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CLOSED GEODESICS
AND STABILITY OF
NEGATIVELY CURVED METRICS
(M, g) closed negatively curved Riemannian manifold

- **Marked length spectrum** \((\mathcal{L}_g)\)
  
  function whose values are lengths of closed geodesics

- **Rigidity**

  \[
  \mathcal{L}_g = \mathcal{L}_{g_0} \implies g = g_0
  \]

- **Stability**

  \[
  \mathcal{L}_g \approx \mathcal{L}_{g_0} \text{ on a finite set} \implies g \approx g_0
  \]
THE MARKED LENGTH SPECTRUM
**Definition:** The *length spectrum* of \((M, g)\) is the set of lengths of all closed geodesics.

Closely related to the Laplace spectrum

“Can you hear the shape of a drum?”
No (Vignéras ‘80, Sunada ‘85)
NEGATIVE CURVATURE

positive curvature

zero curvature

negative curvature
**THE MARKED LENGTH SPECTRUM**

**Setting:** $(M, g)$ closed negatively curved Riemannian manifold, $n = \dim(M)$, $\Gamma = \pi_1(M)$

**Fact:** Every closed curve in $M$ is freely homotopic to a unique closed geodesic

**Definition:**

\[ \mathcal{L}_g : \text{conjugacy class in } \Gamma \rightarrow \text{length of closed geodesic} \]
MARKED LENGTH SPECTRUM
RIGIDITY
Conjecture (Burns–Katok ’85):

If \((M, g)\) and \((N, g_0)\) have \(\pi_1(M) \cong \pi_1(N)\) and \(L_g = L_{g_0}\), then \(g\) is isometric to \(g_0\).

Question: What does \(L_g\) tell us about \(g\)?

\[ L_g = L_{g_0} \implies g = g_0 \]
DYNAMICAL SYSTEM

\[ \mathcal{L}_g = \mathcal{L}_{g_0} \implies g = g_0 \]

\[ \varphi^t : T^1M \to T^1M \text{ geodesic flow} \]

\[ \text{sec}(M) < 0 \implies \varphi^t \text{ is an Anosov flow} \]

periodic orbits of \( \varphi^t \) \hspace{1cm} \text{closed geodesics}

are dense in \( T^1M \)

Anosov closing lemma
DYNAMICAL SYSTEM

$\varphi^t : T^1M \to T^1M$ geodesic flow

$\sec(M) < 0 \implies \varphi^t$ is an Anosov flow

$T(T^1M) \cong \text{flow} \oplus \text{stable} \oplus \text{unstable}$

$\mathcal{L}_g = \mathcal{L}_{g_0} \implies g = g_0$

Anosov closing lemma

periodic orbits of $\varphi^t$ are dense in $T^1M$
DYNAMICAL SYSTEM

\[ \mathcal{L}_g = \mathcal{L}_{g_0} \implies g = g_0 \]

\[ T^1 \tilde{M} \]

\[ \tilde{M} \]

local product structure
**Dynamical System**

\[ \mathcal{L}_g = \mathcal{L}_{g_0} \iff g = g_0 \]

- \( \varphi^t : T^1 M \to T^1 M \) geodesic flow
  - \( \varphi^t v \)
  - \( \sec(M) < 0 \iff \varphi^t \) is an Anosov flow

**Fact:** Let \( \varphi^t, \psi^t \) geodesic flows on \( T^1 M, T^1 N \).

\[ \mathcal{L}_g = \mathcal{L}_{g_0} \implies \varphi^t \) and \( \psi^t \) are conjugate, i.e. there is a homeomorphism \( \mathcal{F} : T^1 M \to T^1 N \) such that

\[ \mathcal{F}(\varphi^t v) = \psi^t \mathcal{F}(v). \]
known answers

\( \mathcal{L}_g = \mathcal{L}_{g_0} \implies g = g_0 \)

- hyperbolic surfaces (constant negative curvature)

\( \mathcal{L}_g = \mathcal{L}_{g_0} \) on a finite set \( \implies g = g_0 \)

\[ \dim \mathcal{T}(S) = 6 \text{ genus}(S) - 6 \]
known answers

\[ \mathcal{L}_g = \mathcal{L}_{g_0} \implies g = g_0 \]

- **dimension 2** (Otal ‘90, Croke ‘91)
- **dimension at least 3, \( g_0 \) locally symmetric** (Hamenstädt ‘97, Besson–Courtois–Gallot ‘95)
- **locally** (Guillarmou–Lefeuvre ‘18)
  
  there exist \( k = k(n) \), \( \varepsilon = \varepsilon(g_0) \) so that if \( \|g - g_0\|_{C^k(M)} < \varepsilon \) and \( \mathcal{L}_g = \mathcal{L}_{g_0} \) then \( g \) is isometric to \( g_0 \)
STABILITY
KEY QUESTION

\[ \mathcal{L}_g = \mathcal{L}_{g_0} \bigg|_{\{y : \mathcal{L}_g(y) \leq L\}} \implies g \approx g_0 \]

\[ \mathcal{L}_g \approx \mathcal{L}_{g_0} \]
SHORT GEODESICS DETERMINE THE MLS APPROXIMATELY

\[ \mathcal{L}_g = \mathcal{L}_{g_0} \bigg|_{\{y \mid \mathcal{L}_g(y) \leq L\}} \implies g \approx g_0 \]

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Theorem (B '22)

Let \((M, g)\) and \((N, g_0)\) have sectional curvatures between \(-\Lambda^2\) and \(-\lambda^2\). There is \(L_0\) so that if \(L \geq L_0\) and

\[ \mathcal{L}_g(\gamma) = \mathcal{L}_{g_0}(\gamma) \]

for all \(\gamma \in \Gamma_L := \{\gamma \in \Gamma \mid \mathcal{L}_g(\gamma) \leq L\}\) then

\[ 1 - CL^{-\alpha} \leq \frac{\mathcal{L}_g(\gamma)}{\mathcal{L}_{g_0}(\gamma)} \leq 1 + CL^{-\alpha} \]

for all \(\gamma \in \Gamma\). The constants \(L_0, C, 0 < \alpha < 1\) depend only on \(n, \Gamma, \lambda, \Lambda, i_N\).
Theorem (B ’22)

Let \((M, g)\) and \((N, g_0)\) have sectional curvatures between \(-\Lambda^2\) and \(-\lambda^2\). There is \(L_0\) so that if \(L \geq L_0\) and

\[
1 - \delta \leq \frac{\mathcal{L}_g(\gamma)}{\mathcal{L}_{g_0}(\gamma)} \leq 1 + \delta
\]

for all \(\gamma \in \Gamma_L := \{\gamma \in \Gamma | \mathcal{L}_g(\gamma) \leq L\}\) then

\[
1 - (CL^{-\alpha} + \delta) \leq \frac{\mathcal{L}_g(\gamma)}{\mathcal{L}_{g_0}(\gamma)} \leq 1 + (CL^{-\alpha} + \delta)
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for all \(\gamma \in \Gamma\). The constants \(L_0, C, 0 < \alpha < 1\) depend only on \(n, \Gamma, \lambda, \Lambda, i_N\).
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APPROXIMATE MLS RIGIDITY

- **Local case** (Guillarmou–Knieper–Lefeuvre ‘19)
  requires $\|g - g_0\|_{C^k}$ small

- **Surfaces**

  **constant curvature** (Thurston ‘98)

  best Lipschitz constant for
  \[
  \varphi : (M, g) \to (N, g_0) \quad \text{homotopic to } f
  \]

  \[
  = \sup_{\gamma \in \pi_1(M)} \frac{\mathcal{L}_{g_0}(f_{\ast}\gamma)}{\mathcal{L}_g(\gamma)}
  \]

  **variable curvature** (B ‘22)

  Lipschitz constant depends continuously on $\mathcal{L}_g/\mathcal{L}_{g_0}$ near 1
**Theorem (B ‘22)**

If \((N, g_0)\) is locally symmetric with \(\dim(N) \geq 3\) and \((M, g)\) satisfies \(-\Lambda^2 \leq \sec(M) < 0\) and

\[
1 - \varepsilon \leq \frac{\mathcal{L}_g}{\mathcal{L}_{g_0}} \leq 1 + \varepsilon
\]

then there is \(F : M \rightarrow N\) and \(C = C(n, \Gamma, \Lambda)\) with

\[
1 - C\varepsilon^{1/8(n+1)} \leq \|dF(v)\| \leq 1 + C\varepsilon^{1/8(n+1)}
\]

for all \(v\) in \(T^1M\).
If the geodesic flow of \((N, g_0)\) has \(C^1\) Anosov splitting, then \(\mathcal{L}_g = \mathcal{L}_{g_0} \implies \text{vol}_g(M) = \text{vol}_{g_0}(N)\).

- **Same volume** (Hamenstädt '97)

- **Isometry** (Besson–Courtois–Gallot '95)
  If \((N, g_0)\) is locally symmetric and \(\dim(N) \geq 3\) then
  \[
  \text{vol}_g(M) = \text{vol}_{g_0}(N) \quad h(g) = h(g_0) \implies (M, g) \text{ isometric to } (N, g_0)
  \]
**APPROXIMATE MLS RIGIDITY**

- **Same volume** (Hamenstädt ‘97)
  
  If the geodesic flow of \((N, g_0)\) has \(C^1\) Anosov splitting, then \(\mathcal{L}_g = \mathcal{L}_{g_0} \implies \text{vol}_g(M) = \text{vol}_{g_0}(N)\)

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  If \((N, g_0)\) is locally symmetric and \(\text{dim}(N) \geq 3\) then
  
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  \text{vol}_g(M) = \text{vol}_{g_0}(N) \implies (M, g) \text{ isometric to } (N, g_0)
  \]
Theorem (B ‘22)

If the geodesic flow of \((N, g_0)\) has \(C^{1+\alpha}\) Anosov splitting and \((M, g)\) satisfies

\[
1 - \varepsilon \leq \frac{\mathcal{L}_g}{\mathcal{L}_{g_0}} \leq 1 + \varepsilon
\]

then there is \(C\) depending on \((\tilde{N}, \tilde{g}_0)\) such that

\[
(1 - C\varepsilon^a)(1 - \varepsilon)^n \leq \frac{\text{vol}_g(M)}{\text{vol}_{g_0}(N)} \leq (1 + C\varepsilon^a)(1 + \varepsilon)^n.
\]

For \(N\) locally symmetric, \((1 \pm C\varepsilon^a)\) becomes \((1 \pm C(n)\varepsilon^2)\)
vol_g(M) is determined by μ(T^1 M), where μ is Liouville measure

dμ = dλ × dt

Liouville current on G˜M
Lebesgue measure on orbits
If the geodesic flow of \((N, g_0)\) has \(C^1\) Anosov splitting, then \(L_g = L_{g_0} \implies \text{vol}_g(M) = \text{vol}_{g_0}(N)\)

**Same volume** (Hamenstädt ’97)

If \((N, g_0)\) is locally symmetric and \(\dim(N) \geq 3\) then
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\]

\[
h(g) = \lim_{T \to \infty} \frac{\log \# \{ \gamma \mid L_g(\gamma) \leq T \}}{T}
\]
**Finiteness** \[ \mathcal{L}_g = \mathcal{L}_{g_0} \big| \{ \gamma \mid \mathcal{L}_g(\gamma) \leq L \} \implies \mathcal{L}_g \approx \mathcal{L}_{g_0} \]

Cover \( T^1M \) with \( m = m(\delta) \) flow boxes.

We say a closed geodesic is *short* if it visits each flow box at most once, and *long* otherwise.

Short geodesics have length at most \( L := m\delta \)

\[ |\mathcal{L}_g(\gamma) - \mathcal{L}_g(\gamma_1) - \mathcal{L}_g(\gamma_2)| < C\delta \]

\[ |\mathcal{L}_{g_0}(\gamma) - \mathcal{L}_{g_0}(\gamma_1) - \mathcal{L}_{g_0}(\gamma_2)| < C'\delta^\beta \]
Theorem (B '22)

Let \((M, g)\) and \((N, g_0)\) have sectional curvatures between \(-\Lambda^2\) and \(-\lambda^2\). There is \(L_0\) so that if \(L \geq L_0\) and

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for all \(\gamma \in \Gamma\). The constants \(L_0, C, 0 < \alpha < 1\) depend only on \(n, \Gamma, \lambda, \Lambda, i_N\).