Optimal Consumption and Portfolio Decisions with Partially Observed Real Prices [*]

Alain Bensoussan
University of Texas at Dallas
School of Management
International Center for
Decision and Risk Analysis
alain.bensoussan@utdallas.edu

Jussi Keppo
University of Michigan
IOE Department
keppo@umich.edu

Suresh P. Sethi
University of Texas at Dallas
School of Management
Center for Intelligent
Supply Networks
sethi@utdallas.edu

Abstract

We consider optimal consumption and portfolio investment problems of an investor who is interested in maximizing his utilities from consumption and terminal wealth subject to a random inflation in the consumption basket price over time. We consider two cases: (i) when the investor observes the basket price and (ii) when he receives only noisy observations on the basket price. We derive the optimal policies and show that a modified Mutual Fund Theorem consisting of three funds holds in both cases. The compositions of the funds in the two cases are the same, but in general the investor's allocations of his wealth into these funds will differ. However, in the particular case when the investor has CRRA utility, his optimal investment allocations into these funds are also the same in both cases.

Key Words. optimal consumption and investment, inflation, stochastic control with partial observations, separation principle, Zakai equation

---

[*] We thank Brian Kim, Tim Maull, Sophie Shive, and Tuomo Vuolteenaho for useful feedback, and Minwei Lian for research assistance. We benefited from seminar feedback at the University of Michigan and Goethe Universitaet. The authors acknowledge financial support for this research from the National Science Foundation.

1 Introduction

We study the optimal portfolio and consumption decision problem of an investor when the consumption basket and real (inflation adjusted) asset prices are partially observed. Traditionally, the investment literature has assumed that the basket prices are fully observed. In reality, the basket price is difficult to measure, as it requires collecting the prices of all the consumption goods in the basket and their weights. Moreover, these prices may not be unique.[1] In other words, inflation in the consumption basket price is not fully observed and, as a consequence, the real asset prices are also incompletely observed.

As a benchmark case, we first consider fully observed basket price. In this case, the real asset market is complete, and the optimal policy can be obtained by solving the Hamilton-Jacobi-Bellman equation for the problem. The real optimal consumption process equals the optimal policy studied in the classical case in Merton (1971), Karatzas et al. (1986), and Sethi (1997). However, since the consumption basket price is also stochastic, its presence affects the optimal portfolio selection. We show that the optimal portfolio can be characterized as a combination of three funds: a risk-free fund and a growth optimal fund as in the classical case, and a fund that arises from the correlation between the inflation uncertainty and the market risk. Every investor uses the first two funds, whereas the composition of the last fund depends on his own consumption basket price dynamics. Thus, if two investors have the same consumption habit, then they will also have the same last fund. Furthermore, in general, the amount invested in each of the three funds depend on their respective consumption basket price and utility functions.

Henceforth, we will use the terms nominal consumption and consumption interchangeably. When we mean real consumption, it will be so specified. The same convention will apply to the terms asset prices, wealth, savings, etc.

---

[1] In contrast to financial asset prices, consumption good prices are not necessarily uniquely defined. For instance, according to Borenstein and Rose (1994), the expected absolute difference in airline fares between two passengers on the same route is 36% of the airline's average ticket price. Because there may be many prices for the same good, it is difficult for the consumer to determine inflation from the price observations. Further, the goods and/or their weights in the basket change over time, and not all the goods are consumed continuously (durable/seasonal goods).

Following the analysis of the benchmark case, we study the situation when the investor receives noisy *signals* on the consumption basket price. Given the signal observations, the investor obtains the conditional probability distribution of the current basket price and, in turn, the conditional distribution of the current real asset prices. In general, the new risk due to the partial observability of the basket price affects the optimal policy.

Interestingly enough, the characterization of the optimal portfolio in the partially observed case is the same as in the fully observed case. Thus, in both cases, the optimal portfolio is a linear combination of the risk-free fund, growth optimal fund, and the fund that arises from the correlation between the inflation uncertainty and the market risk. As before, the composition of the last fund for an investor depends on the nature of his consumption basket, and the relative wealth allocation in the three funds depends on his utility function and the dynamics of the consumption basket price filter, which represents his best estimate given the observations.

In the particular case of the constant relative risk aversion (CRRA) utilities, however, we show that the allocation in the three funds remain the same in both cases. Moreover, in this case the consumption process is not affected by the new risk due to the partial observability. Thus, our model in the CRRA case predicts that the additional measurement uncertainty in the current consumption basket price and, consequently, in the current real asset prices, does not change the optimal behavior of the investor. For asset managers this is a good news, because if they can come up with the consumption basket process parameters for a specified segment of the population of investors, then they can ignore the inflation measurement uncertainty and create the third fund for that segment as discussed earlier by using the correlation between the inflation uncertainty and the market risk.

There have been several studies on consumption measurement. Klenow (2003) discusses how the U.S. government measures consumption growth and how it considers the fact that the consumption basket changes over time. Inflation measurement and the problems with that are considered in Alchian and Klein (1973), Bradley (2001), and Shapiro and Wilcox (1997). Many of the social costs of inflation are caused by its unpredictability. The unpredictability is studied, e.g., by Ungar and

Zilberhard (1993). The results of these studies are consistent with the present paper in the sense that our investor, due to noisy signals/measurements, does not completely observe the consumption goods prices and, therefore, updates his belief from different consumption basket price signals.

The connection between inflation and asset prices are studied by Basak and Yan (2007), Campbell and Vuolteenaho (2004), and Cohen et al. (2005). According to them, the stock market suffers from money illusion, i.e., it incorrectly discounts real cash flows with nominal discount rates. Thus, when the inflation is high, the equity premium is also high and vice versa. In this paper we do not consider money illusion. Optimal portfolio selection under inflation is studied, e.g., by Chen and Moore (1985), Manaster (1979), Brennan and Xia (2002), Solnik (1978), and Munk et al. (2004). Our paper is closest to Brennan and Xia (2002), who consider a more complicated inflation process but assume perfectly observed inflation. In our paper, we emphasize the fact that inflation signals are noisy and, therefore, the current consumption basket price is not completely observed. Portfolio selection with learning is also considered in Xia (2001) and Brennan (1998). In these papers, the investor learns about the stock returns, i.e., about the parameters of the price processes. As explained earlier, in the present paper the investor does not observe the consumption basket price directly, but infers it from the observed inflation signal. Thus, without the perfect information, the current real asset prices are also incompletely observed. In this way, our model differs from the above papers, and it also answers a different economic question: What is the effect of the noisy observations of inflation on the optimal portfolio selection?

The rest of the paper is organized as follows. Section 2 presents the underlying information sets and stochastic processes. The optimal policy under the fully observed consumption basket price is derived in Section 3. Section 4 formulates the model in the partially observed case. Section 5 provides the optimal policy. Section 6 concludes the paper.

2 Model

In this section we provide the investor's information sets and introduce the underlying stochastic processes. The signal process about the consumption basket price is introduced in Section 4.

2.1 Information Sets

Let $(\Omega, \mathcal{F}, P)$ denote a probability space hosting Wiener processes $w_I(t), w_Z(t)$, and $w(t)$. The first two processes are scalar valued, whereas the last one is n dimensional. The process $w_I(t)$ models the random nature of inflation, $w_Z(t)$ models the noise in the signal observed when the consumption basket is not directly observable, and $w(t)$ models the uncertainties in the risky financial assets. There is a correlation between the inflation and the market, i.e., the process $w_I(t)$ and $w(t)$ are correlated. The process $w_Z(t)$ is independent of the others. The sigma algebra

$$\mathcal{F}_t = \sigma\{w_I(s), w_Z(s), w(s), s \leq t\}$$

denotes the basic filtration on which all processes are adapted, except possibly the initial conditions. We shall introduce several processes: $Y_0(t)$ represents the riskless asset, $Y(t)$ is the n dimensional stochastic process representing the prices of the risky assets, $X(t)$ denotes the investor's wealth, $B(t)$ is the process denoting the consumption basket price, $L(t) = \ln B(t)$, and $Z(t)$ is the signal when the process $L(t)$ is not directly observable. In this case, the pair $(w(t), Z(t))$ is observable and we let

$$\mathcal{G}_t = \sigma\{w(s), Z(s), s \leq t\},$$

which denotes the filtration of the observations. This case of partial observation will be treated in Section 4, and it is there that we will define the signal process $Z(t)$.

2.2 Price and Inflation Processes

The evolution of the nominal value of the riskless asset is described by

$$dY_0(t) = r(t)Y_0(t)dt, \qquad y_0(0) = y_0,$$

where $y_0 > 0$ is a given constant and $r(t)$ is the deterministic nominal risk-free interest rate. Since we will assume the instantaneous expected inflation in our model to be constant, we could also assume that the risk-free rate is constant (if the real rate is also constant). The evolution of the nominal risky asset prices is given by

$$dY_i(t) = Y_i(t)\left(\alpha_i(t)dt + \sum_{j=1}^{n} \sigma_{ij}(t)dw_j(t)\right),\ Y_i(0) = y_i,\ i = 1, ...n,$$

where $y_i > 0$ are given constants, and $\alpha_i(t)$ and $\sigma_{ij}(t)$ are deterministic and bounded expected returns and volatility functions, respectively. The volatility matrix $\sigma(t) = (\sigma_{ij}(t))_{1\leq i\leq n, 1\leq j\leq n}$ is invertible for each t, i.e., our market is complete.

The consumption basket of the investor includes several consumption goods along with their weights. For simplicity, we ignore its multidimensional aspect and, thus, consider directly the weighted sum of the prices of the goods in the consumption basket. This sum referred to as the consumption basket price, or simply the basket price, $B(t)$, acts as a numeraire in order to get the real consumption process and the real terminal wealth from the nominal ones. The dynamics of the basket price is given by

$$dB(t) = B(t)\left(Idt + \zeta dw_I(t)\right), B(0) = B_0,$$

in which B_0 represents the initial condition, I and ζ are constants, and the correlation between $w(t)$ and $w_I(t)$ is defined by a vector $\rho = (\rho_1, ..., \rho_n)^\mathsf{T}$, where

$$E\left[dw_i(t)dw_I(t)\right] = \rho_i dt,\ i = 1, ...n,$$

and y^T denotes the transpose of y. The real number I denotes the expected instantaneous inflation and $\zeta > 0$ is the inflation volatility. Since the basket price usually rises, it is natural to assume that $I > 0$. The initial basket price B_0 is known when there is full observation. In the case of partial information for the process $B(t)$, the initial condition can be a random variable independent of $\mathcal{F}_t$.

For convenience, we will work with the ln-basket price $L(t)$ directly, instead of $B(t)$. Since ln is an increasing function, the higher the basket price $B(t)$, the higher is $L(t)$. From the B-dynamics, we get

$$dL(t) = \left(I - \tfrac{1}{2}\zeta^2\right) dt + \zeta dw_I(t), \quad L(0) = \ln B_0. \tag{2.1}$$

The nominal wealth at time t is defined by

$$X(t) = \varpi_f(t)Y_0(t) + \varpi(t)Y(t),$$

where $\varpi_f(t)$ and $\varpi(t)$ denote the amount of riskless and risky assets owned by the investor. Note here that $\varpi = (\varpi_1, ..., \varpi_n)$ and $Y = (Y_1, ..., Y_n)^\mathsf{T}$ are n-dimensional vectors. We assume that the self-financing condition holds so that

$$dX(t) = \varpi_f(t)dY_0(t) + \varpi(t)dY(t) - C(t)dt,$$

where $C(t)$ is the instantaneous nominal consumption rate.

We obtain the *market price of risk* $\theta = (\theta_1, ..., \theta_n)^\mathsf{T}$ by solving the linear system:

$$\alpha_i(t) - r(t) = \sum_{j=1}^{n} \sigma_{ij}(t)\theta_j(t), \ i = 1, ...n. \tag{2.2}$$

Note that a unique θ exists because of the completeness of the market. Now the wealth evolution can be written as

$$dX(t) = r(t)X(t)dt + X(t)\pi(t)\sigma(t)\left[dw(t) + \theta(t)dt\right] - C(t)dt, \tag{2.3}$$

where $\pi = (\pi_1, ..., \pi_n)$ is the proportional wealth in the risky assets, i.e.,

$$\varpi_i(t)Y_i(t) = \pi_i(t)X(t), \ i = 1, ...n.$$

In the next section, we consider the full information case.

3 Fully Observed Consumption Basket Prices

3.1 Objective

In the full information case, the investor observes the process $L(t)$ and his information set is $\mathcal{F}_t$. Thus, the investor continuously follows all the prices in his consumption basket without any friction. We shall consider a problem starting at time t with known initial conditions $X(t) = x$ and $L(t) = L$.

The role of the consumption basket is to discount consumption and wealth. Thus, the real consumption and the real wealth are Ce^{-L} and Xe^{-L}, respectively, where C and X are, respectively, the amount of money spent on consumption and the wealth. The agent gets utility from the real consumption and the real wealth. Therefore, we introduce the respective utility functions $U_1(\cdot)$ and $U_2(\cdot)$ of real consumption and real wealth, respectively. We assume that these are twice differentiable, strictly increasing, and concave. Moreover,

$$U_i'(0) = \infty, \quad U_i'(\infty) = 0, \ i = 1, 2. \tag{3.1}$$

By the utility from real consumption, we have the following derivatives with respect to the nominal consumption C and the ln-basket price L:

$$\begin{aligned} \tfrac{\partial U_1(Ce^{-L})}{\partial C} &= U_1'(Ce^{-L})e^{-L} > 0, \quad \tfrac{\partial U_1(Ce^{-L})}{\partial L} = -U_1'(Ce^{-L})Ce^{-L} < 0 \\ \tfrac{\partial^2 U_1(Ce^{-L})}{\partial C^2} &= U_1''(Ce^{-L})e^{-2L} < 0, \quad \tfrac{\partial^2 U_1(Ce^{-L})}{\partial L^2} = Ce^{-L}\left[U_1'(Ce^{-L}) + U_1''(Ce^{-L})Ce^{-L}\right]. \end{aligned}$$

Note that the second derivative with respect to L is nonnegative if, and only if, $-\frac{U_1''\left(Ce^{-L}\right)Ce^{-L}}{U_1'(Ce^{-L})} \leq 1$, i.e., iff the relative risk aversion is less than one. Differentiating the marginal utility from consumption with respect to L gives

$$\frac{\partial^2 U_1\left(Ce^{-L}\right)}{\partial L \partial C} = -e^{-L}\left[U_1'\left(Ce^{-L}\right) + U_1''\left(Ce^{-L}\right)Ce^{-L}\right] = -\frac{1}{C}\frac{\partial^2 U_1(Ce^{-L})}{\partial L^2}$$

and, by the second derivative above, this is negative if the relative risk aversion is less than one. Thus, in this case, the marginal utility from consumption falls in the consumption basket price. Because of this, we shall see in the examples (Sections 3.3

and 5.3) that, as in Phelps (1962), Stiglitz (1970), Rothschild and Stiglitz (1971), and Mirman (1971), our comparative statics crucially depends on whether the value of the relative risk aversion is greater or less than unity.

The empirical evidence on the value of the relative risk aversion is not conclusive. For instance, Friend and Blume (1974) and Farber (1978) show that on average the relative risk aversion is greater than one. On the other hand, e.g., Schluter and Mount (1976) and Hansen and Singleton (1983) find that the average risk aversion is less than unity. Therefore, we do not make any additional assumption on the value of the relative risk aversion.

Let us now define the objective function as follows

$$\begin{aligned} J\left(C(\cdot),\pi(\cdot);x,L,t\right) &= E\left[\int_t^T e^{-\beta(s-t)}U_1\left(C(s)e^{-L(s)}\right)ds\right. \\ &+ \left. e^{-\beta(T-t)}U_2\left(X(T)e^{-L(T)}\right)|L(t)=L,X(t)=x\right], \end{aligned}$$

where wealth $X(t)$ follows (2.3), terminal time $T > 0$, β is the utility discount rate that may be different from the risk-free rate. The *value function* is defined by

$$V(x,L,t) = \sup_{C(\cdot),\pi(\cdot)} J\left(C(\cdot),\pi(\cdot);x,L,t\right).$$

Hence, the agent selects consumption and investment processes in order to maximize the sum of his expected discounted utilities from real consumption and real terminal wealth. We use the real processes as arguments of the utility functions, since it is from these that the investor derives his enjoyment over time.

3.2 Optimal Policy

As in the classical case treated in Merton (1971) and Sethi (1997), we solve for the value function and the optimal policy by use of the *Hamilton-Jacobi-Bellman* (HJB) equation. In the full observation case, the HJB equation is

$$\begin{aligned}\frac{\partial V}{\partial t} - \beta V + r(t)x\frac{\partial V}{\partial x} + \left(I - \tfrac{1}{2}\zeta^2\right)\frac{\partial V}{\partial L} + \frac{1}{2}\zeta^2\frac{\partial^2 V}{\partial L^2} + \max_C\left\{U_1\left(Ce^{-L}\right) - C\frac{\partial V}{\partial x}\right\}\\ + \max_\pi\left\{x\pi\sigma(t)\left(\theta(t)\frac{\partial V}{\partial x} + \rho\zeta\frac{\partial^2 V}{\partial L\partial x}\right) + \frac{1}{2}x^2\pi a(t)\pi^\intercal\frac{\partial^2 V}{\partial x^2}\right\} = 0 \qquad (3.2)\end{aligned}$$

with the terminal condition $V(x, L, T) = U_2(xe^{-L})$, where $a(t) = \sigma(t)\sigma(t)^\intercal$. As in (3.2), we drop the arguments whenever convenient. To find the optimal policies, we need to solve the two maximization problems appearing in the left-hand side of (3.2).

The problem of solving for the consumption policy in (3.2) is a concave maximization problem. Therefore, the first-order necessary condition is also sufficient, and it is

$$e^{-L}U_1'\left(Ce^{-L}\right) - V_x(x, L, t) = 0,$$

where we have used the subscript x to denote the partial derivative of V with respect to x. From this, we get the optimal feedback consumption policy

$$C^*(x, L, t) = e^L\ell_1\left(e^LV_x(x, L, t)\right), \qquad (3.3)$$

where $l_1(\cdot)$ is the unique inverse of $U_1'(\cdot)$. This means that once we know the value function derivative, we can obtain the optimal consumption policy by (3.3).

The problem of obtaining the portfolio policy π in (3.2) is a quadratic maximization problem. By the assumption of the market completeness, $a = \sigma\sigma^\intercal$ is invertible and we get

$$\pi^*(x, L, t)^\intercal = -\frac{(\sigma(t)^\intercal)^{-1}}{xV_{xx}(x, L, t)}\left[\theta(t)V_x(x, L, t) + \rho\zeta V_{Lx}(x, L, t)\right]. \qquad (3.4)$$

Note that the first term on the right-hand-side is the classical solution and the second term is the effect of the uncertainty in inflation. If the inflation is uncorrelated with all the risky assets, then the second term is zero. The inflation effect depends also on the inflation volatility $\zeta > 0$ and on V_{Lx}, i.e., on the sensitivity of the marginal value V_x with respect to the ln-basket price. If the marginal value of nominal wealth rises (falls) in the basket price, then the higher the correlation, the more (less) does the agent invest in the stock market. This is related to the coefficient of the relative risk aversion as elaborated in subsection 3.3.

Now we state the following modification of the classical Mutual Fund Theorem.

Theorem 1 *With fully observed inflation, the following hold:*
(i) The optimal portfolio involves an allocation between the risk-free fund F^1 *and two risky funds that consist only of risky assets:* $F^2(t) = (\sigma(t)^\intercal)^{-1}\theta(t)$ *and* $F^3(t) = (\sigma(t)^\intercal)^{-1}\rho$*, where the vector* $F^k(t)$ *represents the* k*th portfolio's weights of the risky assets at time* t*,* $k = 2, 3$*.*
(ii) The optimal proportional allocation $\mu^k(t)$ *of wealth in the fund* $F^k(t)$*,* $k = 1, 2, 3$*, at time* t *are given by* $\mu^2(t) = -\frac{V_x(x,L,t)}{X(t)V_{xx}(x,L,t)}$*,* $\mu^3(t) = -\frac{\zeta V_{Lx}(x,L,t)}{X(t)V_{xx}(x,L,t)}$*, and* $\mu^1(t) = 1 - \mu^2(t) - \mu^3(t)$*.*

PROOF: The right-hand-side of (3.4) equals $\mu^2(t)F^2(t) + \mu^3(t)F^3(t)$, where $\mu^k(t)$ is one dimensional and $F^k(t)$ is n-dimensional for $k = 2, 3$. Thus, $\mu^k(t)$ can be viewed as the proportional wealth in the fund $F^k(t)$, $k = 2, 3$. ☐

By Theorem 1, the optimal portfolio can consist of investments in three funds, whereas the classical problem requires only two funds. The first fund is the risk-free asset and the second one is the growth optimum portfolio fund as in the classical problem. The third fund arises from the correlation between the inflation uncertainty and the market risk.

Three-fund theorems are not new. Zhao (2005) considers an optimal asset allocation policy for an investor concerned with the performance of his investment relative to a benchmark. In his case, one of the two risky funds replicates the benchmark portfolio. In the three-fund theorem obtained by Brennan and Xia (2002), one fund replicates real interest rate uncertainty, another one is the classical growth optimal fund, and the last one replicates the fully observed inflation uncertainty. They do not consider partially observed inflation as in the present paper.

By (3.4), we have

$$x^2\pi^* a(t)(\pi^*)^\intercal V_{xx} = -\frac{|\theta(t)V_x + \rho\zeta V_{Lx}|^2}{V_{xx}}.$$

Therefore, the HJB equation (3.2) can be written as

$$
\begin{aligned}
V_t - \beta V + r(t) x V_x + \left(I - \tfrac{1}{2}\zeta^2\right) V_L + \tfrac{1}{2}\zeta^2 V_{LL} + U_1\left(\ell_1\left(e^L V_x\right)\right) - e^L \ell_1\left(e^L V_x\right) V_x& \\
- \frac{1}{2}\frac{|\theta(t)V_x + \rho\zeta V_{Lx}|^2}{V_{xx}} &= 0
\end{aligned}
\tag{3.5}
$$

with the terminal condition $V(x, L, T) = U_2(xe^{-L})$. Equation (3.5) can be solved explicitly in some cases as is shown in the next subsection.

3.3 Example

Let us consider the constant relative risk-aversion (CRRA) utility[2]

$$U_i(y) = \frac{y^{1-\phi}}{1-\phi},\ i = 1, 2,$$

where the relative risk aversion coefficient is $\phi > 0$, $\phi \neq 1$. Note that $y^{1-\phi}$ is increasing in y if $\phi \in (0, 1)$, but decreasing if $\phi > 1$. Therefore, the divisor $(1-\phi)$ in the utility function defined above ensures that the marginal utility is positive for all values of ϕ. Further, note that the third derivative is positive, implying a positive motive for precautionary saving (see, e.g., Leland (1968) and Sandmo (1970)).

From the CRRA utility, we get $U_1'(y) = y^{-\phi}$ and $\ell_1(\lambda) = \lambda^{-1/\phi}$. Therefore, (3.5) can be written as

$$
\begin{aligned}
V_t - \beta V + r(t) x V_x + \left(I - \tfrac{1}{2}\zeta^2\right) V_L + \tfrac{1}{2}\zeta^2 V_{LL} + \tfrac{\phi}{1-\phi}\left(e^L V_x\right)^{\frac{\phi-1}{\phi}}& \\
- \frac{|\theta(t)V_x + \rho\zeta V_{Lx}|^2}{2V_{xx}} &= 0,
\end{aligned}
$$

where $V(x, L, T) = U_2(xe^{-L}) = \frac{1}{1-\phi}\left(xe^{-L}\right)^{1-\phi}$. The solution of this equation is given by

$$V(x, L, t) = \tfrac{1}{1-\phi}\left(xe^{-L}\right)^{1-\phi} g(t), \tag{3.6}$$

---

[2] As, e.g., in Shreve and Soner (1994), the analysis can be extended to consider the log-utility, i.e., the case with $\phi = 1$.

where $g(t)$ solves

$$\begin{aligned} g' + g\Big[(1-\phi)\left(r(t) - I + \tfrac{1}{2}\zeta^2\right) - \beta + \tfrac{1}{2}\zeta^2(1-\phi)^2 \\ + \tfrac{1}{2}\tfrac{1-\phi}{\phi}|\theta(t) - \rho\zeta(1-\phi)|^2\Big] + \phi g^{\frac{\phi-1}{\phi}} = 0 \end{aligned}$$

with the terminal condition $g(T) = 1$. By (3.3) and (3.4), the optimal consumption policy is

$$C^*(x, L, t) = x\Big/\left(g(t)^{\frac{1}{\phi}}\right), \quad \pi^*(x, L, t)^\mathsf{T} = \tfrac{1}{\phi}\left(\sigma(t)^\mathsf{T}\right)^{-1}[\theta(t) - \rho\zeta(1-\phi)]\,. \tag{3.7}$$

Naturally, when the investor is wealthier, his consumption rate is higher. Also, he invests more in the risky assets, the higher the market price of risk and the lower the risks of the risky assets. If $\rho > 0$ and $\phi > 1$, then the investor puts in more in the risky assets than a corresponding investor who does not consider inflation. Note that, by the discussion in Section 3.1, $\phi > 1$ implies $\frac{\partial^2 U_i(ye^{-L})}{\partial L^2} < 0$ for all $y > 0$ and $i = 1, 2$, i.e., $\frac{\partial U_i(ye^{-L})}{\partial L}$ falls when L rises. Thus, the agent, by investing more in the risky assets, hedges the decrease in the future utility from the rising consumption basket price. Conversely, when $\rho > 0$ and $\theta < 1$, then $\frac{\partial^2 U_i(ye^{-L})}{\partial L^2} > 0$, and therefore the agent invests less in the risky assets.

4 Partially Observed Basket Prices

4.1 Observation Processes

In the previous section, we assumed that the investor observes fully the price of his consumption basket. In reality, the basket price is difficult to measure, and we assume in this section that it is only partially observed. Therefore, the process $L(t)$ is not observable, and the investor receives a noisy signal $Z(t)$ on his consumption basket price. Examples of this signal are prices of some consumption goods in the basket, monthly credit card bills, and the consumer price index (CPI). Since the investor also observes financial asset prices, his information set at time t is $\mathcal{G}_t = \sigma\{w(s), Z(s), s \leq t\}$. Furthermore, by way of correlation ρ, the financial asset prices provide information on the consumption basket price.

The signal process is as follows

$$dZ(t) = L(t)dt + m dw_Z(t), \quad Z(0) = 0, \tag{4.1}$$

where $m > 0$ is a constant signal volatility. Recall that the Wiener process $w_Z(t)$ is independent of the other uncertainties implied by $w(t)$ and $w_I(t)$.

The investor also observes the process $w(t)$ from the asset prices in the market, i.e., by the invertibility of the matrix $\sigma(t)$, he recovers the process $w(t)$ from the asset price $Y(t)$. Because the agent's information is described by the filtration $\mathcal{G}_t$, the decisions $C(t)$ and $\pi(t)$ must be adapted to $\mathcal{G}_t$. Moreover, since the objective function depends on $L(t)$, we compute the conditional probability of $L(t)$ given $\mathcal{G}_t$. Therefore, we have a non-linear filtering problem (e.g., Bensoussan (2004, Chapter 4)).

For the filtering problem, we next transform the observation processes into equivalent ones, which are independent Wiener processes on a new probability measure. Indeed, define

$$d\tilde{w}_I(t) = \frac{dw_I(t) - \rho^\mathsf{T} dw(t)}{\sqrt{1 - |\rho|^2}}, \quad d\tilde{w}(t) = dw(t) + \theta(t)dt, \quad d\tilde{Z}(t) = \frac{dZ(t)}{m}, \tag{4.2}$$

with the initial conditions $\tilde{w}_I(0) = \tilde{w}(0) = \tilde{Z}(0) = 0$. The first one is obviously a Wiener process under P, and it is independent of the market. The last two are diffusion processes with drifts under P. Next, we create a new probability measure under which the above three are Wiener processes. For this we define the process $M(t)$ as follows

$$dM(t) = -M(t)\left(\theta(t)^\mathsf{T} dw(t) + \frac{L(t)}{m} dw_Z(t)\right), \quad M(0) = 1.$$

Its solution is given by

$$M(t) = \exp\left(-\int_0^t \left(\theta(s)^\mathsf{T} dw(s) + \frac{L(s)}{m} dw_Z(s)\right) - \frac{1}{2}\int_0^t \left(|\theta(s)|^2 + \frac{L^2(s)}{m^2}\right) ds\right).$$

Since the coefficients in (2.1) are constants and the components of $\theta(t)$ in (2.2) are bounded, $M(t)$ is a $(P, \mathcal{F}_t)$ martingale, and it starts at 1. Therefore, we can define the probability $\tilde{P}$ on $(\Omega, \mathcal{F})$ by the Radon-Nikodym derivative

$$\frac{d\tilde{P}}{dP} = M(t)$$

on $\mathcal{F}_t$. From the Girsanov Theorem (e.g., Øksendal (1998, Theorem 8.6.3)), we get

Lemma 1 *The processes $\tilde{w}_I(t)$, $\tilde{w}(t)$, and $\tilde{Z}(t)$ are independent standard Wiener processes for $\tilde{P}$ and $\mathcal{F}_t$.*

Since the observation processes are independent standard Wiener processes under $\tilde{P}$, it is convenient to use the measure $\tilde{P}$ instead of P in the filtering problem under consideration.

4.2 Conditional Density of Ln-Basket Price

Given a smooth test function $\psi_t(L) = \psi(L, t)$, we want to derive the operator

$$\Pi(t)(\psi_t) = E\left[\psi(L(t), t) | \mathcal{G}_t\right], \tag{4.3}$$

where the notation means that the operator $\Pi(t)$ for each fixed t is a linear operator on functions of L. It is enough to consider the space of continuous bounded functions of L, and, by approximation, we can consider smooth bounded functions of L. This operator is the solution of a functional equation. Since it is a conditional expectation, we may hope that it is obtained via a conditional probability density $p(L, t)$:

$$E[\psi(L(t), t) | \mathcal{G}_t] = \int p(L, t)\psi(L, t) dL$$

for any test function $\psi(L, t)$. The density $p(L, t)$ is the solution of a Kushner equation, which can be obtained from an un-normalized probability density $q(L, t)$ derived from a *Zakai equation* (Zakai (1969)). Indeed, in Appendix A.1, we prove the following lemma.

Lemma 2 *The process of the un-normalized probability density $q(L,t)$ is as follows:*

$$dq = \left[-q_L\left(I - \tfrac{1}{2}\zeta^2\right) + \tfrac{1}{2}\zeta^2 q_{LL}\right]dt + \left(q\theta(t) - q_L\zeta\rho\right)^{\mathsf{T}} d\tilde{w}(t) + q\tfrac{L}{m}d\tilde{Z}(t), \tag{4.4}$$

where $q(L,0) = p_0(L)$ and $\int p(L,t)\psi(L,t)dL = \int q(L,t)\psi(L,t)dL / \int q(L,t)dL$ for any test function $\psi \in C^{2,1}$.

Thus, we assume that $p_0(L)$ is the initial probability density for $\Pi(0)$. In the Appendix A.2, we prove the following important result.[3]

Theorem 2 *Let the initial density be normal with mean L_0 and variance S_0, i.e.,*

$$p_0(L) = \tfrac{1}{\sqrt{2\pi S_0}}e^{-\frac{1}{2}(L-L_0)^2/S_0}.$$

Then, the equation (4.4) has an explicit solution

$$q(L,t) = K(t)e^{-\frac{1}{2}\left(L-\hat{L}(t)\right)^2/S(t)}, \tag{4.5}$$

where $\hat{L}(t) = E[(L(t)/\mathcal{G}_t]$, and the variance $S(t) = E[(L(t)-\hat{L}(t))^2|\mathcal{G}_t]$ is deterministic, given by

$$S(t) = \begin{cases} m\Lambda_1\frac{\Lambda_2\exp(2\Lambda_1 t/m)-1}{\Lambda_2\exp(2\Lambda_1 t/m)+1} & \text{if } S_0 < m\Lambda_1 \\ m\Lambda_1 & \text{if } S_0 = m\Lambda_1 \\ m\Lambda_1\frac{\Lambda_2\exp(2\Lambda_1 t/m)+1}{\Lambda_2\exp(2\Lambda_1 t/m)-1} & \text{if } S_0 > m\Lambda_1, \end{cases} \tag{4.6}$$

where $\Lambda_1 = \zeta\sqrt{1-|\rho|^2}$ and $\Lambda_2 = \left|\frac{m\Lambda_1+S_0}{m\Lambda_1-S_0}\right|$. Furthermore, the belief $\hat{L}$ is the solution to the Kalman filter

$$d\hat{L}(t) = \left(I - \tfrac{1}{2}\zeta^2 - \zeta\rho^{\mathsf{T}}\theta(t)\right)dt + \zeta\rho^{\mathsf{T}}d\tilde{w}(t) + \tfrac{S(t)}{m}\left(d\tilde{Z}(t) - \tfrac{\hat{L}(t)}{m}dt\right), \tag{4.7}$$

---

[3] We refer to Chapters 4 and 6 in Bensoussan (2004) for more general results.

where $\hat{L}(0) = L_0$*. The variable* $K(t)$ *in (4.5) is adapted to* $\mathcal{G}_t$ *and is given by*

(4.8)
$$K(t) = \exp\left(-\frac{1}{2}\int_0^t \left(\frac{\hat{L}^2(s)}{m^2} + |\theta(s)|^2 \right) ds + \int_0^t \frac{\hat{L}(s)}{m} d\tilde{Z}(s) + \int_0^t \theta(s)^{\mathsf{T}} d\tilde{w}(s) \right).$$

Thus, the conditional probability law of $L(t)$ given $\mathcal{G}_t$ is Gaussian with mean $\hat{L}(t)$ and variance $S(t)$. If $S_0 = 0$, then $S(t) = m\Lambda_1 \tanh\left(\frac{\Lambda_1}{m}t\right)$, which is clearly a non-decreasing function of t with $\lim_{t\to\infty} S(t) = m\Lambda_1$.

We conclude this section by stating the following result that will be used in the next section. Its proof appears in Appendix A.3.

Lemma 3 *Let the innovation process* $\tilde{w}_Z(t)$ *be defined by*

$$d\tilde{w}_Z(t) = \tfrac{1}{m}\left(dZ(t) - \hat{L}(t)dt\right), \quad \tilde{w}_Z(0) = 0.$$

Then, $\tilde{w}_Z(t)$ *and* $w(t)$ *together form an* $(n+1)$*-dimensional* $(P, \mathcal{G}_t)$ *Wiener process.*

5 Optimal Consumption and Portfolio Policy

5.1 Objective

By (4.2), (4.7), and Lemma 3, the dynamics of $\hat{L}$ is given by

(5.1)
$$d\hat{L}(t) \quad = \quad \left(I - \tfrac{1}{2}\zeta^2\right) dt + \zeta\rho^{\mathsf{T}} dw(t) + \tfrac{S(t)}{m} d\tilde{w}_Z(t).$$

This dynamics is driven by two independent Wiener processes: $w(t)$ from the wealth process (2.3) and $\tilde{w}_Z(t)$ from the signals (4.1). By definition, the signals give information on the consumption basket price and, on account of the correlation ρ, the asset prices give some information on the basket price.

Let us consider an optimal consumption and investment problem starting at time t with the initial conditions $\hat{L}(t) = \hat{L}$ and $X(t) = x$. The objective function of the investor is given similarly as in subsection 3.1, but now the expectation is with respect to the observation filtration $\mathcal{G}_t$. That is,

$$\begin{aligned}
&\tilde{J}\left(C(\cdot), \pi(\cdot); x, \hat{L}, t\right) \\
&= E\left[\int_t^T e^{-\beta(s-t)} U_1\left(C(s)e^{-L(s)}\right) ds + e^{-\beta(T-t)} U_2\left(X(T)e^{-L(T)}\right) | \mathcal{G}_t\right] \\
&= E\left[\int_t^T e^{-\beta(s-t)} E\left[U_1\left(C(s)e^{-L(s)}\right) | \mathcal{G}_s\right] ds + ce^{-\beta(T-t)} E\left[U_2\left(X(T)e^{-L(T)}\right) | \mathcal{G}_T\right] | \mathcal{G}_t\right].
\end{aligned}$$

Hence, we integrate and calculate first the expected utilities:

$$\begin{aligned}
\tilde{U}_1(C, \hat{L}, s) &= \tfrac{1}{\sqrt{2\pi}} \int U_1\left(Ce^{-\hat{L}-y\sqrt{S(s)}}\right) e^{-\frac{1}{2}y^2} dy, \\
\tilde{U}_2(X, \hat{L}, s) &= \tfrac{1}{\sqrt{2\pi}} \int U_2\left(Xe^{-\hat{L}-y\sqrt{S(s)}}\right) e^{-\frac{1}{2}y^2} dy,
\end{aligned}$$

which are, thus, the expected utilities over the distribution of L. Then, the objective function can be written as

$$\begin{aligned}
\tilde{J}\left(C(\cdot), \pi(\cdot); x, \hat{L}, t\right) &= E\left[\int_t^T e^{-\beta(s-t)} \tilde{U}_1\left(C(s), \hat{L}(s), s\right) ds\right. \\
&+ \left. e^{-\beta(T-t)} \tilde{U}_2\left(X(T), \hat{L}(T), T\right) | \hat{L}(t) = \hat{L}, X(t) = x\right]
\end{aligned}$$

and the value function as

$$\tilde{V}(x, \hat{L}, t) = \sup_{C(\cdot), \pi(\cdot)} \tilde{J}\left(C(\cdot), \pi(\cdot); x, \hat{L}, t\right).$$

5.2 Solution

As in subsection 3.2, we first write the HJB equation, which is now given as

$$\begin{aligned}
(5.2) \qquad & \tilde{V}_t - \beta\tilde{V} + r(t)x\tilde{V}_x + \left(I - \tfrac{1}{2}\zeta^2\right)\tilde{V}_{\hat{L}} + \tfrac{1}{2}\left(\zeta^2|\rho|^2 + \tfrac{S^2(t)}{m^2}\right)\tilde{V}_{\hat{L}\hat{L}} \\
&+ \sup_C\left\{\tilde{U}_1(C, \hat{L}, t) - C\tilde{V}_x\right\} + \sup_\pi\left\{x\pi\sigma(t)\left(\theta(t)\tilde{V}_x + \rho\zeta\tilde{V}_{\hat{L}x}\right) + \tfrac{1}{2}x^2\pi a(t)\pi^\mathsf{T}\tilde{V}_{xx}\right\} = 0
\end{aligned}$$

with $\tilde{V}(x, \hat{L}, T) = \tilde{U}_2(x, \hat{L}, T)$, where $|\rho| = \sqrt{\rho^\mathsf{T}\rho}$ and $a(t) = \sigma(t)\sigma(t)^\mathsf{T}$. Note that we get (3.2) if we assume perfect observation ($\rho = 1$).

By the first-order conditions, the optimal consumption and portfolio strategies

$(\hat{C}, \hat{\pi})$ satisfy

$$
\begin{aligned}
(5.3) \qquad \tfrac{\partial \tilde{U}_1(C,\hat{L},t)}{\partial C}\Big|_{C=\hat{C}(x,\hat{L},t)} &= \tilde{V}_x(x,\hat{L},t) \\
\hat{\pi}(x,\hat{L},t)^{\mathsf{T}} &= -\frac{(\sigma(t)^{\mathsf{T}})^{-1}}{x\tilde{V}_{xx}(x,\hat{L},t)}\left[\theta(t)\tilde{V}_x(x,\hat{L},t)+\rho\zeta\tilde{V}_{\hat{L}x}(x,\hat{L},t)\right].
\end{aligned}
$$

We can now state the following three-fund theorem.

Theorem 3 *Under the partially observable basket price process, Theorem 1 holds with a modified proportional allocations of wealth between the funds:* $\hat{\mu}^2(t) = -\frac{\tilde{V}_x(x,\hat{L},t)}{X(t)\tilde{V}_{xx}(x,\hat{L},t)}$, $\hat{\mu}^3(t) = -\frac{\zeta\tilde{V}_{\hat{L}x}(x,\hat{L},t)}{X(t)\tilde{V}_{xx}(x,\hat{L},t)}$, *and* $\hat{\mu}^1(t) = 1-\hat{\mu}^2(t)-\hat{\mu}^3(t)$, *where* $\hat{\mu}^k(t)$ *is the proportional wealth invested in the* k*th fund at time* t.

PROOF: Follows directly from Theorem 1 and (5.3). □

Theorems 1 and 3 imply that the components of the funds are the same under the fully observed and partially observed basket price, only the relative allocations of the wealth invested in these funds are different. Thus, in both cases the optimal portfolio is a linear combination of the risk-free fund, the growth optimum fund, and the fund that arises from the correlation between the inflation uncertainty and the market risk. The proportions of the wealth invested in these funds are different because the investor's belief on the consumption basket price is not the same under different information sets, i.e., because $\hat{L} \neq L$. Thus, the noisy signals affect the optimal solution through the value function derivatives.

Brennan (1998) and Xia (2001) consider the effects of uncertainty in the stock return predictability on the optimal dynamic portfolio choice. That is, in their models, the expected returns are unknown and are learned from market variables. The model in this section is quite different from theirs, since in the present paper we have uncertainty on the deflator, i.e., on the consumption basket price. Our investor is only interested in the real prices $\{Y_i/B\}$, but he does not observe the values of B, which implies that the current real prices are random with their distributions depending on the observations made thus far. Thus, in our model there is no uncertainty on the expected returns, but on the current real prices.

From (5.2) and (5.3), we get

$$\begin{aligned}&\tilde{V}_t - \beta\tilde{V} + r(t)x\tilde{V}_x + \left(I - \tfrac{1}{2}\zeta^2\right)\tilde{V}_{\hat{L}} + \tfrac{1}{2}\left(\zeta^2|\rho|^2 + \tfrac{S^2(t)}{m^2}\right)\tilde{V}_{\hat{L}\hat{L}} \\ &+\tilde{U}_1(\hat{C},\hat{L},t) - \hat{C}\tilde{V}_x - \frac{|\theta(t)\tilde{V}_x + \rho\zeta\tilde{V}_{x\hat{L}}|^2}{2\tilde{V}_{xx}} = 0,\end{aligned}$$

where $\tilde{V}(x,\hat{L},T) = \tilde{U}_2(x,\hat{L},T)$.

The issues of the existence and the regularity of a solution to the HJB equation are beyond the scope of this paper. One can consult Fleming and Rishel (1975), e.g., for details. For our purpose, if we find a sufficiently smooth solution of the HJB equation, then the standard verification argument applies and it proves that the solutions is the value function. Furthermore, it provides an optimal feedback policy for the problem. Indeed, in the next subsection, we treat a special case of interest with CRRA utility, where we obtain the requisite smooth solution explicitly.

5.3 Example

We use the CRRA utility function introduced in subsection 3.3 and, therefore,

$$\begin{aligned}\tilde{U}_1(C,\hat{L},t) &= \tfrac{1}{1-\phi}C^{1-\phi}e^{-(1-\phi)\hat{L}+\frac{1}{2}(1-\phi)^2S(t)}, \\ \tilde{U}_2(x,\hat{L},T) &= \tfrac{1}{1-\phi}x^{1-\phi}e^{-(1-\phi)\hat{L}+\frac{1}{2}(1-\phi)^2S(T)},\end{aligned}$$

where the relative risk aversion is $\phi > 0$, $\phi \neq 1$. From (5.3) we get

$$\hat{C}(x,\hat{L},t) = \tilde{V}_x^{-1/\phi}e^{\frac{\phi-1}{\phi}\hat{L}+\frac{1}{2}\frac{(1-\phi)^2}{\phi}S(t)}.$$

Then the HJB equation (5.2) reduces to

$$\begin{aligned}&\tilde{V}_t - \beta\tilde{V} + r(t)x\tilde{V}_x + \left(I - \tfrac{1}{2}\zeta^2\right)\tilde{V}_{\hat{L}} + \tfrac{1}{2}\left(\zeta^2|\rho|^2 + \tfrac{S^2(t)}{m^2}\right)\tilde{V}_{\hat{L}\hat{L}} \\ &+\frac{\phi}{\phi-1}\left(e^{\hat{L}-\frac{1}{2}(1-\phi)S(t)}\tilde{V}_x\right)^{\frac{\phi-1}{\phi}} - \frac{|\theta(t)\tilde{V}_x + \rho\zeta\tilde{V}_{x\hat{L}}|^2}{2\tilde{V}_{xx}} = 0\end{aligned}$$

with $\tilde{V}(x, \hat{L}, T) = \frac{1}{1-\phi} x^{1-\phi} e^{-(1-\phi)\hat{L} + \frac{1}{2}(1-\phi)^2 S(T)}$. Its solution is given by

$$\tilde{V}(x, \hat{L}, t) = \tfrac{1}{1-\phi} \left(x e^{-\hat{L}} \right)^{1-\phi} h(t), \tag{5.4}$$

where $h(t)$ solves

$$\begin{aligned} & h' + h \Big[(1-\phi)(r(t) - I + \tfrac{1}{2}\zeta^2) - \beta + \tfrac{1}{2}(1-\phi)^2 \left(\zeta^2 |\rho|^2 + \tfrac{S^2(t)}{m^2} \right) \\ & + \tfrac{1}{2} \tfrac{1-\phi}{\phi} |\theta(t) - \rho\zeta(1-\phi)|^2 \Big] + \phi h^{\frac{\phi-1}{\phi}} e^{\frac{1}{2} \frac{(1-\phi)^2}{\phi} S(t)} = 0 \end{aligned}$$

with $h(T) = e^{\frac{1}{2}(1-\phi)^2 S(T)}$. In fact it is easy to check that

$$h(t) = g(t) e^{\frac{1}{2}(1-\phi)^2 S(t)}. \tag{5.5}$$

Therefore, we get

$$\begin{aligned} \hat{C}(x, \hat{L}, t) &= x / (g(t))^{\frac{1}{\phi}}, \\ \hat{\pi}(x, \hat{L}, t)^{\mathsf{T}} &= \tfrac{1}{\phi} \left(\sigma(t)^{\mathsf{T}} \right)^{-1} \left[\theta(t) - \rho\zeta(1-\phi) \right]. \end{aligned} \tag{5.6}$$

By comparing with (3.7), we get the result below. Since these formulas are not affected by L they are the same in the fully observed and partially observed cases.

Proposition 1 *Let the investor have the CRRA utilities. Then the optimal consumption and investment decisions are not affected by the uncertainty in the knowledge of the current consumption basket price.*

Let us consider the following numerical example. There are only two financial assets: a risk-free asset and a risky asset which has the correlation (two cases) $\rho = 0.3$ (resp. -0.3) with the inflation. The risky asset has the annual volatility $\sigma = 0.15$ and the market price of risk $\theta = 0.5$. The risk-free interest rate $r = 0.05$. The expected annual inflation rate $I = 0.02$ and the inflation volatility $\zeta = 0.05$. The signal volatility $m = 0.3$. The agent's initial wealth $x = 1$, his relative risk aversion $\phi = 2$, the utility discount rate $\beta = 0.05$, and the terminal time $T = 1$ year. By Proposition 1, the optimal policy in the fully observed case is the same

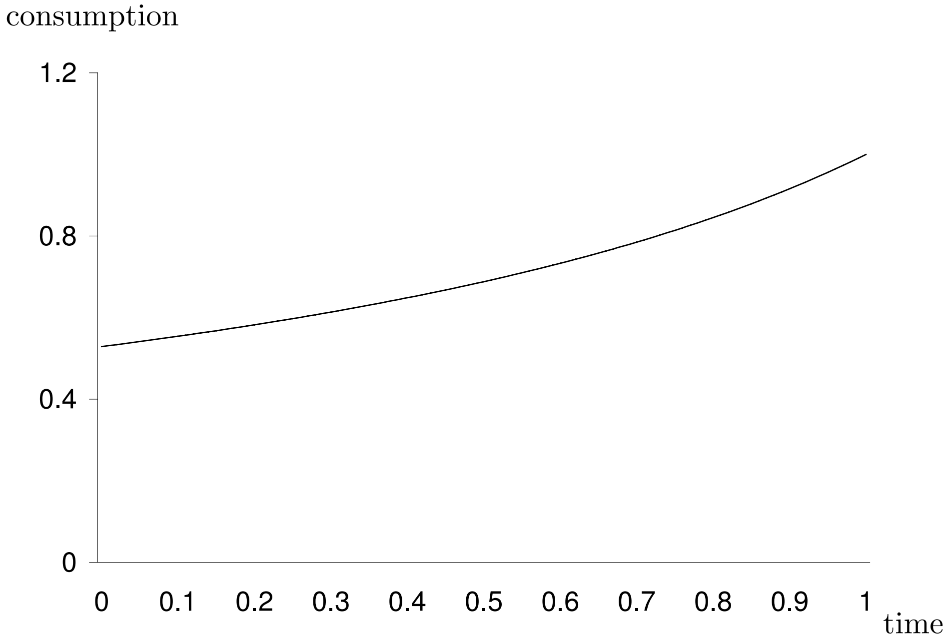

Figure 1: Optimal nominal consumption. Parameter values: $\rho = 0.3$, $\sigma = 0.15$, $\theta = 0.5$, $r = 0.05$, $I = 0.02$, $\zeta = 0.05$, $m = 0.3$, $\hat{L}(0) = 0$, $x = 1$, $\phi = 2$, $\beta = 0.05$, and $T = 1$.

as in the partially observed case. By using our numerical parameter values in (3.7), we see that the optimal relative investment in the risky asset is $\pi^* = 1.72$ when the correlation is $\rho = 0.3$, and $\pi^* = 1.62$ when $\rho = -0.3$, and the investor borrows 0.72 and 0.62 times his wealth, respectively. Thus, when the correlation is high the investor hedges the consumption basket price variance by investing more in the risky asset, as well as borrows more money. Figure 1 illustrates the consumption path. As can be seen, the investor raises the consumption rate over time. Correlation ρ does not change the consumption policy significantly and, therefore, we use only $\rho = 0.3$ in the figure.

6 Conclusions

We have formulated a new optimal portfolio and consumption decision model under partially observed real prices. The investor observes noisy signals on the consumption basket price over time. Based on these, he updates his estimates of the consumption basket and real asset prices at any given moment in time, and then decides on his portfolio and his consumption rate at that time.

We show that a modified mutual fund theorem consisting of three funds holds. The funds are a risk-free fund, a growth optimum fund, and a fund that arises from the correlation between the inflation uncertainty and the market risk. In general, the wealth invested in these funds depends on the investor's utility function and on his beliefs about the consumption basket price. However, the funds are robust over different information sets on the consumption basket price. That is, the investor uses the same three funds regardless of the noise in observing the consumption basket price.

We solve our model explicitly for the CRRA utility functions. In this case, the additional measurement uncertainty in the current consumption basket price does not change the optimal policy. This simplifies asset management since if the portfolio managers know the consumption basket process parameters then they can ignore the measurement uncertainty in the portfolio optimization.

Appendix A.1: Proof of Lemma 2

Here we follow the steps in Bensoussan (2004, Chapter 4).

Step 1: Un-Normalized Conditional Probability. Let us introduce a new information filtration

$$\widetilde{\mathcal{G}}_t = \sigma\{\tilde{w}(s), \tilde{Z}(s), s \leq t\}.$$

Obviously, $\mathcal{G}_t = \widetilde{\mathcal{G}}_t$ and, therefore,

$$\Pi(t)(\psi_t) = E\left[\psi(L(t), t)|\mathcal{G}_t\right] = E\left[\psi(L(t), t)|\widetilde{\mathcal{G}}_t\right],$$

where $\psi \in C^{2,1}$ is a test function.

It is convenient to use $\tilde{P}$ instead of P, since the observation processes under $\tilde{P}$

are Wiener processes. Therefore, we need also the Radon-Nikodym derivative

$$\frac{dP}{d\tilde{P}} = \frac{1}{M(t)} = \eta(t)$$

on $\mathcal{F}_t$. Now we have

$$\Pi(t)(\psi_t) = E\left[\psi(L(t),t)|\widetilde{\mathcal{G}}_t\right] = \frac{\tilde{E}\left[\psi(L(t),t)\eta(t)|\widetilde{\mathcal{G}}_t\right]}{\tilde{E}\left[\eta(t)|\widetilde{\mathcal{G}}_t\right]},$$

where $\tilde{E}$ is the expectation with respect to $\tilde{P}$. This formula leads to the introduction of the un-normalized conditional probability defined by

$$p(t)(\psi_t) = \tilde{E}\left[\psi(L(t),t)\eta(t)|\widetilde{\mathcal{G}}_t\right].$$

Step 2: Zakai Equation. To proceed, we note that

$$\begin{aligned} dL(t) &= \left(I - \tfrac{1}{2}\zeta^2\right)dt + \zeta dw_I(t) \\ &= \left(I - \tfrac{1}{2}\zeta^2 - \zeta\rho^\mathsf{T}\theta(t)\right)dt + \zeta\left[\sqrt{1-|\rho|^2}d\tilde{w}_I(t) + \rho^\mathsf{T}d\tilde{w}(t)\right] \end{aligned}$$

and

$$\begin{aligned} d\eta(t) &= \eta(t)\left[\theta(t)^\mathsf{T}dw(t) + \frac{L(t)}{m}dw_Z(t)\right] + \eta(t)\left[\theta(t)^\mathsf{T}\theta(t) + \frac{L^2(t)}{m^2}\right]dt \\ &= \eta(t)\left[\theta(t)^\mathsf{T}d\tilde{w}(t) + \frac{L(t)}{m}d\tilde{Z}(t)\right], \end{aligned}$$

because

$$\begin{aligned} dw_I(t) &= \sqrt{1-|\rho|^2}d\tilde{w}_I(t) + \rho^\mathsf{T}dw(t), \\ dw(t) &= d\tilde{w}(t) - \theta(t)dt, \\ dw_Z(t) &= d\tilde{Z}(t) - \tfrac{L(t)}{m}dt. \end{aligned}$$

Therefore, by Ito's lemma

$$\begin{aligned}
d\left[\eta(t)\psi(L(t),t)\right] &= \eta(t)\left[\frac{\partial\psi}{\partial t} + \left(I - \frac{1}{2}\zeta^2 - \zeta\rho^\mathsf{T}\theta(t)\right)\frac{\partial\psi}{\partial L} + \frac{1}{2}\zeta^2\frac{\partial^2\psi}{\partial L^2}\right]dt \\
&+ \eta(t)\zeta\frac{\partial\psi}{\partial L}\left[\sqrt{1-|\rho|^2}d\tilde{w}_I(t) + \rho^\mathsf{T}d\tilde{w}(t)\right] \\
&+ \eta(t)\psi\left[\theta(t)^\mathsf{T}d\tilde{w}(t) + \frac{L(t)}{m}d\tilde{Z}(t)\right] + \eta(t)\zeta\frac{\partial\psi}{\partial L}\rho^\mathsf{T}\theta(t)dt \\
&= \eta(t)\left\{\left[\frac{\partial\psi}{\partial t} - \mathcal{A}\psi\right]dt + \zeta\frac{\partial\psi}{\partial L}\sqrt{1-|\rho|^2}d\tilde{w}_I(t)\right. \\
&+ \left.\left(\psi\theta(t) + \frac{\partial\psi}{\partial L}\zeta\rho\right)^\mathsf{T}d\tilde{w}(t) + \psi\frac{L(t)}{m}d\tilde{Z}(t)\right\},
\end{aligned}$$

where the second order differential operator is given by

$$\mathcal{A} = -\left(I - \frac{1}{2}\zeta^2\right)\frac{\partial}{\partial L} - \frac{1}{2}\zeta^2\frac{\partial^2}{\partial L^2}.$$

To compute the conditional expectation, we use test functions which are $\widetilde{\mathcal{G}}_t$-measurable. Because the generating processes are Wiener processes, it is sufficient to test with stochastic processes of the form (see Bensoussan (2004, p. 81))

$$d\gamma(t) = i\gamma(t)\left(\beta_1(t)^\mathsf{T}d\tilde{w}(t) + \beta_2(t)d\tilde{Z}(t)\right),\ \gamma(0) = 1,$$

where $i = \sqrt{-1}$, and $\beta_1(t) \in \mathbb{R}^n$ and $\beta_2(t) \in \mathbb{R}$ are arbitrary deterministic bounded functions. Then, by Ito's lemma and expectation, we get

$$\begin{aligned}
&\tilde{E}\left[\gamma(t)\eta(t)\psi(L(t),t)\right] = \tilde{E}[\psi(L(0),0)] + \tilde{E}\left[\int_0^t \gamma(s)\eta(s)\left\{\frac{\partial\psi}{\partial s} - \mathcal{A}\psi\right.\right. \\
&+ \left.\left. i\beta_1(s)^\mathsf{T}\left(\psi(L(s),s)\theta(s) + \rho\zeta\frac{\partial\psi}{\partial L}\right) + i\beta_2(s)\psi(L(s),s)\frac{L(s)}{m}\right\}ds\right].
\end{aligned}$$

By the process of $\eta(t)\psi(L(t),t)$, the definition of $p(t)(\psi_t)$, and $\tilde{E}\left[\tilde{w}_I(t)|\widetilde{\mathcal{G}}_t\right]=0$,

$$\begin{aligned}\tilde{E}\left[\gamma(t)p(t)(\psi_t)\right] &= \tilde{E}\left[\gamma(t)\Pi(0)(\psi_0)\right]+\tilde{E}\left[\gamma(t)\left\{\int_0^t p(s)\left(\frac{\partial\psi}{\partial s}-\mathcal{A}\psi\right)ds\right.\right.\\ &+ \int_0^t p(s)\left(\psi(L(s),s)\theta(s)+\frac{\partial\psi}{\partial L}\zeta\rho\right)^{\mathsf{T}} d\tilde{w}(s)\\ &+ \left.\left.\int_0^t p(s)\left(\psi(L(s),s)\frac{L(s)}{m}\right)d\tilde{Z}(s)\right\}\right].\end{aligned}$$

Because this relation holds for all $\gamma(t)$ (defined above), we get the Zakai equation

$$\begin{aligned}\text{(A.1)}\quad p(t)(\psi_t) &= \Pi(0)(\psi_0)+\int_0^t p(s)\left(\frac{\partial\psi}{\partial s}-\mathcal{A}\psi\right)ds\\ &+\int_0^t p(s)\left(\psi(L(s),s)\theta(s)+\frac{\partial\psi}{\partial L}\zeta\rho\right)^{\mathsf{T}} d\tilde{w}(s)+\int_0^t p(s)\left(\psi(L(s),s)\frac{L(s)}{m}\right)d\tilde{Z}(s).\end{aligned}$$

Step 3: Un-Normalized Density. We look for a density that solves (A.1), i.e., $q(L,t)$ such that

$$p(t)(\psi_t)=\int q(L,t)\psi(L,t)dL.$$

From (A.1) we get

$$\begin{aligned}\int q(L,t)\psi(L,t)dL &= \int q(L,0)\psi(L,0)dL+\int_0^t\int q(L,s)\left(\frac{\partial\psi}{\partial s}-\mathcal{A}\psi\right)dLds\\ &+ \int_0^t\int q(L,s)\left(\psi(L(s),s)\theta(s)+\frac{\partial\psi}{\partial L}\zeta\rho\right)^{\mathsf{T}} dLd\tilde{w}(s)\\ &+ \int_0^t\int q(L,s)\left(\psi(L(s),s)\frac{L(s)}{m}\right)dLd\tilde{Z}(s).\end{aligned}$$

Using integration by parts in t and L, we get

$$\int\left[dq+\mathcal{A}^*qdt-\left(q\theta(t)-q_L\zeta\rho\right)^{\mathsf{T}}d\tilde{w}-q\tfrac{L}{m}d\tilde{Z}\right]\psi dL=0,$$

where $\mathcal{A}^*$, the adjoint of $\mathcal{A}$, is given as

$$\mathcal{A}^*=\left(I-\frac{1}{2}\zeta^2\right)\frac{\partial}{\partial L}-\frac{1}{2}\zeta^2\frac{\partial^2}{\partial L^2}.$$

This gives the stochastic partial differential equation for the density. $\square$

Appendix A.2: Proof of Theorem 2

We show that equations (4.5)–(4.8) give (4.4) and then, by Lemma 2, we have the solution to the Zakai equation (A.1).

STEP 1: UN-NORMALIZED DENSITY PROCESS. We postulate

$$q(L,t) = \exp\left(-\tfrac{1}{2}\left[\Gamma(t)L^2 - 2v(t)L + b(t)\right]\right), \tag{A.2}$$

where $\Gamma(t)$ is deterministic, and $v(t)$ and $b(t)$ are Ito processes. So, we write

$$dv = v_0 dt + v_1^{\top} d\tilde{w} + v_2 d\widetilde{Z} \quad \text{and} \quad db = b_0 dt + b_1^{\top} d\tilde{w} + b_2 d\widetilde{Z}. \tag{A.3}$$

By Ito's lemma, we get from (A.2) and (A.3),

$$dq = q\left[-\tfrac{1}{2}\dot{\Gamma}L^2 dt + L dv - \tfrac{1}{2}db + \tfrac{1}{2}|Lv_1 - \tfrac{1}{2}b_1|^2 dt + \tfrac{1}{2}\left(Lv_2 - \tfrac{1}{2}b_2\right)^2 dt\right]. \tag{A.4}$$

Note that by (A.2), $q_L = q(-L\Gamma + v)$ and $q_{LL} = q(-L\Gamma + v)^2 - q\Gamma$. Using (A.3) and equating the diffusion terms of (4.4) and (A.4), we get

$$v_1 = \zeta\Gamma\rho, \quad -\tfrac{1}{2}b_1 = \theta - \zeta v\rho, \quad v_2 = 1/m, \quad b_2 = 0. \tag{A.5}$$

By (A.3), (A.5), q_L, and q_{LL}, the drift terms of (4.4) and (A.4) give

$$\begin{aligned} &-\tfrac{1}{2}\dot{\Gamma}L^2 + Lv_0 - \tfrac{1}{2}b_0 + \tfrac{1}{2}\left[\tfrac{L^2}{m^2} + |(L\Gamma - v)\zeta\rho + \theta|^2\right] \\ &= (L\Gamma - v)\left(I - \tfrac{1}{2}\zeta^2\right) + \tfrac{1}{2}\zeta^2\left[(L\Gamma - v)^2 - \Gamma\right]. \end{aligned} \tag{A.6}$$

We select our parameters so that (A.6) holds as well, and then (A.2) leads to (4.4).

STEP 2: VARIANCE. Equating the coefficients of L^2 in (A.6) gives

$$-\dot{\Gamma} + \tfrac{1}{m^2} + \Gamma^2\zeta^2\left(|\rho|^2 - 1\right) = 0.$$

By Lemma 2 and Theorem 2, $q(L,0) = p_0(L) = \frac{1}{\sqrt{2\pi S_0}} e^{-(L-L_0)^2/(2S_0)}$, which with (A.2) gives the initial condition $\Gamma(0) = 1/S(0)$. Thus, by setting $S(t) = 1/\Gamma(t)$, we obtain the following Riccati equation for the variance:

$$S'(t) = -\tfrac{1}{m^2} S^2(t) + \zeta^2 \left(1 - |\rho|^2\right), \ \ S(0) = S_0, \tag{A.7}$$

which gives $m^2 dS(t) / \left[m^2 \zeta^2 \left(1 - |\rho|^2\right)\right] = dt$ if the denominator is nonzero. This gives (4.6). Thus, (4.6) is the solution to the Riccati equation (A.7).

Step 3: Kalman Filter. By equating the multipliers of L on the left- and right-hand sides, we can identify the coefficient v_0 of L in (A.6) to be

$$v_0 = \zeta^2 v \Gamma(|\rho|^2 - 1) + \Gamma\left(I - \tfrac{1}{2}\zeta^2 - \zeta \rho^\intercal \theta\right),$$

which with (A.3) and (A.5) give the dynamics of the process $v(t)$:

$$dv + \left(1 - |\rho|^2\right) \zeta^2 \Gamma v dt = \Gamma\left(I - \tfrac{1}{2}\zeta^2 - \zeta \rho^\intercal \theta\right) dt + \zeta \Gamma \rho^\intercal d\tilde{w} + \tfrac{1}{m} d\widetilde{Z}.$$

Let $\hat{L}(t) = v(t) S(t)$. This selection will help us to write (A.2) as (A.10). By Ito's lemma, we obtain the Kalman filter (4.7). Because $q(L,0) = p_0(L)$ and (A.2) imply $v(0) = L_0/S_0$, we get the initial condition $\hat{L}(0) = L_0$.

Step 4: Conditional Probability Density. Identifying the terms independent of L in (A.6) permits us to compute b_0. This gives

$$b_0 = |\theta|^2 + v^2\zeta^2(|\rho|^2 - 1) + v(2I - \zeta^2) + \zeta^2 \Gamma - 2v\zeta\theta^\intercal \rho.$$

From this and equations (A.3) and (A.5), we get the dynamics of the process $b(t)$:

$$db = \left[|\theta|^2 + v^2\zeta^2\left(|\rho|^2 - 1\right) + v\left(2I - \zeta^2\right) + \zeta^2\Gamma - 2v\zeta\theta^\intercal\rho\right] dt + 2(\zeta v \rho - \theta)^\intercal d\widetilde{w} \tag{A.8}$$

By $q(L,0) = p_0(L)$ and (A.2), we have the initial condition

$$e^{-b(0)/2} = \tfrac{1}{\sqrt{2\pi S_0}} e^{-L_0^2/(2S_0)}. \tag{A.9}$$

Using $\Gamma(t) = 1/S(t)$ and $v(t) = \hat{L}(t)/S(t)$, we can now write (A.2) as follows:

$$q(L,t) = \frac{K(t)}{\sqrt{2\pi S(t)}} \exp\left(-\frac{1}{2S(t)}\left(L - \hat{L}(t)\right)^2\right), \tag{A.10}$$

where $K(t) = \sqrt{2\pi S(t)} e^{\frac{1}{2}\left(-b(t)+\hat{L}^2(t)/S(t)\right)} \equiv e^{\eta(t)}$, i.e., $\eta(t) = \frac{1}{2}\left(-b(t) + \hat{L}^2(t)/S(t)\right) + \log\left(\sqrt{2\pi S(t)}\right)$. Then from (A.9) and the fact that $\hat{L}(0) = L_0$, we get $\eta(0) = 0$. By Ito's lemma and equations (4.7), (A.7), and (A.8), we get

$$d\eta = -\tfrac{1}{2}\left(\hat{L}^2/m^2\right) dt + \theta^{\mathsf{T}} d\tilde{w} + (\hat{L}/m) d\tilde{Z},$$

which gives (4.8). Thus, the un-normalized density in Theorem 2 equals (A.2) and (A.10). Then, by $\Gamma(t)$, $v(t)$, and $b(t)$, equation (4.4) holds. Now using Lemma 2 we get that the density solves the Zakai equation $(A.1)$. □

Appendix A.3: Proof of Lemma 3

By (4.1), we have $d\tilde{w}_Z(t) = (\varepsilon(t)/m)dt + dw_Z(t)$, where $\varepsilon(t) = L(t) - \hat{L}(t)$. In order to solve the distribution of $(w_Z(t), w(t))$ under P, we analyze characteristic function

$$\begin{aligned} \varphi(t) &= E\left[\exp\left(i\int_0^t [\varrho(s) d\tilde{w}_Z(s) + \varsigma(s) dw(s)]\right) |\mathcal{G}_0\right] \\ &= E\left[\exp\left(i\int_0^t [(\varrho(s)\varepsilon(s)/m)dt + \varrho(s) dw_Z(s) + \varsigma(s) dw(s)]\right) |\mathcal{G}_0\right], \end{aligned}$$

where $i = \sqrt{-1}$, and $\varrho(t) \in \mathbb{R}$ and $\varsigma(t) \in \mathbb{R}^n$ are arbitrary deterministic bounded functions. Let us define

$$H(t) = \exp\left(i\int_0^t [(\varrho(s)\varepsilon(s)/m)dt + \varrho(s) dw_Z(s) + \varsigma(s) dw(s)]\right).$$

By Ito's lemma, iterated expectation, and the fact that $\varepsilon(s)$ is independent of $\mathcal{G}_s$, we have

$$\begin{aligned}
\varphi(t) &= E\left[H(0)|\mathcal{G}_0\right] + E\left[\int_0^t dH(s)|\mathcal{G}_0\right] \\
&= \varphi(0) + i\int_0^t \varrho(s)E\left[H(s)\varepsilon(s)/m|\mathcal{G}_0\right]ds + i\int_0^t \varsigma(s)E\left[H(s)dw(s)|\mathcal{G}_0\right] \\
&- \tfrac{1}{2}\int_0^t \left(\varrho^2(s) + |\varsigma(s)|^2\right)E\left[H(s)|\mathcal{G}_0\right]ds \\
&= 1 + i\int_0^t \varrho(s)E\left[\tfrac{H(s)}{m}E\left[\varepsilon(s)|\mathcal{G}_s\right]|\mathcal{G}_0\right]ds - \tfrac{1}{2}\int_0^t \left(\varrho^2(s) + |\varsigma(s)|^2\right)E\left[H(s)|\mathcal{G}_0\right]ds \\
&= 1 - \tfrac{1}{2}\int_0^t \left(\varrho^2(s) + |\varsigma(s)|^2\right)E\left[H(s)|\mathcal{G}_0\right]ds = \exp\left(-\tfrac{1}{2}\int_0^t \left(\varrho^2(s) + |\varsigma(s)|^2\right)ds\right).
\end{aligned}$$

A comparison of this with the characteristic function of the standard $(n+1)$-dimension Wiener process (e.g., Øksendal (1998, Chapter 2)) completes the proof.

□

References

Alchian, A.A., and B. Klein (1973). “On a Correct Measure of Inflation”, *Journal of Money, Credit, and Banking*, **5**: 173-191.

Basak, S., and H. Yan (2007). “Equilibrium Asset Prices and Investor Behavior in the Presence of Money Illusion”, working paper, Yale School of Management.

Bensoussan, B. (2004). “Stochastic Control of Partially Observable Systems. Cambridge University Press, Cambridge, UK.

Borenstein, S., and N.L. Rose (1994). “Competition and Price Dispersion in the U.S. Airline Industry”, *Journal of Political Economy*, **102**: 653-683.

Bradley, R. (2001). “Finite Sample Effects in the Estimation of Substitution Bias in the Consumer Price Index”, *Journal of Official Statistics*, **17**: 369-390.

Brennan, M.J. (1998). “The role of learning in dynamic portfolio decisions”, *European Economic Review*, **1**: 295-306.

Brennan, M.J., and Y. Xia (2002). “Dynamic Asset Allocation under Inflation”, *Journal of Finance*, **57**: 1201-1238.

Campbell, J., and T. Vuolteenaho (2004). “Inflation Illusion and Stock Prices”, *American Economic Review Papers and Proceedings*, **94**: 1923.

Chen, S.-N., and W.T. Moore (1985). “Uncertain Inflation and Optimal Portfolio Selection: A Simplified Approach”, *The Financial Review*, **20**: 343-356.

Cohen, R., C. Polk, and T. Vuolteenaho (2005). “Money Illusion in the stock market: The Modigliani-Cohn hypothesis”, *Quarterly Journal of Economics*, **120**: 639-668.

Farber, H.S. (1978). “Individual Preferences and Union Wage Determination: The case of the United Mine Workers”, *Journal of Political Economy*, **86**: 923942.

Fleming, W.H., and R.W. Rishel (1975). Deterministic and Stochastic Optimal Control. Springer, New York.

Friend, I., and M. Blume (1974). “The Demand for Risky Assets”, *American Economic Review*, **65**: 900-922.

Hansen, L.P., and K.J. Singleton (1983). “Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns”, *Journal of Political Economy*, **91**: 249-265.

Karatzas, I., J.P. Lehoczky, S.P. Sethi, and S.E. Shreve (1986). “Explicit Solution of a General Consumption/Investment Problem,” *Mathematics of Operations Research*, **11** 2: 261-294.

Klenow, P.J. (2003). “Measuring Consumption Growth: The Impact of New and Better Products”, *Federal Reserve Bank of Minneapolis Quarterly Review*, **27**: 10-23.

Leland, H. E. (1968). “Saving and Uncertainty: The Precautionary Demand for Saving”, *Quarterly Journal of Economics*, **82**: 465-473.

Liu, H., and G. Zhou (2006). “Limited Participation, Consumption, and Saving Puzzles: A Simple Explanation”, working paper, Olin School of Business, Washington University in St. Louis.

Manaster, S. (1979). “Real and Nominal Efficient Sets”, *Journal of Finance*, **34**: 93-102.

Merton, R. (1971). “Optimum Consumption and Portfolio Rules in a Continuous Time Model”, *Journal of Economic Theory*, **3**: 373-413.

Mirman, L. (1971). “Uncertainty and Optimal Consumption Decision”, *Econometrica*, **39**: 179-185.

Munk, C., C. Srensen, and T. Nygaard Vinther (2004). “Dynamic Asset Allocation under Mean-reverting Returns, Stochastic Interest Rates and Inflation Uncertainty”, *International Review of Economics & Finance*, **13**: 141-166.

Phelps, E. (1962). “The Accumulation of Risky Capital: A Sequential Analysis”, *Econometrica*, **30**: 729-743.

Øksendal, B. (1998). Stochastic Differential Equations: An Introduction with Applications. Springer, 5^{th} edition, New York.

Rothschild, M., and J. Stiglitz (1971). “Increasing Risk II: Its Economic Consequences”, *Journal of Economic Theory*, **3**: 66-84.

Sandmo, A. (1970). “The Effect of Uncertainty on Saving Decisions”, *Review of Economic Studies*, **37**: 353-360.

Schluter, M.G., and T.D. Mount (1976). “Some Management Objectives of Risk Aversion in the Choice of Cropping Patterns, Surat District, India”, *Journal of Development Studies*, **12**: 246-261.

Sethi, S.P. (1997). Optimal Consumption and Investment with Bankruptcy. Kluwer Academic Publishers, Norwell, MA.

Shapiro, M.D., and D.W. Wilcox (1997). “Alternative Strategies for Aggregating Prices in the CPI”, *Federal Reserve Bank of St. Louis Review*, **79**: 113-125.

Shreve, S.E., and H.M. Soner (1994). “Optimal Investment and Consumption with Transaction Costs”, *Annals of Applied Probability*, **4**: 609-692.

Solnik, B.H. (1978). “Inflation and Optimal Portfolio Choice”, *Journal of Financial and Quantitative Analysis*, **13**: 903-925.

Stiglitz, J. (1970). “A Consumption-Oriented Theory of the Demand for Financial Assets and the Term Structure of Interest Rates”, *Review of Economic Studies*, **37**: 321-351.

Ungar, M., and B.-Z. Zilberhard (1993). “Inflation and Its Unpredictability – Theory and Empirical Evidence”, *Journal of Money, Credit and Banking*, **25**: 709-720.

Xia, L. (2001). “Learning about Predictability: The Effects of Parameter Uncertainty on Dynamic Asset Allocation”, *Journal of Finance*, **56**: 205-246.

Zakai, M. (1969). “On the optimal filtering of diffusion processes”, *Z. Wahrsch. verw. Gebiete*, **11**: 230-243.

Zhao, Y. (2005). “A Dynamic Model of Active Portfolio Management and Mutual Fund Performance Evaluation”, Working paper, Nanyang Business School, Singapore.