Delbaen, F. and W. Schachermayer, 1994, “A general version of the fundamental theorem of asset pricing,” Mathematische Annalen, 300, 463-520.

Delbaen, F. and W. Schachermayer, 1998, “The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes,” Mathematische Annalen, 312, 215-250.

Deng, S.-J., Johnson, B. and A. Sogomonian, 2001, “Exotic electricity options and the valuation of electricity generation and transmission assets,” *Decision Support Systems*, 30, 383–392.

Duffie, D., 1992, *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton.

Garman, M.B. and A. Barbieri, 1997, “Ups and Downs of Swings,” *Energy and Power Risk Management*, 1(2).

Geman, H., 1989, “The importance of the forward neutral probability in a stochastic approach of interest rates,” Working paper, ESSEC.

Gjelsvik, A., T. Røtting, and J. Røynstrand, 1992, “Long-term scheduling of hydro-thermal power systems,” Hydropower, 539-546.

Goll T. and J. Kallsen, (2000), “Optimal portfolios for logarithmic utility,” Stochastic Processes and their Applications, 89, 31-48.

Harrison, J. and S. Kreps, 1979, “Martingales and Arbitrage in Multiperiod Securities Markets,” *Journal of Economic Theory*, 20, 381 - 408.

Harrison, J. and S. Pliska, 1981, “Martingales and Stochastic Integrals in the Theory of Continuous Trading,” *Stochastic Processes and Their Applications*, 11, 215 - 260.

Heath, D., Jarrow, R. and A. Morton, 1992, “Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation,” *Econometrica*, 60, 77 - 106.

Henderson, J.S., 1986, "Price Determination Limits in Relation to the Death Spiral," *The Energy Journal*, 7(3), 150-200.

Hillier, F.S. and G.J. Lieberman, 2001, *Introduction to Operations Research*, 7th edition, McGraw-Hill, Columbus.

Ikeda, N. and S. Watanabe, 1981, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, New York.

Jaillet, P., E. Ronn, and S. Tompaidis, 2001, "Valuation Commodity Based Swing Options," Working paper, UT Austin.

Keppo, J. and M. Räsänen, 1999, "Pricing of Electricity Tariffs in Competitive Markets," *Energy Economics*, 21, 213-223.

Keppo, J. and M. Räsänen, 2000, "Pricing of Fixed Budget Contingent Claims in Competitive Electricity Markets," *Central European Journal of Operations Research*, 8, 163-171.

Keppo, J., 2002, "Optimality with Hydropower System," IEEE Transactions on Power Systems, 3, 583-589.

Koekebakker, S. and F. Ollmar, 2001, "Forward curve dynamics in the Nordic electricity market," Preprint, Norwegian School of Economics and Business Administration.

Pereira, M., 1989, "Optimal stochastic operations scheduling of large hydroelectric systems," International Journal of Electrical Power & Energy Systems, 11, 161-169.

Pilipovic, D. and J. Wengler, 1998, "Getting into the Swing," *Energy and Power Risk Management*, 2(10).

Protter, P.E., 1992, *Stochastic integration and differential equations: a new approach*, Springer, New York.

Schweizer, M., 2002, "On Bermudan Options," in: K. Sandmann and P. J. Schönbucher (eds.), *Advances in Finance and Stochastics. Essays in Honour of Dieter Sondermann*, Springer, 257-269.

Taha, H.A., 1997, *Operations Research*, Prentice-Hall International, New Jersey.

Thompson, A.C., 1995, "Valuation of Path-Dependent Contingent Claims with Multiple Exercise Decisions over Time: The Case of Take-or-Pay," *Journal of Financial and Quantitative Analysis*, 30(2), 271-293.

Wolf, A., 1982, "Fundamentals of Commodity Options on Futures," *Journal of Futures Markets*, 2, 391 – 408.

Appendix: Proof of Proposition 1.

In this appendix we prove Proposition 1. From Lemma 1 and equation (6) we get by using dynamic programming [see e.g. Hillier and Lieberman (2001)] since p_C is a function of p_S

$$
\begin{aligned}
\text{(A1)}\quad z(t,T_0,T_1) &= \int_{T_0}^{T_1} e^{-r(y-t)}\left(f(t,y)-K\right)p_{low}(y)dy + \sup_{p_S\in A_S}\left\{E^Q\left[\int_{T_0}^{T_1} e^{-r(y-t)}\left(S(y)-K\right)p_S(y)dy\,\middle|F_t\right]\right.\\
&\left. + \sup_{p_C\in A_C(p_S)} E^Q\left[\int_{T_0}^{T_1} e^{-r(y-t)}\mathbf{1}\left\{S(y)\geq K\right\}\left(S(y)-K\right)p_C(y)dy\,\middle|F_t\right]\Big]\right\}
\end{aligned}
$$

where $t < T_0$,

$$
z(t,T_0,T_1) \leq \int_{T_0}^{T_1}\left\{e^{-r(y-t)}\left[f(t,y)-K\right]p_{low}(y) + \mathbf{1}\left\{e_{up} > e_{low}\right\}C(t,y,K)\left[p_{up}(y)-p_{low}(y)\right]\right\}dy < \infty
$$

and the T-maturity European call option price with strike price K at time t is $C\left(t,T,K\right) = e^{-r(T-t)}E^Q\left[\mathbf{1}\left\{S(T)\geq K\right\}\left(S(T)-K\right)\middle|F_t\right]$. Note that the optimal consumption $p_{low} + p_S^* + p_C^*$ depends on the power and energy constraints as well as on the forward curve and its dynamics.

Next we represent equation (A1) under the electricity forward measures. That is, by using the absolute value of T-maturity forward contract that has forward price equal to K as a numeraire with the second term of (A1) we get [see e.g. Geman (1989) and Björk (1998)]

$$
\begin{aligned}
\frac{\pi_S(t,T)}{e^{-r(T-t)}\left|f(t,T)-K\right|} &= E^{Q_{f(t,T)}}\left[\frac{\left(S(T)-K\right)p_S(y)}{\left|S(T)-K\right|}\right]\\
\text{(A2)}\qquad &= E^{Q_{f(t,T)}}\left[\left(\mathbf{1}\left\{S(T)\geq K\right\} - \mathbf{1}\left\{S(T)<K\right\}\right)p_S(T)\right] = p_S(t,T,K)\\
&\qquad \text{for all}\quad t\in[0,T],\quad T\in\left[T_0,T_1\right],
\end{aligned}
$$

where $\pi_S(t,T) = e^{-r(T-t)}E^Q\left[p_S(T)\left(S(T)-K\right)\middle|F_t\right]$, $f(t,T) = E^Q\left[S(T)\middle|F_t\right]$, $P\left(\left|S(T)-K\right|>0\right)=1$, and the numeraire forward contract (forward price K) is equal at time t to a portfolio of one long forward contract with forward price $f(t,T)$ and $e^{-r(T-t)}\left(f(t,T)-K\right)$ in the risk free asset. Note first that if $f(t,T)=K$ in (A2) then we set $\pi_S(t,T)=0$. Second, we can use the absolute value of the forward contract as a numeraire since $\left|f(y,T)-K\right|>0$, with probability one, for each $y\in(t,T]$. In the same way the third term of (A1)

$$\text{(A3)}\quad \begin{aligned}\frac{\pi_C(t,T)}{e^{-r(T-t)}\left|f(t,T)-K\right|} &= E^{Q_{f(t,T)}}\left[\frac{\mathbf{1}\{S(y)\geq K\}\left(S(y)-K\right)p_C(y)}{\left|S(T)-K\right|}\middle|F_t\right]\\ &= E^{Q_{f(t,T)}}\left[\mathbf{1}\{S(T)\geq K\}\,p_C(T)\middle|F_t\right]=p_C(t,T,K)\quad\text{for all}\quad t\in[0,T],\quad T\in[T_0,T_1],\end{aligned}$$

where $\pi_C(t,T)=e^{-r(T-t)}E^Q\left[p_C(T)\mathbf{1}\{S(T)\geq K\}\left(S(T)-K\right)\middle|F_t\right]$. That is, $p_S(t,t,K)=p_S(t)\left(\mathbf{1}\{S(t)\geq K\}-\mathbf{1}\{S(t)<K\}\right)$ and $p_C(t,t,K)=p_C(t)\mathbf{1}\{S(t)\geq K\}$. From (A1) we now get by changing the order of the integrations [we assume technical conditions for this, see e.g. Ikeda and Watanabe (1981)]

$$\text{(A4)}\quad \begin{aligned} z(t,T_0,T_1) &= \int_{t\vee T_0}^{T_1} e^{-r(y-t)}p_{low}(y)\left[f(t,y)-K\right]dy\\ &+\sup_{p_S\in A_S,\,p_C\in A_C(p_S)}\int_{t\vee T_0}^{T_1}\left[p_S(t,y,K)+p_C(t,y,K)\right]e^{-r(y-t)}\left|f(t,y)-K\right|dy\end{aligned}$$

where $t\vee T=\max[t,T]$ and $t<T_1$.

This optimization problem is hard to solve because it is path dependent and also depends on the Radon-Nikodym derivative $\frac{dQ_{f(t,T)}}{dP}$ as well as on the forward curve dynamics. However, because the objective function is linear with respect to p_S and p_C, $p_S^*(t),p_C^*(t)\in\left[0,p_{up}(t)-p_{low}(t)\right]$, and because if $\int_0^{t-}p(y)dy<e_{up}$ then instantaneous consumption $p(t)$ does not violate this cumulative constraint ($p_{up}(t)$ is bounded), we can characterize the optimal terms $p_S^*(t)=p_S^*(t,\omega)$ and $p_C^*(t)=p_C^*(t,\omega)$ by using indicator functions as follows

$$\text{(A5)}\qquad \begin{aligned} &p_S^*(t,\omega) = \left[p_{up}(t) - p_{low}(t)\right]\mathbf{1}\left\{p_S^*(t,\omega) > 0\right\} \\ &p_C^*(t,\omega) = \left[p_{up}(t) - p_{low}(t)\right]\mathbf{1}\left\{p_C^*(t,\omega) > 0\right\} \quad \text{for all} \quad t \in [T_0, T_1], \end{aligned}$$

where we assume that we know the sets $\left\{\omega \middle| p_S^*(t,\omega) > 0\right\}$ and $\left\{\omega \middle| p_C^*(t,\omega) > 0\right\}$ for all $t \in [T_0, T_1]$ and where we have written the consumption terms as functions of event $\omega \in \Omega$ to emphasize the fact that electricity consumption is a stochastic process. However, as we will see, the main problem in the swing option pricing is to solve these sets. As mentioned earlier, these sets depend on the forward curve and its uncertainties at time t as well as the energy used up to time t.

By using equations (A4) and (A5) we can characterize (A1) as follows

$$\text{(A6)}\qquad \begin{aligned} z(t,T_0,T_1) = &\int_{t\vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y)\left(f(t,y) - K\right)dy + \int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right] \\ &\cdot e^{-r(y-t)}\left|f(t,y) - K\right|\left[Q_{f(t,y)}\left(p_S^*(y) > 0, S(y) \geq K\right) - Q_{f(t,y)}\left(p_S^*(y) > 0, S(y) < K\right)\right. \\ &\left. + Q_{f(t,y)}\left(p_C^*(y) > 0, p_S^*(y) = 0\right)\right]dy \end{aligned}$$

where $Q_{f(t,T)}(x > 0, y > 0) = E^{Q_{f(t,T)}}\left[\mathbf{1}\{x > 0\}\mathbf{1}\{y > 0\}\middle|F_t\right]$ is the probability under $Q_{f(t,T)}$ that x and y are strictly positive,

$$\int_{t\vee T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{p_S^*(y) > 0\right\}dy = e_S(t),$$

$$\int_{T_0}^{T_1} \left[p_{up}(y) - p_{low}(y)\right]\mathbf{1}\left\{p_C^*(y) > 0\right\}dy \leq e_C(t)$$

$$e_S(t) = e_{low} - \int_{T_0}^{T_1} p_{low}(y)dy - \int_{T_0}^{t\vee T_0} p_S(y)dy, \text{ and}$$

$$e_C(t) = e_{up} - e_{low} - \int_{T_0}^{t \vee T_0} p_C(y)dy\,.$$

This gives Proposition 1. *Q.E.D.*

Appendix: Proof of Corollary 1.

In this appendix we prove Corollary 1 and we proceed in the same way as in Proposition 1.

By using the electricity call option as a numeraire with the third term of equation (A1) we get

$$\text{(A7)} \qquad \frac{\pi_C(t,T)}{C(t,T,K)} = E^{Q_{C(t,T,K)}}\left[p_C(T)\middle|F_t\right] = p_C(t,T) \quad \text{for all} \quad t \in [0,T], \quad T \in [T_0, T_1],$$

where $\pi_C(t,T) = e^{-r(T-t)}E^Q\left[p_C(T)\mathbf{1}\left\{S(T) \geq K\right\}\left(S(T)-K\right)\middle|F_t\right]$ and European call option $C(t,T,K) = e^{-r(T-t)}E^Q\left[\mathbf{1}\left\{S(T) \geq K\right\}\left(S(T)-K\right)\middle|F_t\right]$. That is, $p_C(t,t) = p_C(t)$. Note that in (A3) the indicator $\mathbf{1}\left\{S(T) \geq K\right\}$ is inside the expectation, because the numeraires of (A3) and (A7) are different. Because $C(t,T,K) > 0$ for all $T > t$, we have $\pi_C(t,T) \geq 0$. As in the proof of Proposition 1, we can use the value of the call option as a numeraire since it is always positive.

From (A1), (A2), and (A7) we get by changing the order of the integrations

$$
\begin{aligned}
z(t,T_0,T_1) = {} & \int_{t\vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y)\big(f(t,y)-K\big)\,dy + \sup_{p_S\in A_S}\left\{ \int_{t\vee T_0}^{T_1} p_S(t,y,K) e^{-r(y-t)} \big|f(t,y)-K\big|\,dy \right. \\
& \left. + \sup_{p_C\in A_C(p_S)} \int_{t\vee T_0}^{T_1} p_C(t,y) C(t,y,K)\,dy \right\},
\end{aligned}
\tag{A8}
$$

where $t < T_1$. By using (A5) and (A8) we get

$$
\begin{aligned}
z(t,T_0,T_1) = {} & \int_{t\vee T_0}^{T_1} e^{-r(y-t)} p_{low}(y)\big(f(t,y)-K\big)\,dy \\
& + \int_{t\vee T_0}^{T_1} \big[p_{up}(y) - p_{low}(y)\big] e^{-r(y-t)} \big|f(t,y)-K\big| \Big[Q_{f(t,y)}\big(p_S^*(y) > 0, S(y) \geq K\big) \\
& - Q_{f(t,y)}\big(p_S^*(y) > 0, S(y) < K\big)\Big]\,dy \\
& + \int_{t\vee T_0}^{T_1} \big[p_{up}(y) - p_{low}(y)\big] C(t,y,K) Q_{C(t,y,K)}\big(p_C^*(y) > 0, p_S^*(y) = 0\big)\,dy
\end{aligned}
\tag{A9}
$$

where

$$
\int_{t\vee T_0}^{T_1} \big[p_{up}(y) - p_{low}(y)\big] \mathbf{1}\big\{p_S^*(y) > 0\big\}\,dy = e_S(t)
$$

and, because the value of an option is always nonnegative, we have now the following equality constraint for p_C^*

$$
\int_{t\vee T_0}^{T_1} \big[p_{up}(y) - p_{low}(y)\big] \mathbf{1}\big\{p_C^*(y) > 0\big\}\,dy = e_C(t).
$$

This gives Corollary 1. *Q.E.D.*