OPTION PRICING WITH AN EXPONENTIAL EFFECT FUNCTION

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Abstract. We consider option hedging and pricing for a large agent. The large agent affects the market's demand-supply equilibrium and, therefore, the market prices of financial instruments. By assuming a specific large agent's effect function for the underlying asset we derive the corresponding effect function for call options on that asset. As we show, the price of a call option in our model is the solution to a Black-Scholes partial differential equation with a modified terminal condition. Finally we estimate our model parameters from option market data and compare our model with other large agent models.

0. Introduction

An agent (or trader) in a financial market is considered *large* if her trading activities affect market prices. This typically happens if the agent's trading volume is a significant proportion of the total market volume. Examples include large investment banks in a regular financial market, energy companies in the electricity market, and any trader in an illiquid market. A group of small agents behaving in the same way, e.g. a group of delta hedging option writers, may also have a strong impact and thus be considered as an (aggregate) large agent.

The existence of a large agent price effect has been described empirically in several papers. Loeb [Lo]; Kraus and Stoll [KS]; Holthausen, Leftwich and Mayers [HML]; and Keim and Madhavan [KM] show that the price impact of block trades is significant in stocks with small market capitalization, and is systematically related to trade size. According to [Lo] the large agent impact often exceeds 15 percent for stocks with market capitalization less than 25 million dollars. In liquid large market-cap stocks Loeb estimated smaller market impacts, as low as one percent.

The large agent effect has ramifications for option pricing and the purpose of this paper is to extend the Black-Scholes option pricing model to a situation where the large agent affects the underlying asset price in an exponential way as described in Keppo [Ke]. In our model, which is consistent with the empirical studies cited above, there is a single parameter measuring the impact of the large agent on the market price of the underlying asset. This parameter can be viewed as the price elasticity with respect to the large agent's asset holding. Using our option pricing model, we estimate the implied large agent effect from IBM and GM option prices in 1995 and show that our elasticity parameter is significant. This indicates that the large agent effect is priced in the option market. Further, we show that our model fits market data better than those in Jonsson and Keppo [JK] and Liu and Yong [LY], which also have explicit option pricing equations.

In traditional asset pricing models, the underlying price process is assumed to be an exogenous variable and, therefore, derivative prices can be calculated in terms of the underlying asset price alone (see e.g. Black and Scholes [BS], Harrison and Kreps [HK], Harrison and Pliska [HP] and Kreps [Kr]). The presence of a large agent implies that market prices are no longer exogenous, but rather found endogenously within the model. Thus the large agent's option price (i.e. the initial premium needed for a hedging strategy implemented by a large agent) has to be expressed not only in terms of the underlying asset price but also in terms of the number of options hedged. That is, the number of options hedged by the large agent impacts the unit option price.

Our model describes the price dynamics in two stages. First there is a "small agent price" process s that describes the dynamics of the asset price had there been no large agent. We assume that s follows geometric Brownian motion, as in the Black-Scholes model. The observed market price S (which is

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the price *any* agent faces) is then derived from s and from the current asset holding of the large agent through the exponential effect function.

We perform a careful Black-Scholes type analysis and show that the large agent option price (as a function of time t and the "small agent" price s) is the solution to the Black-Scholes partial differential equation, with a terminal condition that reflects the exponential effect function. Further, the unit option price is an increasing function of the number of options sold, at least when the small agent price s is held fixed. In practice, if all option sellers follow delta hedging then the option's open interest increases the option price and, therefore, this information should be used in option pricing.

Option hedging under a large trader has been considered by several authors. One approach has been to divide the market players into two groups: large (program) traders and small (reference) traders. The underlying asset price dynamics is then derived from: 1) the market shares for these two groups, 2) the inverse demand curve of small traders, and 3) the stochastic risk factor of the small traders' demand. This is the approach taken in Frey [F], Sircar and Papanicolaou [SP], and Schönbucher and Wilmott [SW]. Another possibility, and the one explored in the current paper, is to directly model how the large agent impacts the underlying asset price and compute option prices on these assets. This approach can be found in Jarrow [J], Cvitanic and Ma [CM], Bank and Baum [BB], Jonsson and Keppo [JK], Cetin, Jarrow, and Protter [CJP], Cetin, Jarrow, Protter, and Warachka [CJPW], and Liu and Yong [LY].

Another distinction between large agent models comes from how the large agent's trading activity influences asset prices. In [CM], [JK] and [LY], the trading only affects the parameters in the stochastic differential equations describing the asset price process, whereas in the other papers mentioned above, as well as the present one, the activities of the large trader has a direct influence on the asset price.

The main way in which the current paper differs from previous work lies in the wealth dynamics. We assume that the large agent is always able to split block trades into infinitesimal packages. This assumption significantly simplifies the wealth dynamics of the large agent and in the end leads to a linear pricing PDE with an explicit solution. The idea of splitting trades is not new, and can be found in [BB] and [CJP], as well as in [SW]. However, in [BB] and [CJP], the splitting is only used as a book-keeping tool: the actual block trades (if any) are not split. As a consequence, the wealth dynamics in these papers is more complicated and options can in general only be asymptotically replicated.

We use IBM and GM option market data to compare our model to that of [JK] and [LY]. The main reason for picking these models is that they also admit explicit pricing equations. The results show that our current model provides the best fit. This indicates that the large agent effect mainly affects the current asset prices and not the stochastic process parameters.

The rest of the paper is organized as follows: in Section 1 we define the framework used in the paper. In particular we discuss the wealth process of the large agent. Section 2 contains the derivation of the partial differential equation for the call option price, and Section 3 an analysis of the solution. In Section 4 we compute market implied parameters. Section 5 concludes and Appendix A outlines the derivation of the wealth process.

1. The market and the wealth of a large agent

We consider a market consisting of a risk-free asset B (bond) and a risky asset (stock). The bond follows the dynamics

$$\frac{dB}{B} = r\,dt,$$

where risk-free rate r is constant. The dynamics of the stock is complicated by the presence of a (single) *large agent* whose trading activities affect the market price of the stock. This large agent may be an agent whose trading constitutes a significant proportion of the whole market volume. It can also be a group of agents that behave in the same way, such as delta-hedging option writers. The latter case is the one considered in the empirical analysis in Section 4.

The market stock price dynamics derives from two sources. First, there is an underlying risk factor process s which follows geometric Brownian motion, i.e.

$$\frac{ds}{s} = \mu\, dt + \sigma\, dW,$$

where μ and σ are constant and W is Brownian motion. We will sometimes refer to s as the *small agent stock price*, for reasons that will be apparent shortly. The stock price S that is observed in the market is then given by

$$S = se^{ga}, \tag{1.1}$$

where a is the number of stocks currently held by the large agent and $g \geq 0$ is an effect parameter. We will assume $g > 0$ so that the trading activities of the large agent has a definite effect on the market price. Thus, the more the agent buys the more she pays, and the more the agent sells the lower the price received. This effect function is consistent with previous empirical studies (Loeb [Lo]; Kraus and Stoll [KS]; Holthausen, Leftwich and Mayers [HML]; Keim and Madhavan [KM]). According to e.g. Loeb [Lo], the large agent parameter g depends on the market capitalization and the liquidity of the market. In the limiting case $g = 0$, the "large" agent is just another agent and $S = s$, explaining the name small agent stock price. We emphasize that even in the presence of a large agent, all agents face the same market price for the stock. In particular, a small agent also has to pay S and not s.

As, e.g., in Cetin, Jarrow, Protter and Warachka [CJPW] equation (1.1) implies that in contrast to usual asset pricing models, the large trader faces a supply curve that depends on her asset holding. The relationship in (1.1) can of course be generalized; we keep this simple form for tractability.

In order to avoid obvious arbitrage opportunities, we assume that the large agent is always on the "wrong side of the trade", meaning that if she instantaneously sells (buys) a block of shares, causing the stock price to drop (rise), then the transaction price is the one *after* the stock price has dropped (risen).

This assumption in fact implies that it is disadvantageous for the large agent to execute block trades. Indeed, suppose for instance that the large agent holds $a > 0$ stocks and decides to sell them all at once. The proceeds from the sale is then as (and not ase^{ga}). On the other hand, if the large agent is able to split her block order into n consecutive smaller (but still instantaneous) orders, each of size a/n, then the proceeds would be $as\frac{1}{n}\sum_{i=0}^{n-1} e^{gai/n}$, which is clearly larger than as (but less than aS). We shall in fact assume that the large agent can make use of this strategy to the extreme, and split all block trades into *infinitesimal packages* so that the proceeds from the (still instantaneous) trade above is $s\int_0^a e^{gy}\, dy$.

Under this assumption, the wealth dynamics of the large agent becomes quite tractable. To be precise, we define the *wealth* V of the large agent to be the wealth after all stock positions have been cleared. Then we have the following formula for V assuming the large agent holds a stocks at market value S and b bonds at market value B:

$$\begin{aligned} V &= bB + S\int_{-a}^{0} e^{gy}\, dy = bB + \frac{S}{g}\left(1 - e^{-ga}\right) \\ &= bB + s\int_0^a e^{gy}\, dy = bB + \frac{s}{g}\left(e^{ga} - 1\right) \\ &= bB + \widetilde{a}s, \end{aligned}$$

where

$$\widetilde{a} = \frac{1}{g}\left(e^{ga} - 1\right). \tag{1.2}$$

Moreover, if the large agent trades in a self-financing way, then we have

$$dV = \widetilde{a}\, ds + b\, dB = \widetilde{a}\, ds + rbB\, dt; \tag{1.3}$$

see Appendix A for a derivation of this formula.

Remark 1.1. An interesting interpretation of (1.3) is that the wealth dynamics of a large agent following a self-financing strategy (a, b) is the same as that of an agent applying the self-financing strategy $(\widetilde{a}, b)$ in a market without large agents and where the stock price is given by the risk factor s.

Remark 1.2. Our wealth dynamics is quite simple and in particular less complicated than the one derived in [BB]. The reason is that we assume that all trades (including block trades) are always split into infinitesimal packages. Bank and Baum [BB], on the other hand, only use the splitting of orders when computing the value of the portfolio after all positions have been cleared. In particular, an instantaneous block trade by the large agent does not affect her wealth in our model, but does lead to decreased wealth in the model in [BB]. See Appendix A for more details.

2. Option pricing I: derivation of the PDE

We now derive a partial differential equation for the unit option price in our large agent framework. In the next section we analyze some properties of its solutions.

2.1. **Assumptions.** We assume that the large agent is short $k \geq 0$ European call options with strike price K and expiration date T. Due to the large agent effect it is necessary to be precise about the meaning of the call option contract: it means that the holder of the option has the right to buy h stocks at time T from the large agent for a payment of $\$hK$, where $h \in [0, k]$ is a number chosen by the holder of the options.

We also need to specify how the delivery of the h stocks takes place. If the stocks are handed over in one package, then the market stock price will instantaneously drop from S to Se^{-gh}, seemingly leaving the holder of the options short-changed. We shall therefore assume that the delivery of the h stocks is taking place in infinitesimal packages, in the same way as the trading of the large agent is assumed to happen. Thus the option holder is able to sell these packages as they are handed over, thus receiving a total of $S\int_0^h e^{-gy}\,dy$ in cash. Equivalently, we may assume that the large agent is selling the h stocks on the market and handing over the proceeds to the holder of the option.

To get a unique price of the option we have to make further assumptions. The large agent hedges the call options by trading in the underlying asset (stock) and risk-free instrument. We assume that at any given time, all of the agent's stock holding is used for the hedging and we further assume that the large agent is Δ-neutral at any time $t \in [0, T[$. We also require that at the terminal time, immediately before exercise, the large agent owns exactly h stocks, where $h \in [0, k]$ is the number of options that the holder chooses (optimally) to exercise.

To summarize, we look for a self-financing trading strategy for the large agent that accomplishes the following:

- the large agent is Δ-neutral at any time $t \in [0, T[$;
- immediately before exercise the large agent holds $h \in [0, k]$ stocks;
- at exercise the large agent sells the h stocks on the market and gives the cash proceeds $s\int_0^h e^{gy}\,dy - hK$ (notice that $S = se^{gh}$) to the option holder;
- the number h maximizes the value of the option $s\int_0^h e^{gy}\,dy - hK$ on the interval $[0, k]$.

We will show that such a trading strategy exists (uniquely) and thus determines the price of the option. Generally speaking we will find the option price $C(s, t)$ as a function of the small agent stock price s, rather than the observed stock price S.

It will be demonstrated below that the large agent effect depends only on the product of the large agent parameter and the number of options hedged. Therefore we introduce

Definition 2.1. The *total large agent parameter* γ is defined by $\gamma = gk$, where g is the large agent parameter and k is the number of options being hedged.

Remark 2.2. The large agent does not directly affect option prices, but only indirectly, when she is hedging the options by trading in the underlying. This effect is significant only if $k \gg 0$, or more accurately $\gamma \gg 0$, because then the large agent hedges many options, thus influences the underlying asset price.

2.2. **The terminal condition.** Let us first find the price of the option at the terminal time T, immediately before exercise. By our assumptions the large agent then holds h stocks, with $h \in [0, k]$. Hence $S = se^{gh}$. At exercise the large agent sells the h stocks on the market and hands over the proceeds, which amount to $s\int_0^h e^{gy}\,dy$, to the holder of the options. After doing so, the large agent holds no stocks, so $S = s$.

It follows that the unit price of the k options is given by $\phi(h^*) = \max\{\phi(h) \mid h \in [0,k]\}$, where

$$\phi(h) = \frac{s}{k}\int_0^h e^{gy}\,dy - \frac{h}{k}K.$$

We see that $\phi''(h) = \frac{sh}{k}e^{gh} \geq 0$, so ϕ is convex, implying that the maximum of ϕ must occur at one of the boundary points $h = 0$ or $h = k$. In fact it is easy to see that the maximum occurs at $h = 0$ if $s < \frac{K\gamma}{e^\gamma - 1}$ and at $h = k$ if $s > \frac{K\gamma}{e^\gamma - 1}$, where $\gamma = gk$ is the total large agent parameter (see Definition 2.1). Notice that $\phi(0) = 0$ and $\phi(k) = \frac{s}{\gamma}(e^\gamma - 1) - K$. Therefore, the unit price of the options at time T is given by

$$C(s,T;\gamma) = \left(s\frac{e^\gamma - 1}{\gamma} - K\right)^+. \tag{2.1}$$

Notice that when $\gamma \to 0$, i.e. $g \to 0$ or $k \to 0$, then $C(s,T;\gamma) \to (s-K)^+$, the Black-Scholes terminal condition. We also see that it is always optimal for the holder of the option to exercise all or none of the options, depending on whether the quantity in (2.1) is positive or negative.

2.3. **The PDE.** We now proceed to derive the PDE for the unit option price $C(s,t;\gamma)$, where $t < T$ and $\gamma = gk$ is the total large agent parameter. To find an equation for C, we try to find an instantaneously riskless, self-financing portfolio Π containing the short option positions and $k\Delta = k\Delta(s,t)$ stocks. The value of such a portfolio is

$$\Pi = -kC + \widetilde{k\Delta}s,$$

where $\widetilde{k\Delta}$ is given by (1.2). By Itô, and by (1.3) this portfolio follows the dynamics

$$\begin{aligned} d\Pi &= -k\,dC + \widetilde{k\Delta}ds \\ &= -k\left(\frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial s}\,ds + \frac{1}{2}\frac{\partial^2 C}{\partial s^2}\,ds^2\right) + \widetilde{k\Delta}\,ds \\ &= -k\left(\frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial s}\,ds + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 C}{\partial s^2}\,dt\right) + \widetilde{k\Delta}\,ds. \end{aligned} \tag{2.2}$$

This is riskless if and only if the ds-terms cancel, which gives $\widetilde{k\Delta} = k\frac{\partial C}{\partial s}$, amounting to

$$\Delta = \frac{1}{\gamma}\log\left(1 + \gamma\frac{\partial C}{\partial s}\right), \tag{2.3}$$

where $\gamma = gk$ as before. Further, if the ds-terms in (2.2) cancel, then we are left with

$$d\Pi = \left(-k\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 C}{\partial s^2}\right)\right)dt$$

The no-arbitrage condition $d\Pi = r\Pi\,dt$ then implies that

$$r\left(-kC + \widetilde{k\Delta}s\right) = -k\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 C}{\partial s^2}\right)$$

or, after rearranging, using (2.3)

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 C}{\partial s^2} + r\left(s\frac{\partial C}{\partial s} - C\right) = 0. \tag{2.4}$$

In other words, $C(s,t)$ satisfies the usual Black-Scholes PDE. Let us summarize:

Proposition 2.3. *Under the assumptions above, the unit option price $C(s,t)$ of k European call options with strike price K and expiration date T is given by the solution to the Black-Scholes PDE* (2.4) *with the terminal condition*

$$C(s,T) = \left(s\frac{e^\gamma - 1}{\gamma} - K\right)^+,$$

where $\gamma = gk$ is the total large agent parameter. The hedging ratio is given by (2.3)*.*

Notice that s is not the market price of the stock, but the small agent price. As $\gamma \to 0$, i.e. $g \to 0$ or $k \to 0$, then $\tilde{\Delta} = \Delta = C_s$ and the situation is exactly standard Black-Scholes.

3. Option pricing II: properties of the solution

We now analyze the properties of the solution to the option pricing PDE derived above. By linearity of the Black-Scholes equation we may express the function $C(s, t; \gamma)$ in terms of the standard Black-Scholes solutions. If $C^*(s, t) = C^*(s, t; K)$ denotes the (small agent) price of a call option, then we have

$$C(s, T; \gamma) = \alpha C^*(s, T; \frac{K}{\alpha}), \tag{3.1}$$

where

$$\alpha = \frac{1}{k} \int_0^k e^{gy} \, dy = \frac{1}{\gamma} \left(e^{\gamma} - 1 \right).$$

and hence

$$C(s, t; \gamma) = \alpha C^*(s, t; \frac{K}{\alpha}),$$

for all $t \leq T$ by linearity of the Black-Scholes PDE. Also, from (2.3) we get

$$\Delta(s, t; \gamma) = \frac{1}{\gamma} \log \left(1 + \alpha \gamma \Delta^*(s, t; \frac{K}{\alpha}) \right),$$

where Δ^* denotes the Black-Scholes Delta.

3.1. **The option price as a function of the small agent stock price.** We now investigate how the unit option price C and hedging ratio Δ depend on the parameter $\gamma = gk$. We consider the small agent stock price s as fixed (which means that the market stock price S is *not* necessarily fixed).

Proposition 3.1. *The unit option price $C(s, t; \gamma)$ is an increasing function of the total large agent parameter $\gamma = gk$.*

Proof. This follows from the maximum principle [E, PW] and the fact that the terminal condition

$$\left(\frac{s}{\gamma} \left(e^{\gamma} - 1 \right) - K \right)^+$$

is an increasing function of γ. □

Remark 3.2. Proposition 3.1 deserves two comments. First, the higher the large agent effect in the stock market, the larger the (indirect, through delta hedging) impact on option prices. Second, an important case of a large agent made up by option hedgers that follow a delta-hedging strategy. Assuming that all, or at least most, option writers are doing this, we have that a large open interest (and hence a large γ) leads to high option prices.

In a similar vein we have the following result, which implies Proposition 3.1

Proposition 3.3. *The unit hedging ratio $\Delta(s, t) = \Delta(s, t; \gamma)$ is an increasing function of the total large agent parameter $\gamma = gk$.*

The proof is based on (3.1) and the following computational lemma.

Lemma 3.4. *For any $a \in (0, 1)$ the function $\gamma \mapsto \gamma^{-1} \log(1 + a(e^{\gamma} - 1))$ is strictly increasing in γ.*

Proof of Proposition 3.3. The case when $t = T$ is easy and left to the reader. Let us assume $t < T$. By (3.1) and the Black-Scholes formulas we have

$$\Delta = \frac{1}{\gamma} \log \left(1 + \alpha \gamma N(d_1) \right),$$

where

$$d_1 = d_1(\gamma) = \frac{1}{\sigma\sqrt{T-t}}\left(\log\frac{\alpha s}{K} + (r + \frac{1}{2}\sigma^2(T-t))\right).$$

If $0 < \gamma < \gamma'$ then $N(d_1(\gamma)) < N(d_1(\gamma'))$ and hence Lemma 3.4 with $a = N(d_1(\gamma'))$ gives

$$\Delta(s,t;\gamma) = \frac{1}{\gamma}\log\left(1 + \gamma N(d_1)\right)) < \frac{1}{\gamma}\log\left(1 + a\gamma\right)) = \Delta(s,t;\gamma').$$

This completes the proof. □

Proof of Lemma 3.4. The function $\phi(\xi) = (1+\xi)\log(1+\xi)$ is convex for $\xi > 0$ and satisfies $\phi(0) = 0$. Hence $\xi \mapsto \phi(\xi)/\xi$ is strictly decreasing for $\xi > 0$. Now write $v(\gamma) = \gamma^{-1}\log(1+a(e^\gamma - 1))$ and $\xi = e^\gamma - 1$. Then

$$\begin{aligned}\frac{\gamma}{a}v'(\gamma) &= \frac{e^\gamma}{1+a(e^\gamma-1)} - \frac{\log\left(1+a(e^\gamma-1)\right)}{a\gamma} \\ &= \frac{1+\xi}{1+a\xi} - \frac{\log\left(1+a\xi\right))}{a\log(1+\xi)} = \frac{\xi}{(1+a\xi)\log(1+\xi)}\left(\frac{\phi(\xi)}{\xi} - \frac{\phi(a\xi)}{a\xi}\right) > 0,\end{aligned}$$

which completes the proof. □

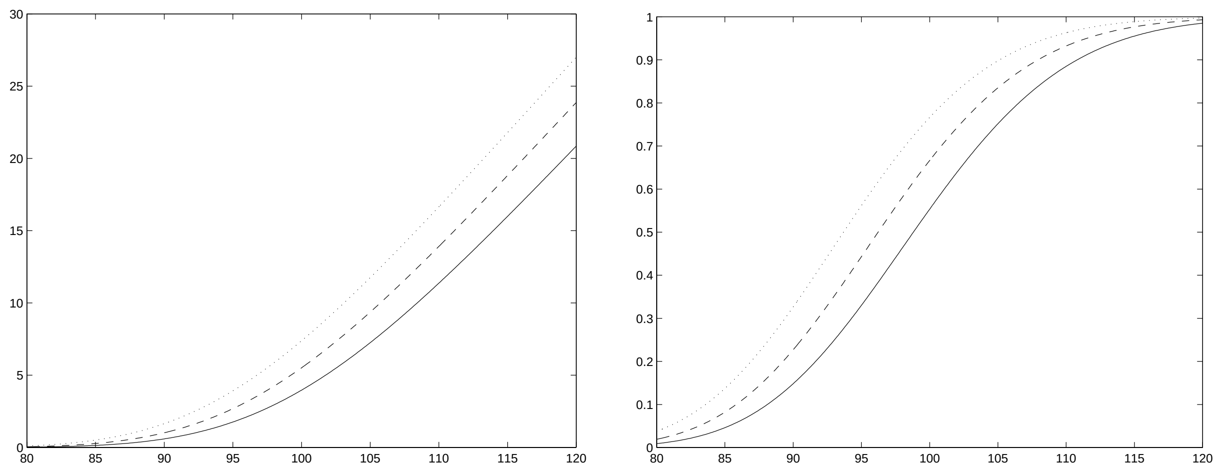

FIGURE 1. *Option pricing for a large agents.* The three different curves show the unit option price $C(s,t;\gamma)$ (left) and unit hedging ratio $\Delta(s,t;\gamma)$ (right), where $\gamma = gk$, as a function of the small agent price s for three different values of γ: $\gamma = 0$ (solid), $\gamma = 0.05$ (dashed) and $\gamma = 0.1$ (dotted). The other parameters were $T-t = 0.2$, $r = 0.04$, $\sigma = 0.2$, $K = 100$. Notice that both C and Δ are increasing functions of γ. See Propositions 3.1 and 3.3.

3.2. **The option price as a function of the market stock price.** As we have seen in Proposition 3.1, the option price $C(s,t)$ is an increasing function of $\gamma = gk$ when the small agent stock price s is held fixed. We emphasize again that s is not the observed stock price. Rather, this price is given by $S = se^{\gamma\Delta(s,t)}$, where $\Delta(s,t)$ is gotten from (2.3). We obtain

$$S = s(1+\gamma C_s). \tag{3.2}$$

Now, for fixed t,γ we have the following properties of $C(s,t)$: $0 < C_s < \alpha$, $C_s \to 0$ as $s \to 0$, and $C_s \to \infty$ as $s \to \infty$. This implies that the function $s \mapsto s(1+\gamma C_s)$ maps $[0,\infty)$ bijectively onto $[0,\infty)$. Hence, given an S we may find the corresponding s and compute the unit option price $C(S,t)$.

Observe that since for fixed s, both S and C increase with γ it is not obvious that $C(S,t)$ will be increasing in γ *for fixed* S. Indeed, the numerical computation illustrated in Figure 2 shows that in fact $C(S,t)$ has a complicated dependence on γ. The same is true for the unit hedging ratio $\Delta(S,t;\gamma)$.

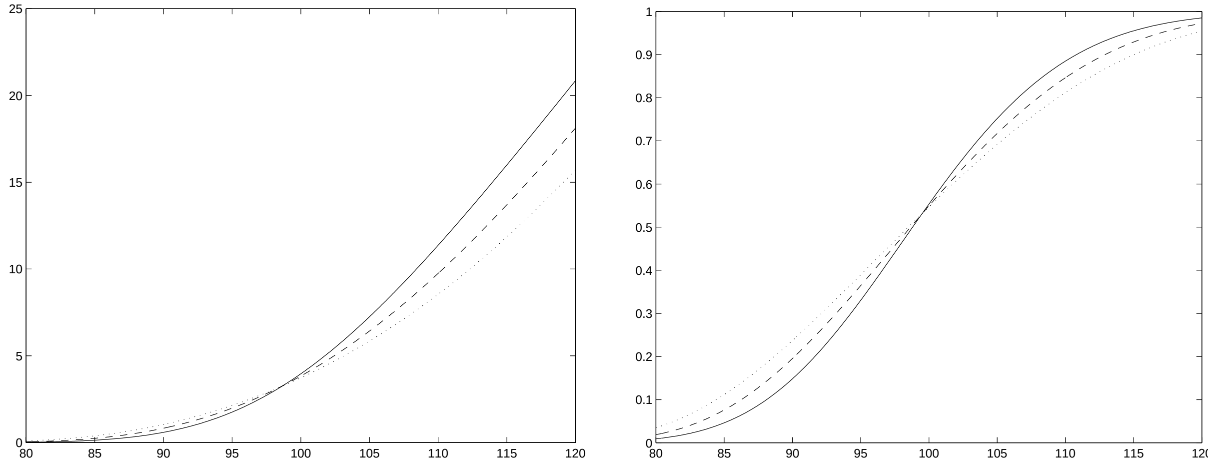

FIGURE 2. *Option pricing for a large agents.* The three different curves show the unit option price $C(S, t; \gamma)$ (left) and unit hedging ratio $\Delta(S, t; \gamma)$ (right), where $\gamma = gk$, as a function of the observed market price S for three different values of γ: $\gamma = 0$ (solid), $\gamma = 0.05$ (dashed) and $\gamma = 0.1$ (dotted). The other parameters were $T - t = 0.2$, $r = 0.04$, $\sigma = 0.2$, $K = 100$.

4. Empirical Analysis

The existence of the large agent price effect has already been empirically proved in several papers, mostly using stock price data (see e.g. [Lo]). Here we estimate the large agent effect from option market data by using our pricing model and we compare it with the models in [JK] and [LY]. We use these two models as they have explicit pricing equations. They differ from the present paper because they model the large agent effect through the parameters in the SDE for the portfolio value.

In our analysis, the large agent consists of a group of delta-hedging option writers. More precisely, we assume that all option writers (for all put and call options) use delta hedging and consequently can be modeled as one large aggregate agent. The stock holding of this agent is the aggregate hedging position of all option writers. For each option series, the number of written options equals the open interest on that series.

The parameter estimation is done in two parts. In the first part, we ignore the open interest, and instead calculate the implied total large agent parameters (see Definition 2.1) from option prices. In this analysis we use IBM and GM option prices from 1995 and compare the current model to both [JK] and [LY]. In the second part, we do use the open interest data from IBM options and compare our model with [JK]. Here we cannot make the comparison to [LY], as that model does not use the open interest as an input variable.

We use least squares for calculating the model's implied parameters. More precisely, suppose we are given the prices at times $t_1, \ldots, t_n$ of M options. Let θ be the vector of parameters that define our model. We divide the n times into $m \leq n$ consecutive periods, each of which contains $n_l > 0$ $(l = 1, 2, \ldots, m)$ distinct times. Our goal is to come up with an estimate for θ for each group l.

Let $\hat{C}_j(t_i)$ denote the observed price of the j-th option at time t_i. Let $C_j(t; \theta)$ denote the model price of the j-th option with parameter value θ. For each period $l \in \{1, 2, \ldots, m\}$, we define the best fit parameter set, $\hat{\theta}_l$, by

$$\sum_{i=n_{l-1}+1}^{n_l} \left(\hat{C}_j(t_i) - C(t_i; \hat{\theta}_l)\right)^2 = \min_{\theta_l} \left\{ \sum_{i=n_{l-1}+1}^{n_l} \left(\hat{C}_j(t_i) - C(t_i; \theta_l)\right)^2 \right\}. \tag{4.1}$$

Repeating this m times gives us a sequence of parameter sets $\hat{\theta}_1, \ldots, \hat{\theta}_m$. Note that we use both call and put options. Call prices are solved from Proposition 2.3 and put prices from put-call-parity.

Model	Our Model		[JK]		[LY]	
Parameter	γ	σ	γ_{JK}	σ_{JK}	λ	σ_{LY}
Mean	0.2465	0.1963	0.1613	0.1503	0.3086	0.1366
Median	0.1526	0.1842	0.2668	0.1466	0.3568	0.1402
Variance	0.0546	0.0071	0.1069	0.0061	0.1352	0.0126
Min	4.24e-4	0.1275e-4	-0.5434	0.0103	-0.6175	0.0101
Max	0.8906	0.4800	0.5602	0.2874	0.9614	0.2907

Table 1. Estimated implied parameters for IBM options.

We use IBM and GM call and put options traded on the Chicago Board Options Exchange during 1995 to estimate the implied model parameters. However, we do not consider options that were out-of-money during the month of maturity. We use a one day time interval and divide each year's data into 30 periods, each containing about 8 consecutive trading days. Thus, for both IBM and GM we obtain a total of 30 sets of estimated parameters. The risk free rate is assumed to be 3%, approximating the rate on the 4-week treasury bill during 1995.

The option price in our model depends on the risk factor dynamics and the parameter γ. Now γ in turn depends on the large agent parameter g and the number of options hedged. The parameter g is the same for all the options during the same period $l \in \{1, 2, \ldots, m\}$ but the number of options is usually different for different options. Thus, we set

$$\theta = \{\sigma, \gamma_1, \ldots, \gamma_M, \},$$

where σ is the constant volatility of s and for $1 \leq i \leq M$ the variable $\gamma_i = gk_i$ is the total large agent effect parameter (see Definition 2.1) for option i. That is, in each period we calculate the implied volatility and the total large agent effect of each option in the market. Using least squares estimation 30 times (m=30) we end up with 30 implied volatilities and γ-vectors. We analyze the elements of all the γ-vectors. These elements represent the option interests multiplied by the large agent parameter, g.

In order to compare our model with [JK] and [LY], we estimate their implied parameters by using the same IBM and GM data and the same least square method. As in the present paper, these papers use only one parameter to model the large agent effect: in [JK] the large agent parameter is denoted by γ (we use γ_{JK}) and in [LY] by λ. We apply a finite difference method to solve their pricing PDEs.

We first analyze the IBM case. Table 1 presents descriptive statistics for the implied large agent parameters and volatilities of the present paper, [JK], and [LY]. This table shows that all the γ-values in our model are positive and that the variances of our parameter estimates are quite low. The average square errors from the IBM data and equation (4.1) are reported in Table 4. According to this table all the models fit the data significantly better than the Black-Scholes model. Further, according to Table 4 the present paper fits the data better than the other models and [JK] better than [LY]. With both [JK] and [LY] the large agent effect estimates have some negative values. Note that our model gives a higher volatility estimate than the other models. Tables 1 and 4 imply that the large agent effect is significant in the IBM option market. As we will see later, the existence of this large agent effect is due to the fact that the size of IBM option market is significant compared with IBM stock market.

Figure 3 contains the histogram of γ and it shows that all the estimated γ-parameters are greater than 0. Figure 3 also shows that less than 90% of the estimated γ_{JK} and λ values are greater than 0.

We do the same empirical analysis with European call and put options on GM stock prices in 1995. Table 2 presents the summary of the parameters. As with with the IBM data, all γ-values are positive in Table 2. Note again that the variances of our model parameters are quite low. In contrast to the IBM case now all the λ-values are positive but some γ_{JK}-values are still negative. These results are also illustrated in Figure 4.

Finally, we proceed with the empirical analysis taking into account the open interest data of IBM options (source: Reuters DDE database; we do not have open interest data for GM). Since we have data for all IBM options, we can estimate directly the large agent effect parameter g, instead of γ. We use IBM call and put options traded on the Chicago Board Options Exchange between December 31, 2003

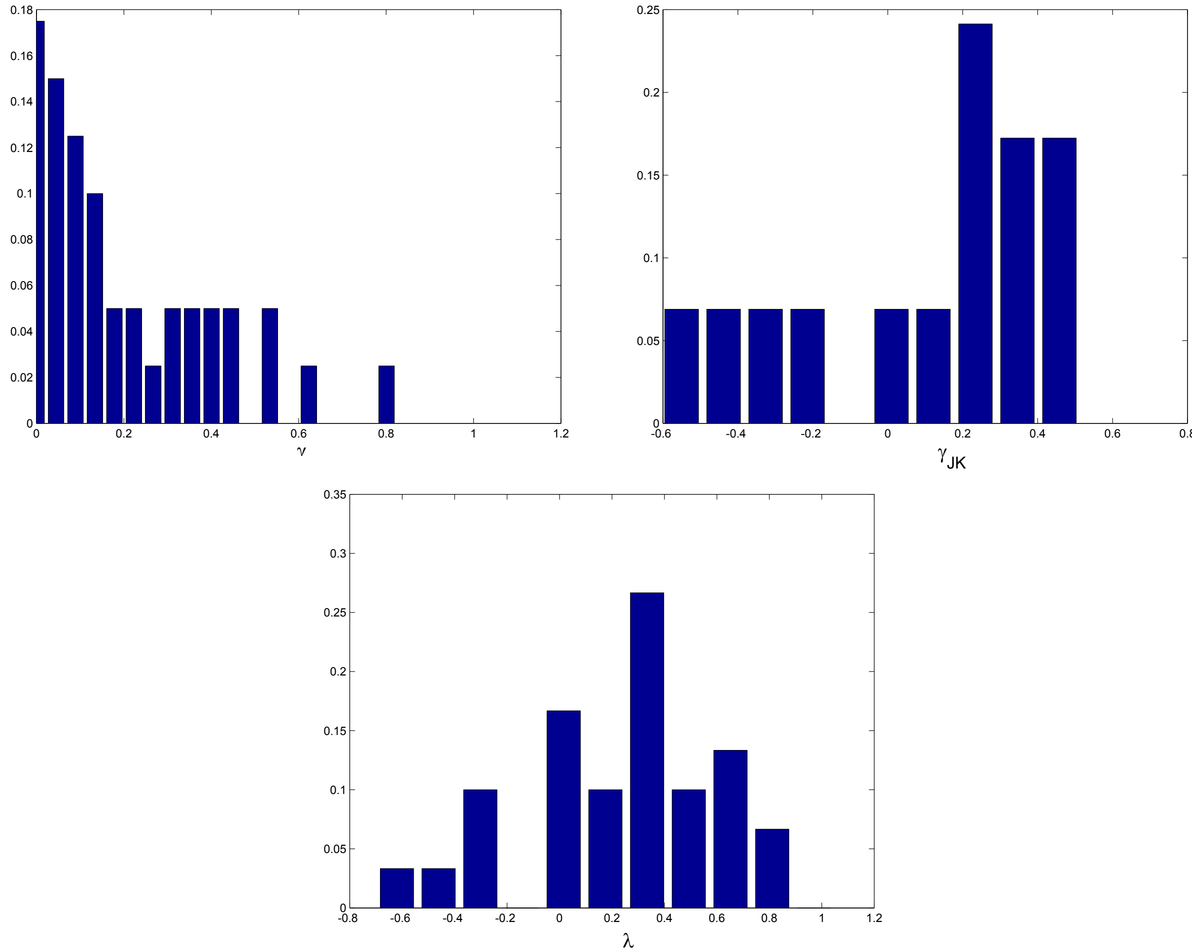

Figure 3. *Histograms of γ (left), γ_{JK} (right) and λ (middle) for IBM options.*

Model	Our Model		[JK]		[LY]	
Parameter	γ	σ	γ_{JK}	σ_{JK}	λ	σ_{LY}
Mean	0.1146	0.2507	0.0968	0.1796	0.2536	0.1470
Median	0.1107	0.2528	0.1601	0.2048	0.1043	0.0892
Variance	0.0031	0.0068	0.0385	0.0089	0.8177	0.0166
Min	0.0213	0.0288	-0.3261	0.0114	0.0013	0.0104
Max	0.2582	0.4636	0.3361	0.2926	0.9183	0.3770

Table 2. Estimated implied parameters for GM options.

and February 10, 2004. Since [LY] does not include open interest in their model, we only compare our model with [JK] here. Table 3 presents the summary of the estimation. Almost all g-parameter values are positive and the variance is low. This is also illustrated in Figure 5 which contains the histogram of g. In our model, more than 90% of the estimated g-values are positive while [JK] has only about 65% positive values.

Table 4 presents the average square errors corresponding to the IBM data and (4.1). According to this table, our model has the smallest error term. This and the results in Table 3 imply that our model fits the data better than [JK] and [LY]. Note that the average square errors corresponding to the data without the open interest are smaller than those with the open interest data. This is because, without the open interest data, the estimation have $2+M$ parameters (g, σ and open interests of each option, where M is the number of total options), while for the case with the open interests data, there are only

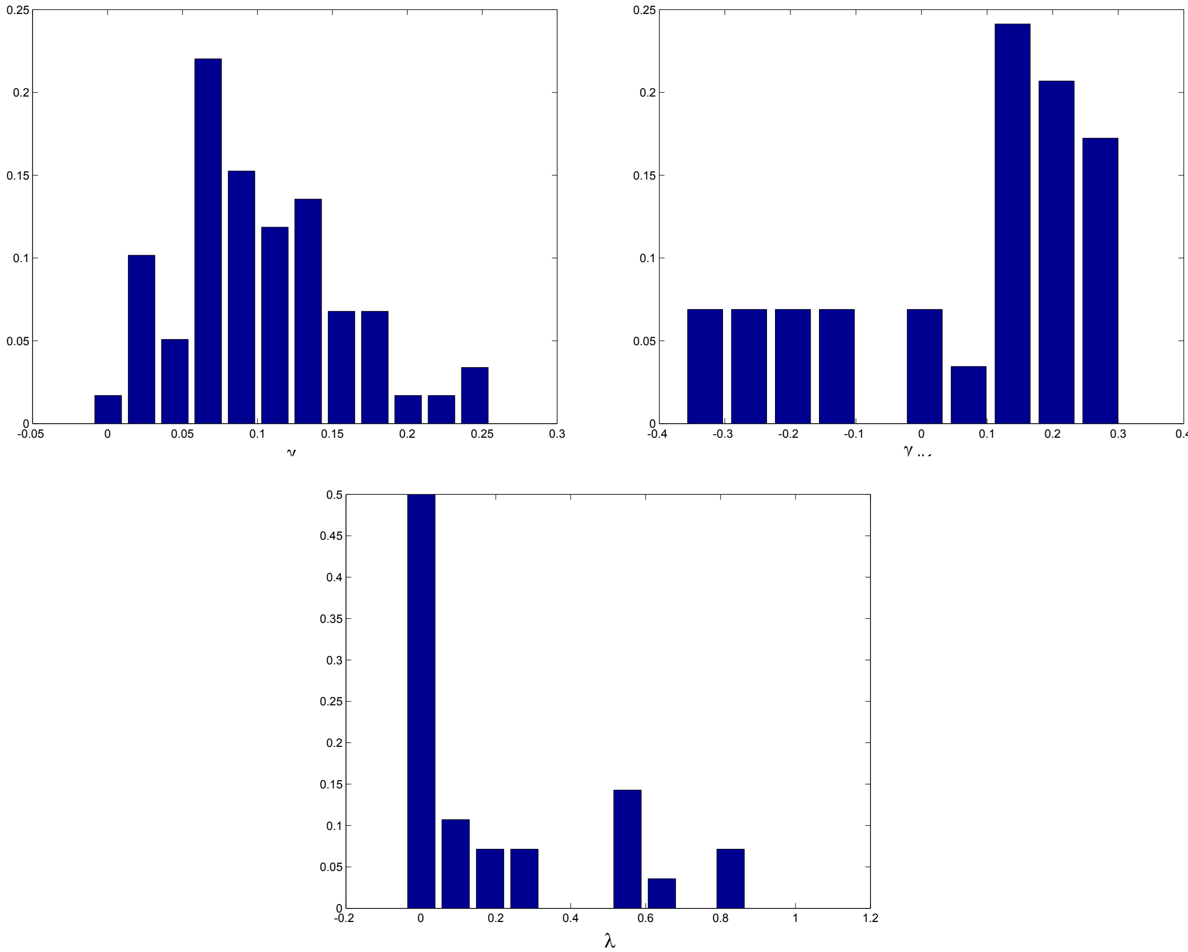

FIGURE 4. *A histogram of γ, γ_{JK} and λ for GM options.*

Model	Our Model		[JK]	
Parameter	g	σ	g_{JK}	σ_{JK}
Mean	3.87E-09	0.201	1.07E-09	0.167
Median	2.96E-09	0.201	1.36E-09	0.161
Variance	1.34E-17	4.24E-05	6.37E-18	4.94E-05
Min	-3.87E-09	0.1886	-6.41E-09	0.115
Max	1.39E-08	0.2125	6.78E-09	0.179

TABLE 3. Estimated parameters for IBM options with open interests data.

Data	With Open Interests		Without Open Interests			
Model	Our Model	[JK]	Our Model	[JK]	[Liu]	Black-Scholes
Average Square Error	0.4667	0.5287	0.2381	0.3792	0.5892	0.8473

TABLE 4. Average square errors from IBM data.

2 parameters (g and σ). Table 4 also shows that the Black-Scholes model has the highest average square error and this is due to the fact that it has only one parameter (σ). From Tables 3 and 4 we see that, as our model predicts, the open interest does influence the unit option price; our model explains this relationship quite well.

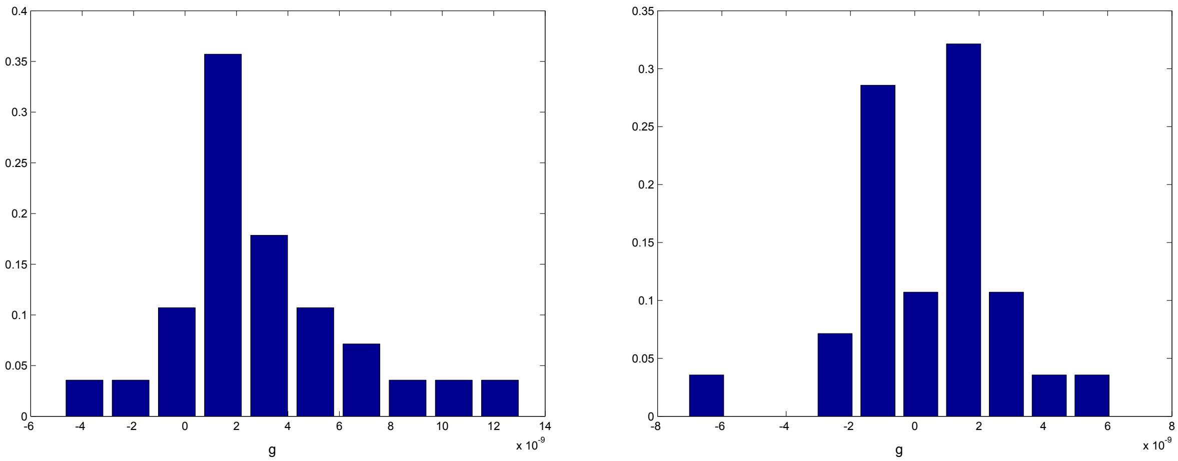

FIGURE 5. *A histogram of g and g_{JK} for IBM options with open interest data.*

Let us illustrate the large agent effect in our model. The open interest of all options on IBM (all IBM put and call options together) on 2/2/2004 was 327,263 (the underlying asset of each option is 100 IBM stocks). The underlying stock quantity of these options corresponds to about 20% of the whole IBM stock market. Table 3 shows that the average large agent effect g is about $3.87 \cdot 10^{-9}$. Thus, the large agent effect is low but, on the other hand, the open interest is high. If we assume that all the IBM options are delta hedged, then a total of 175,600 stocks are bought on 2/2/2004, corresponding to about 5% of the IBM stock volume on that day. The delta hedging accordingly changes the underlying asset price by about 0.1% on 2/2/2004: this is the total large agent effect on IBM stock price from the option market.

Combining our empirical results we see that our present model fits the data better and yields a more significant large agent parameter than those of [LY] and [JK]. Since [LY] and [JK] model the large agent effect through the stochastic process parameters, our empirical analysis indicates that the large agent effect mainly influences the current asset prices and not their stochastic process parameters.

5. Conclusions

We have analyzed pricing and hedging of call options in a model where a large agent influences market prices through an exponential price effect function. In this model, the option price is a function of not only the underlying asset price but also of the number of options sold (and hedged).

Using a hedging argument we showed that the price of the call option in our model is a solution to the usual Black-Scholes partial differential equation, but with a modified terminal condition, which takes into effect the option amount and the parameter measuring the "size" of the large agent. Put option prices can then be calculated using put-call-parity.

We analyzed the solution to the partial differential equation. In particular we showed that if we fix the "small agent" price of the asset, then the unit option price and the unit hedging ratio are increasing functions of the number of options hedged, as well as of the large agent effect parameter. On the other hand, if we freeze the actual (large agent) market price, then the dependence is more complicated.

Finally we estimated the model parameters from IBM and GM option market data and showed that the market implied parameters are significant. This is consistent with the earlier empirical papers that have analyzed the large agent effect using stock prices. We also compared our model with two other large agent models that model large agent effect through the parameters in the SDE for the portfolio value. According to this analysis our model fits option market data better.

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Appendix A. Derivation of the wealth dynamics

Let us indicate how to derive the wealth dynamics (1.3) under our assumptions. We do this first in a discrete time setting. Assume that the large agent holds $a_{t-\Delta t}$ stocks and $b_{t-\Delta t}$ bonds during the time interval $[t-\Delta t, t)$. At time t she rebalances the portfolio, after which she holds a_t stocks and b_t bonds. These are then held during the time interval $[t, t+\Delta t)$, etc.

The rebalancing consists of buying $a_t - a_{t-\Delta t}$ shares. For this, the large agent has to pay $s_t \int_{a_{t-\Delta t}}^{a_t} e^{gy}\,dy$, so if the strategy is self-financing, then this quantity must equal $(b_{t-\Delta t} - b_t)B_t$. Using the notation (1.2), we obtain

$$V_t = \widetilde{a}_t s_t + b_t B_t = \widetilde{a_{t-\Delta t}} s_t + b_{t-\Delta t} B_t,$$

which leads to

$$\begin{aligned}
\Delta V_t &= V_t - V_{t-\Delta t} \\
&= (\widetilde{a_{t-\Delta t}} s_t + b_{t-\Delta t} B_t) - (\widetilde{a_{t-\Delta t}} s_{t-\Delta t} + b_{t-\Delta t} B_{t-\Delta t}) \\
&= \widetilde{a_{t-\Delta t}} \Delta s_t + b_{t-\Delta t} \Delta B_t. \qquad \text{(A.1)}
\end{aligned}$$

As in [B], the continuous-time limit of (A.1) is (1.3).

In fact, the passing to the limit in this context deserves some comments. In a real market, stocks are not infinitely divisible and can only be bought and sold in packages of a definite size. In particular, splitting a trade into infinitesimal packages is not possible.

However, in many cases it is feasible to split a large order into many smaller pieces. Indeed, in practice there is a market order book (see e.g. [Ro]) and large trades are typically split into smaller trades (often with different counterparties). Usually these smaller trades have different prices and are ordered by price. In this situation, (A.1) is a reasonable approximation.

Similarly, trading strictly speaking takes place in discrete rather than continuous time. Of course, for many markets, the continuous-time approximation is fairly accurate and certainly convenient.

Our assumptions leading to the wealth dynamics (1.3) is then, roughly speaking, that the order-splitting approximation "survives" the passage to the continuous time limit. We do not wish to state a precise result characterizing when this is mathematically justified. Suffice it to say that (1.3) is an approximation which leads to a tractable model amenable to calibration.

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