SUPERCUSPIDAL $L$-packets

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Abstract

Let $F$ be a non-archimedean local field and let $G$ be a connected reductive group defined over $F$. We assume that $G$ splits over a tame extension of $F$ and that the residual characteristic $p$ does not divide the order of the Weyl group. To each discrete Langlands parameter of the Weil group of $F$ into the complex $L$-group of $G$ we associate explicitly a finite set of irreducible supercuspidal representations of $G(F)$, and relate its internal structure to the centralizer of the parameter. We give evidence that this assignment is an explicit realization of the local Langlands correspondence.

CONTENTS

1 Introduction 2

2 Non-singular Deligne-Lusztig packets over finite fields 10
  2.1 The basic bicharacter ........................................... 11
  2.2 The Deligne-Lusztig packet as a torsor ......................... 12
  2.3 Natural intertwining operators .................................. 13
  2.4 Geometric intertwining operators ................................ 16
  2.5 A parameterization of Deligne-Lusztig packets ............... 20

3 Non-singular Deligne-Lusztig packets over local fields 22
  3.1 The basic bicharacter again .................................... 24
  3.2 A parameterization of $\kappa(S,\theta)$ ........................ 25
  3.3 On the existence of normalized intertwining operators ...... 27
  3.4 A parameterization of $\text{Ind}_{\kappa(S,\theta)}$ .................. 29
  3.5 A parameterization of the depth-zero Deligne-Lusztig packet . 31
  3.6 General depth ................................................... 33
  3.7 Remarks on the character formula ............................. 35

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1 Introduction

According to the conjectural local Langlands correspondence, the set of isomorphism classes of irreducible admissible representations of the group $G(F)$ of $F$-points of a connected reductive group $G$ defined over a non-archimedean local field $F$ should be partitioned into finite subsets, called $L$-packets, and each $L$-packet should correspond to a Langlands parameter, which is a homomorphism $W_F \times \text{SL}_2(\mathbb{C}) \to {}^L G$ from the Weil-Deligne group of $F$ into the Langlands $L$-group of $G$, subject to certain conditions. The $L$-packet is expected to be in bijection with the set of representations of a certain finite group computed explicitly in terms of $\varphi$. When the image of the parameter does not lie in a proper parabolic subgroup of $^L G$, the $L$-packet is expected to consist of essentially discrete series representations. When furthermore the parameter restricts trivially to $\text{SL}_2(\mathbb{C})$, the packet is expected to consist of supercuspidal representations – this expectation was formulated in [DR09, §3.5] and is a special case of the the more precise conjecture of [AMS]. We shall call such parameters and packets supercuspidal for short.
In [Kal19] we constructed a correspondence between supercuspidal parameters and supercuspidal L-packets under the following assumptions: \( G \) splits over a tamely ramified extension of \( F \), the residual characteristic \( p \) of \( F \) is not a bad prime for the root system of \( G \), and the Langlands parameter satisfies a certain regularity assumption. The construction works in both directions – from parameters to packets and conversely – and has the important feature of being explicit.

In this paper we extend this construction to the case of arbitrary supercuspidal parameters, i.e. we drop the regularity assumption imposed on the parameters in [Kal19]. We do this at the cost of a slightly stricter assumption on \( p \), which we now require to not divide the order of the Weyl group of \( G \). In fact, when \( p \) is not a bad prime for the root system of \( G \), but possibly divides the order of the Weyl group, the construction given here still works and handles many non-regular supercuspidal parameters, but possibly not all of them. More precisely, we call a Langlands parameter \( \text{torally wild} \) if it maps wild inertia into a torus inside of the dual group. When \( G \) splits over a tame extension and \( p \) does not divide the order of the Weyl group, all supercuspidal Langlands parameters are torally wild. In this paper we construct the L-packets associated to torally wild supercuspidal parameters when \( G \) splits over a tame extension and \( p \) is not a bad prime for \( G \) and does not divide the connection index of any simple factor. This is in particular the case when \( G \) splits over a tame extension and \( p \) does not divide the order of the Weyl group.

The following table gives for each Dynkin type the sets of primes that are bad or divide the connection index in the first row, and those that divide the order of the Weyl group in the second row.

<table>
<thead>
<tr>
<th>( A_n )</th>
<th>( B_n )</th>
<th>( C_n )</th>
<th>( D_n )</th>
<th>( E_6 )</th>
<th>( E_7 )</th>
<th>( E_8 )</th>
<th>( F_4 )</th>
<th>( G_2 )</th>
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<tbody>
<tr>
<td>( p</td>
<td>n + 1 )</td>
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<td>( 2 )</td>
<td>( 2 )</td>
<td>( 2,3 )</td>
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<tr>
<td>( p \leq n + 1 )</td>
<td>( p \leq n )</td>
<td>( p \leq n )</td>
<td>( p \leq n )</td>
<td>( 2,3,5 )</td>
<td>( 2,3,5,7 )</td>
<td>( 2,3,5,7 )</td>
<td>( 2,3 )</td>
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</tr>
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</table>

As in the regular case treated in [Kal19], the construction given here goes in both directions, and is explicit. An essential new phenomenon in the non-regular case is that the group \( S_\varphi \) and its variations \( \pi_0(S_\varphi) \) and \( \pi_0(S_\varphi^+) \) that control the structure of the L-packet are often non-abelian. If one considers non-split groups then this already happens for the inner form of \( SL_2 \) and is a classical example discussed by Läbesse and Langlands [LL79]. But there are also examples for split groups of classical type, such as the split group \( Sp_{10} \). This makes the internal structure of the resulting L-packets considerably more subtle.

Before we describe the complications that arise in the non-regular case and our strategy to handle them, we first review the construction in the regular case. A supercuspidal parameter \( \varphi : W_F \to LG \) is called strongly regular if \( \text{Cent}(\varphi(I_F), \hat{G}) \) is abelian. The notion of regularity is slightly weaker and more complicated to state. From a regular supercuspidal parameter we are able to extract the information necessary to write down a formula for the Harish-Chandra character of those supercuspidal representations that should populate the L-packet. On the other hand, we introduce the notion of a regular supercuspidal representation, classify all such, and give a formula for their Harish-Chandra characters, by reinterpreting the works of Adler, DeBacker, and Spice [AS09], [DS18]. Each of the formulas extracted from a regular supercuspidal parameter then uniquely specifies a regular supercuspidal representation.
Extracting from the Langlands parameter $\varphi$ the information for the Harish-Chandra character is done by showing that $\varphi$ specifies an algebraic torus $S$ defined over $F$, then choosing Langlands-Shelstad $\chi$-data $(\chi_\alpha)_\alpha$ for the root system $R(S,G)$, using $(\chi_\alpha)_\alpha$ to obtain an embedding $\iota : S \to \hat{G}$ through which $\varphi$ factors and gives a Langlands parameter $\varphi_{S,\chi}$ for $S$, and hence a character $\theta_\chi$ of $S(F)$, and then using $(\chi_\alpha)_\alpha$ and $\theta_\chi$ together to write down the character formula. The resulting formula is independent of the choice of $(\chi_\alpha)_\alpha$. It specifies for each embedding $j : S \to G$ a regular supercuspidal representation $\pi_j$ of $G(F)$ and the $L$-packet is the set of these representations. Essential for this procedure is that the regularity of $\varphi$ implies the regularity of $\theta_\chi$.

Our work in the non-regular case begins with the observation that when $\varphi$ is a torally wild supercuspidal parameter then $\text{Cent}(\varphi(I_F), \hat{G})$, while in general not abelian, has abelian connected component. This property is certainly weaker than regularity, but it still allows to obtain from $\varphi$ an algebraic torus $S$ over $F$, and after choosing $\chi$-data also a character $\theta_\chi$ of $S(F)$. Moreover, while $\theta_\chi$ is in general not regular, it is still rather constrained. For example, when $\varphi$ is trivial on wild inertia (i.e. it is of depth zero), then $\theta_\chi$ is non-singular in a sense similar to that defined by Deligne-Lusztig [DL76] in the setting of finite groups of Lie type. For finite groups of Lie type, the Deligne-Lusztig induction of a non-singular character of an elliptic maximal torus is a usually reducible cuspidal representation. Its components were studied by Lusztig [Lus88]. It would be natural to expect that the structure of the corresponding reducible depth zero supercuspidal representations can be related, via the results of Moy and Prasad [MP96], to the structure of the cuspidal representation over the finite field, and that the situation of general depth can be reduced to depth zero using Yu's construction [Yu01] and the Howe factorization process introduced in [Kal19], neither of which assumes regularity.

With these observations made, the road to success seems mapped out. However, once one has set foot on that road one encounters a number of serious and initially rather unexpected challenges. This explains the length of this paper and the fact that two essential technical discussions have been relegated to the auxiliary papers [Kala] and [FKS], and a key argument is borrowed from a third paper [Kalb] that had a rather different purpose there.

The first serious obstacle concerns the depth-zero case. Let $R_\theta$ denote the cuspidal representation of a finite group of Lie type obtained via Deligne-Lusztig induction of a non-singular character $\theta$ of an elliptic maximal torus. Lusztig has shown that $R_\theta$ has multiplicity one – it is a direct sum of pairwise inequivalent irreducible cuspidal representations – and the set of these irreducible factors, which we shall denote by $[R_\theta]$, is acted upon simply transitively by a finite abelian group – the Pontryagin dual of the stabilizer in the Weyl group of the non-singular character $\theta$. Work of Moy and Prasad relates depth-zero supercuspidal representations of a $p$-adic group to cuspidal representations of its reductive parahoric quotients. One would thus optimistically expect that results similar to Lusztig’s hold for the depth-zero supercuspidal representations related to non-singular Deligne-Lusztig representations, and that simple Clifford theory would suffice in describing them. This is not the case. In fact, already the multiplicity one statement fails, although constructing an example takes quite a bit of effort, since the $p$-adic group has to be ramified and cannot be simple, simply connected, or adjoint. The culprit is the difference between the parahoric subgroup $G(F)_{x,0}$ and the stabilizer $G(F)_x$ of the point $x$. Lusztig’s results concern the finite group of Lie type $G(F)_{x,0}/G(F)_{x,0+}$ and by inflation the compact open group $G(F)_{x,0}$, while Moy-Prasad theory classifies depth-
zero supercuspidal representations in terms of irreducible representations of $G(F)_x$. It is in the passage from $G(F)_{x,0}$ to $G(F)_x$ where the complications arise. In order to deal with these complications one needs to have a strong handle on the irreducible pieces of a non-singular Deligne-Lusztig representation. For this we view Deligne-Lusztig induction as a geometric analog of parabolic induction and draw inspiration from the classical theory that decomposes principal series representations in terms of intertwining operators and the $R$-group. A geometric analog of the classical intertwining operators was recently introduced in the work of Bonnafe-Dat-Rouquier [BDR17]. It gives a naturally defined $G$-equivariant isomorphism $H^*_c(Y_{B_1}, \mathbb{Q})_\theta \to H^*_c(Y_{B_2}, \mathbb{Q})_\theta$ between the middle-degree compact cohomology groups of the Deligne-Lusztig varieties associated to two Borel subgroups $B_1$ and $B_2$. This isomorphism can be thought of as the geometric analog of the classical integral intertwining operator between the parabolic inductions from two different Borel subgroups containing the same maximal split torus. Just like the case of the classical integral intertwining operators, these geometric intertwining operators do not compose correctly and need to be renormalized. We are able to derive a result analogous to Arthur’s result [Art89] on the normalization of classical intertwining operators for $p$-adic groups, namely that there exists a normalization, without being able to specify a canonical one. We then prove the analogs of Harish-Chandra’s Commuting Algebra Theorem and Basis Theorem in our setting – if $\pi_{(S, \theta)}$ is the reducible depth zero supercuspidal representation obtained from a non-singular depth zero character $\theta$ of a maximally unramified elliptic maximal torus $S$, then the set of self-intertwining operators on $\pi_{(S, \theta)}$, indexed by the elements of $\Omega(S, G)(F)_\theta/S(F)$, forms a basis of the algebra of $G$-endomorphisms, where $N(S, G)(F)_\theta$ are the elements of $G(F)$ that normalize the pair $(S, \theta)$. This implies that there is a bijection

$$[\pi_{(S, \theta)}] \leftrightarrow \text{Irr}(N(S, G)(F)_\theta, \theta),$$

where on the left side we have the set of irreducible constituents of $\pi_{(S, \theta)}$ and on the right side we consider all irreducible representations of $N(S, G)(F)_\theta$ whose restriction to $S(F)$ is $\theta$-isotypic. Equivalently, the right-hand side is the set of $\theta$-projective representations of $\Omega(S, G)(F)_\theta$. This bijection preserves multiplicities – the multiplicity of an irreducible constituent of $\pi_{(S, \theta)}$ is equal to the dimension of the corresponding representation $\rho$ of $N(S, G)(F)_\theta$, equivalently to the multiplicity of $\theta$ in $\rho|_{S(F)}$. These results stand in remarkable analogy with the classical theory on the decomposition of principal series representations in terms of the $R$-group, even though we are dealing here with the opposite end of the spectrum – elliptic tori and supercuspidal representations.

The problem of finding a canonical normalization of the geometric intertwining operators remains so far unsolved. It is a natural problem – given a connected reductive group over a finite field, an elliptic maximal torus, a non-singular character of that torus, and a Borel subgroup containing that torus, there is a 2-cocycle determined by this data. Its cohomology class is independent of the Borel subgroup. We have proved that this class is trivial. Choosing a normalization of the intertwining operators amounts to choosing a trivialization of this 2-cocycle. Given the naturality of the 2-cocycle, it is to be expected that there will be a natural trivialization. This would lead to a canonification of the bijection (1.1), which depends on the choice of normalization of intertwining operators.

The failure of multiplicity one for the representation $\pi_{(S, \theta)}$ has its reflection on the dual side as well. We recall that when the Langlands parameter $\varphi$ is regular there is a canonical isomorphism $S_\varphi \cong S^\Gamma$ between the centralizer of
the parameter and the Galois-fixed points of the torus dual to $S$. At the same
time, each representation $\pi_{(jS,j\theta)}$ is irreducible, and the $L$-packet $\Pi_\varphi$ is the set
of $\pi_{(jS,j\theta)}$ for all possible embeddings of $j : S \rightarrow G$. The internal structure
of the $L$-packet is then an immediate consequence of Tate-Nakayama duality
which describes the set of embeddings of $S$ into all inner forms of $G$ as a torsor
under the finite abelian group dual to $\pi_0(S^T)$ and its variations. In the non-
regular case the isomorphism $S_\varphi \rightarrow S^T$ is replaced by an exact sequence
\begin{equation}
1 \rightarrow S^T \rightarrow S_\varphi \rightarrow \Omega(S,G)(F)_{\theta} \rightarrow 1,
\end{equation}
where $\Omega(S,G)$ is the absolute Weyl group of the torus $S$. The group $\pi_0(S_\varphi)$ is
often non-abelian, which makes the structure of its irreducible representations
more complicated. At the same time, since the representation $\pi_{(jS,j\theta)}$ is often
reducible, the $L$-packet $\Pi_\varphi$ is now the union of the sets $[\pi_{(jS,j\theta)}]$ of irreducible
constituents of $\pi_{(jS,j\theta)}$, for all possible embeddings of $j : S \rightarrow G$. We refer
to each subset $[\pi_{(jS,j\theta)}]$ of $\Pi_\varphi$ as a Deligne-Lusztig packet. Tate-Nakayama duality
is no longer sufficient to describe the internal structure of the $L$-packet $\Pi_\varphi$, but it reduces it to establishing a bijection between the Deligne-Lusztig packet
$[\pi_{(jS,j\theta)}]$ corresponding to a particular embedding $j : S \rightarrow G$ and the set of
those irreducible representations of $\pi_0(S_\varphi)$ whose restriction to $\pi_0(S^T)$ contains
a specific character $\chi$ related to the embedding of $j$ (this is in the setting of pure
inner forms; a slight generalization is needed for rigid inner forms). It turns out
that the extension (1.2) also doesn’t have multiplicity one. Again this precludes
the use of simple Clifford theory to study its representations. What one needs
is a relationship between the extension
\begin{equation}
1 \rightarrow S(F) \rightarrow N(jS,G)(F)_{j\theta} \rightarrow \frac{N(jS,G)(F)_{j\theta}}{S(F)} \rightarrow 1,
\end{equation}
which encodes the structure of $[\pi_{(jS,j\theta)}]$ according to (1.1), and the extension
\begin{equation}
1 \rightarrow S^T \rightarrow S_\varphi,\chi \rightarrow \Omega(S,G)(F)_{\theta,\chi} \rightarrow 1,
\end{equation}
which encodes the the appropriate representations of $\pi_0(S_\varphi)$. It is easy to see
that the cokernels of these extensions are canonically isomorphic. It turns out,
rather miraculously, that the push-out of the first extension along $\theta : S(F) \rightarrow C^*$ is isomorphic to the push-out of the second extension along $\chi$. The argument is an amplification of an argument used in a different context, namely to
establish the validity of a suitable statement of the local Langlands correspondence
for disconnected groups whose connected component is a torus [Kalb].
It relies on the cohomological pairings for complexes of tori of length 2 [KS99,
Appendix A.3] and their extension to the setting of rigid inner forms. The choice
of isomorphism between the two extensions appears to be related to the
normalization of intertwining operators.

We now discuss the difficulties with positive depth representations. From the parameter $\varphi$ we obtain, as discussed above, the torus $S$ and after choosing $\chi$-data $(\chi_\alpha)_\alpha$ we also obtain the character $\theta_\chi$. In the case when $\varphi$ is regular,
treated in [Kall9, §3], we write the formula
\begin{equation}
e(\varphi) e\left(\frac{1}{2} X^*(T)_C - X^*(S)_C, A\right) \sum_{w \in N(S,G)(F) / S(F)} \Delta_{H}[a,\chi](\gamma^w) \theta_\chi(\gamma^w)
\end{equation}
and use it to select for each embedding $j : S \rightarrow G$ a regular supercuspidal
representation $\pi_j$ whose character on shallow elements of $S(F)$ is given by this
formula. Of course we need to know that such a representation exists. This
uses the material of [Kal19, §3], where a bijection is established between the set of pairs $(S, \theta)$ consisting of a tame elliptic maximal torus $S$ and a regular character $\theta$ and the set of regular supercuspidal representation $\pi(S, \theta)$, as well as the material of [Kal19, §4], where the Adler-DeBacker-Spice character formula of [AS09] and [DS18] is reinterpreted in the case of $\pi(S, \theta)$ as the formula

$$e(G)e(\frac{1}{2} X^* T(T)_C - X^* (S)_C, \lambda) \sum_{w \in N(S(G)(F)/S(F))} \Delta_{T}^{abs}(a, \chi'_{w}) e_{f,\text{ram}}(\gamma_{w}) e_{\text{ram}}(\gamma_{w}) \theta(\gamma_{w}),$$

where now $(\chi'_{w})_{\alpha}$ is $\chi$-data computed in terms of $\theta$, and $\gamma \in S(F)$ is shallow. The two formulas (1.3) and (1.4) look very similar, except for the occurrence of [AS09] and [DS18] is reinterpreted in the case of $\pi(S, \theta)$. The material of [Kal19, §4] applies to this representation, so we have (1.4). It is easy to see that multiplication by $e_{f,\text{ram}}$ preserves non-singularity. But multiplication by $e_{\text{ram}}$ does not. So we cannot simply replace $\theta$ by $\theta' = \theta \cdot e_{f,\text{ram}} \cdot e_{\text{ram}}$ as in the regular case.

Consider now the case when $\theta$ is no longer regular, but is still non-singular. One can combine the material of [Kal19, §3] with the results on non-singular supercuspidal representations of depth zero from this paper to obtain a (usually reducible) positive depth supercuspidal representation $\pi(S, \theta)$. The material of [Kal19, §4] applies to this representation, so we have (1.4). It is easy to see that multiplication by $e_{f,\text{ram}}$ preserves non-singularity. But multiplication by $e_{\text{ram}}$ does not. So we cannot simply replace $\theta$ by $\theta' = \theta \cdot e_{f,\text{ram}} \cdot e_{\text{ram}}$ as in the regular case.

There is a parallel phenomenon on the Galois side. In the depth-zero case, the character $\theta_{\chi}$ obtained from a general supercuspidal parameter $\varphi$ after factoring through an $L$-embedding $L_{j_{\chi}} : L^{S} \rightarrow L^{G}$ constructed from minimally ramified $\chi$-data is non-singular. But in the positive depth case this is no longer true. What is now needed is to quantify the failure of $\theta_{\chi}$ to be non-singular and relate it to $e_{\text{ram}}$.

This is done as follows. A parameter $\varphi$ of positive depth determines a tame twisted Levi subgroup $G^{0}$ of $G$. We can choose a tame $L$-embedding $L_{j_{G^{0},G}} : L^{G^{0}} \rightarrow L^{G}$ and obtain a factorization $\varphi = L_{j_{G^{0},G}} \circ \varphi_{G^{0}}$. The parameter $\varphi_{G^{0}} : W_{F} \rightarrow L^{G^{0}}$ is tamely ramified modulo center and we can apply to it the depth-zero discussion. In particular, we can choose minimally ramified $\chi$-data for $R(S, G^{0})$ (in fact there is only one possible choice), obtain an embedding $L_{j_{S,G^{0}}} : L^{S} \rightarrow L^{G^{0}}$ and a factorization $\varphi_{G^{0}} = L_{j_{S,G^{0}}} \circ \varphi_{S,G^{0}}$. The parameter $\varphi_{S,G^{0}} : W_{F} \rightarrow L^{S}$ leads to the non-singular character $\theta_{\chi,G^{0}}$ of $S(F)$. On the other hand we can choose minimally ramified $\chi$-data for $R(S, G)$, obtain an $L$-embedding $L_{j_{S,G}} : L^{S} \rightarrow L^{G}$, a factorization $\varphi = L_{j_{S,G}} \circ \varphi_{S,G}$ and hence a character $\theta_{\chi,G}$ of $S(F)$. The difference between $\theta_{\chi,G^{0}}$ and $\theta_{\chi,G}$ stems from the difference between $L_{j_{S,G}}$ and $L_{j_{G^{0},G}} \circ L_{j_{S,G^{0}}}$. This difference is measured by a character $\delta$ of $S(F)$, so that $\theta_{\chi,G} = \theta_{\chi,G^{0}} \cdot \delta$. Since $\theta_{\chi,G^{0}}$ is non-singular, the failure of $\theta_{\chi,G}$ to be non-singular is measured by $\delta$.

If we replace $L_{j_{G^{0},G}}$ by a different $L$-embedding then $\delta$ is multiplied by a character of $G^{0}(F)$ and the failure of $\theta_{\chi,G}$ to be non-singular is unaffected. This failure is therefore an invariant of the $\chi$-data for $R(S, G) \leftarrow R(S, G^{0})$. But in order to study it, we do need to make $\delta$, and therefore the choice of the $L$-embedding $L_{j_{G^{0},G}}$, explicit.
This we do in a separate paper [Kala], where we show that one can apply the Langlands-Shelstad $\chi$-data construction in a relative setting, namely to a set $R(G^0_{\text{rad}}, G)$ closely related to the set of weights for the action of $Z(\hat{G}^0)$ on $\mathfrak{g}$. We furthermore give a recipe that produces $\chi$-data for $R(S, G)$ from $\chi$-data for $R(G^0_{\text{rad}}, G)$ and $R(S, G^0)$. In that case, we prove that

$$L_j S, G = L_j G^0, G \circ L_j S, G^0.$$  \hfill (1.5)

This equation seems to suggest that the character $\delta$ above is trivial, and that therefore $\theta_{\chi, G}$ should be non-singular. But there is a subtle point here: The $\chi$-data on $R(S, G)$ produced by combining $\chi$-data for $R(G^0_{\text{rad}}, G)$ and $R(S, G^0)$ that are both minimally ramified, may fail to be itself minimally ramified. There is a canonical minimally ramified $\chi$-data associated to it, and the discrepancy between the two versions of $L_j S, G$ is measured by the character $\delta$. In [Kala, §5.4] we derive an explicit formula for this character, which turns out to be independent of the choices of $\chi$-data, and to only depend on the triple $S, G^0, G$.

Now that $\delta$ has been quantified, one can ask for its relationship with $e_{\text{ram}}$. It turns out that in general the two characters $\delta$ and $e_{\text{ram}}$ are not equal. However, their product always extends to a character of $G^0(F)$. This is proved in [FKS]. Therefore, if $\theta$ is non-singular, then so is $\theta'' = \theta \cdot e_{\text{ram}} \cdot e_{\text{ram}} \cdot \delta$. As a consequence we see that the formula (1.3) does specify a non-singular (reducible) supercuspidal representation. Indeed, if we choose $(\chi'_\alpha)_\alpha$ to be minimal, while $(\chi_\alpha)_\alpha$ to be obtained from minimal data for $R(G^0_{\text{rad}}, G)$ and $R(S, G^0)$, then $\theta_{\chi}$ will be non-singular and equal to $\delta \cdot \theta_{\chi'}$, while the formula (1.4) applied to $\pi(\theta_{\chi'})$ will read

$$e(G) e \left( \frac{1}{2}, X^*(T)_C - X^*(S)_C, \Lambda \right) \sum_{\gamma \in \Omega(S, G)(F)/S(F)} \Delta_{\text{abs}}^H \left[ a, \chi \right](\gamma^w) \theta''_{\chi}(\gamma^w).$$

We now discuss the structure of this paper. In Section 2 we consider a connected reductive group defined over a finite field $k$, an elliptic maximal torus $S \subset G$, and a non-singular character $\theta : S(k) \to \bar{\mathbb{Q}}_l^\times$ in the sense of Deligne-Lusztig. Let $N(S, G)(k)_\theta$ resp. $\Omega(S, G)(k)_\theta$ be the stabilizers of $\theta$ in the $k$-points of the normalizer of $S$ in $G$ resp. the $k$-points of the Weyl group. In Subsection 2.2 we review Lusztig’s results that the Deligne-Lusztig virtual character $R^S_{\theta}$, which in this case is a cuspidal representation of $G(k)$, has multiplicity one and the set of its irreducible components receives a natural simply transitive action of the Pontryagin dual of the abelian group $\Omega(S, G)(k)_\theta$. In Subsection 2.3 we define the concept of a natural intertwining operator $H^S_\chi(Y_{B_1}, \bar{\mathbb{Q}}_l)_\theta \to H^S_\chi(Y_{B_2}, \bar{\mathbb{Q}}_l)_\theta$ between the $\theta$-isotypic components of the middle degree compact cohomologies of the Deligne-Lusztig varieties associated to two Borel subgroups $B_1$ and $B_2$ containing $S$. Such an operator is well-defined up to a scalar. This definition is elementary, but it allows us to study the existence of normalized collections of such operators. We are especially interested in normalized collections that are equivariant with respect to the action of the group of automorphisms of $G$ that preserve $S$ and $\theta$, or some subgroup thereof. In Subsection 2.4 we review the work of Bonnafé-Dat-Rouquier [BDR17], which gives rise to an equivariant collection of natural intertwining operators, which is however not normalized. In Subsection 2.5 we prove that any normalized collection of natural intertwining operators that is equivariant with respect to the action of $N(S, G)(k)$ provides a bijection

$$[R^S_{\theta}] \leftrightarrow \text{Irr}(N(S, G)(k)_\theta, \theta)$$ \hfill (1.6)
between the set of irreducible constituents of $R^\theta_\theta$ and the set of representations of $N(S,G)(k)_\theta$ whose restriction to $S(k)$ is $\theta$-isotypic. This is the finite field analog of the bijection (1.1). We do this by obtaining from the operators $\mathcal{H}_\epsilon^\theta(Y_{B_1}, \mathbb{Q}_l)_\theta \rightarrow \mathcal{H}_\epsilon^\theta(Y_{B_2}, \mathbb{Q}_l)_\theta$ a collection of self-intertwining operators of $\mathcal{H}_\epsilon^\theta(Y_{B_1}, \mathbb{Q}_l)_\theta$ indexed by elements of $N(S,G)(k)_\theta$, which gives us an action of $N(S,G)(k)_\theta$ on $\mathcal{H}_\epsilon^\theta(Y_{B_1}, \mathbb{Q}_l)_\theta$ that extends the action of $S(k)$ on the right given by $\theta$. We then decompose $\mathcal{H}_\epsilon^\theta(Y_{B_1}, \mathbb{Q}_l)_\theta$ under the action of $N(S,G)(k)_\theta$.

Lusztig’s multiplicity one result is reflected in the structure of $N(S,G)(k)_\theta$ as follows: We show in Subsection 2.2 that the character $\theta$ extends (non-canonically) to a character of $N(S,G)(k)_\theta$. Therefore the set of representations (a-fortiori characters) of $N(S,G)(k)_\theta$ whose restriction to $S(k)$ is $\theta$-isotypic is a torsor for $\Omega(S,G)(k)_\theta^\theta$. Thus the existence of a bijection (1.6) follows quite readily from Lusztig’s results. The additional information we obtain by considering intertwining operators is that specifying a normalized collection of intertwining operators is equivalent to specifying a bijection (1.6). This information turns out to be crucial for the relationship between the finite field and the $p$-adic field.

In Section 3 we consider a connected reductive group defined over a non-archimedean local field $F$, an elliptic maximally unramified maximal torus $S \subset G$, and a depth zero character $\theta : S(F) \rightarrow \mathbb{C}^\times$ that is non-singular in the sense of Definition 3.0.1. This definition implies that (but is stronger than) the character $\theta$ of $S(k) = S(F)_{0,0+}$ obtained from $\bar{\theta}$ is non-singular with respect to the finite group of Lie type $G_\theta^\theta(k) = G(F)_0,0,\theta$ in the sense of Deligne-Lusztig, where $x$ is the point associated to $S$. Therefore we obtain the cuspidal representation $R^\theta_\theta$ of $G(k)$ and denote by $\kappa_{(S,\bar{\theta})}$ its inflation to $G(F)_0,\theta$. The discussion over the finite field $k$ implies a bijection between $[\kappa_{(S,\bar{\theta})}]$ and $\text{Irr}(N(S,G,F)_0,\theta,\theta)$. In Subsection 3.2 we extend $\kappa_{(S,\bar{\theta})}$ to a representation $\kappa_{(S,\theta)}$ of $S(F)G(F)_0,\theta$. This extension is performed in the same way as in the regular case [Kal19, §3.4.4]. It is possible that distinct irreducible constituents of $\kappa_{(S,\theta)}$ might fuse together to a single irreducible constituent of $\kappa_{(S,\bar{\theta})}$. This turns out to be governed by a bi-character introduced in Subsection 3.1. Using it and the discussion over the finite field we obtain a bijection between $[\kappa_{(S,\theta)}]$ and $\text{Irr}(N(S,F)G(F),0,\theta,\theta)$.

Earlier in the introduction we mentioned the difficult combinatorics exhibited by the irreducible representations when passing from $G(F)_0,\theta$ to $G(F)$. There is the intermediate subgroup $S(F)G(F)_0,\theta$. We have just handled the passage from $G(F)_0,\theta$ to $S(F)G(F)_0,\theta$, but the actual difficulties are encountered in the passage from $S(F)G(F)_0,\theta$ to $G(F)_0,\theta$. In order to handle them we need a collection of normalized intertwining operators for the finite-field situation ($G_0^\circ, S, \bar{\theta}$) that is equivariant for $N(S,G,F)_0,\theta$. So far we have used a collection equivariant for $N(S,G,F)_0,\theta$, whose existence follows directly from Lusztig’s results. The group $N(S,G,F)_0,\theta$ acts on $G_\theta^\circ$ preserving $S$ and $\theta$, but it can act by outer automorphisms of $G_\theta^\circ$. The existence of a collection equivariant for $N(S,G,F)_0,\theta$ is thus not automatic. It is proved in Subsection 3.3 using a devisage argument that is partly already prepared in Subsection 2.3 and ultimately boils down to some elementary but detailed analysis in the case of root systems of type $D_{2n}$. Once the intertwining operators are available, we establish in Subsection 3.4 the multiplicity preserving bijection between $[\text{Ind}_{S(F)G(F)}^{G(F)}]_{\kappa_{(S,\theta)}}$ and $\text{Irr}(N(S,F,G,F)_0,\theta,\theta)$. The results of Moy and Prasad then immediately translate this bijection to the desired bijection (1.1). This is done in Subsection 3.5. In Subsection 3.6 we combine these results with the Howe factorization algorithm of [Kal19, §3.6] and Yu’s construction to obtain a multiplicity preserving.
bijection between $[\pi_{(S,\theta)}]$ and $\text{Irr}(N(S,G)(F)_{\theta},\theta)$ in the case where $S \subset G$ is a tame elliptic maximal torus and $\theta : S(F) \to \mathbb{C}^\times$ is a positive depth character that is non-singular.

The construction of $L$-packets is the subject of Section 4. In Subsection 4.1 we discuss how to extract from a supercuspidal parameter $\varphi$ and $\chi$-data a tame torus $S$ and a character $\theta_\chi$ of $S(F)$. The arguments are similar to [Kal19, §5.2], but we have to pay attention to the subtleties introduced by non-singularity that we discussed above. The construction of the $L$-packets takes place in Subsection 4.2. We then proceed with the study of their internal structure. In Subsection 4.3 we give an example that $S_\varphi$ can be finite and non-abelian even for a classical root system of type $B_4$, establish the exact sequence (1.2), show that it has multiplicity one when $\varphi$ has depth zero and $G$ is unramified or simply connected, and then give an example where it does not have multiplicity one. In Subsection 4.4 we use Tate-Nakayama duality to reduce the internal structure of the $L$-packet to that of a single Deligne-Lusztig packet of depth zero. That latter case is dealt with in Subsection 4.5. In the final Subsection 4.6 we sketch an argument showing that the $L$-packet with this internal parameterization satisfies stability and some cases of endoscopic transfer. The details of this argument and its generalization to all cases of endoscopic transfer will be the subject of a forthcoming paper.

There are a number of appendices to this paper containing information of more technical nature. Some of them review, and possibly extend, known material in a way convenient for our purposes. Such are §A containing an overview of basic Clifford theory, §B containing an abstract version of the Harish-Chandra basis and commuting algebra theorems, §C containing a discussion of representations of extensions with abelian quotients that may fail the multiplicity one property, §D describing the behaviour of Deligne-Lusztig induction under homomorphisms of algebraic groups with abelian kernel and cokernel. Others contain results about Bruhat-Tits theory, such as the compatibility of parahoric subgroups with restriction of scalars §F, or the concept of absolutely special vertices §G generalizing the concept of hyperspecial vertices, as well as that of superspecial vertices of [Kal19, Definition 3.4.8]. In §H we extend to the case of ramified groups the results of [DR09, §6.1] about genericity of depth zero supercuspidal representations. Finally, §I contains technical results about the root system $D_{2n}$.

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## 2 Non-singular Deligne-Lusztig packets over finite fields

Let $G$ be a connected reductive group defined over a finite field $k$, $S \subset G$ an elliptic maximal torus, $\theta : S(k) \to \mathbb{Q}_l^\times$ a character. Assume that $\theta$ is non-singular, in the sense of [DL76, Definition 5.15]. Recall [Kal19, Lemma 3.4.14] that this is equivalent to demanding that for each $\alpha \in R(S,G)$ the character $\theta \circ N \circ \alpha^\vee$ of $(k')^\times$ is non-trivial, where $k'/k$ is some finite extension splitting $S$ and $N : S(k') \to S(k)$ is the norm map.

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Let $R^G_S \theta$ be the virtual representation of $G(k)$ obtained from $(S, \theta)$ via Deligne-Lusztig induction. Let $\sigma(G)$ and $\sigma(S)$ be the $k$-ranks of $G$ and $S$, respectively. According to [DL76, Theorem 8.3], $R_\theta := (-1)^{\sigma(G)-\sigma(S)} R^G_S \theta$ is an actual cuspidal representation of $G(k)$. The representation $R_\theta$ need not be irreducible.

Let $N(S, G)(k)_\theta$ and $\Omega(S, G)(k)_\theta$ be the stabilizers of $\theta$ in the normalizer of $S$ in $G$, and the Weyl group, respectively. In this section we will review results of Lusztig implying that $\Omega(S, G)(k)_\theta$ is abelian and that there is a natural simply transitive action of its character group $\Omega(S, G)(k)_\theta^*$ on the set $[R_\theta]$ of irreducible constituents of $R_\theta$. We will then refine this action to a bijection between the irreducible constituents of $R_\theta$ and the extensions of $\theta$ to $N(S, G)(k)_\theta$, employing recent results of Bonnafé-Dat-Rouquier. This bijection will depend on a choice of normalization of certain intertwining operators. It will be convenient to refer to $[R_\theta]$ as a non-singular Deligne-Lusztig packet.

### 2.1 The basic bicharacter

Recall from [Lus88] the notion of a regular embedding. It is an embedding $G \rightarrow G'$ of connected reductive groups defined over $k$ that induces an isomorphism on the level of adjoint groups, and such that $G'$ has connected center. We will identify $G'_\text{ad}$ with $G_\text{ad}$. By Lang’s theorem $G'(k) \rightarrow G_\text{ad}(k)$ is surjective. There is a unique maximal torus $S' \subset G'$ containing $S$, and again the natural map $S'(k) \rightarrow S_\text{ad}(k)$ is surjective.

**Lemma 2.1.1.** 1. Let $w \in \Omega(S, G)(k)_\theta$ and $s_\text{ad} \in S_\text{ad}(k)$. Choose a lift $s_\text{sc} \in S_\text{sc}(k)$. Then the element $ws_\text{sc}w^{-1}s_\text{sc}^{-1} \in S_\text{sc}(k)$ belongs to $S_\text{sc}(k)$ and is independent of the choice of $s_\text{sc}$.

2. The map

$$\Omega(S, G)(k)_\theta \times \text{cok}(S(k) \rightarrow S_\text{ad}(k)) \rightarrow \tilde{Q}_l^\times, \quad (w, s_\text{ad}) \mapsto \theta(ws_\text{sc}w^{-1}s_\text{sc}^{-1})$$

is well-defined and bi-multiplicative.

3. Its left kernel is trivial, i.e. the induced map

$$\Omega(S, G)(k)_\theta \rightarrow \text{cok}(S(k) \rightarrow S_\text{ad}(k))^*$$

is injective.

4. If $G \rightarrow G'$ is a regular embedding and $\theta' : S'(k) \rightarrow \tilde{Q}_l^\times$ any extension of $\theta$, then the map

$$\Omega(S, G)(k)_\theta \times \text{cok}(S(k) \rightarrow S'(k)) \rightarrow \tilde{Q}_l^\times, \quad (w, s') \mapsto \theta'(ws'w^{-1}s'^{-1})$$

is well-defined and bi-multiplicative, and equals the composition of the above pairing with the natural map $\text{cok}(S(k) \rightarrow S'(k)) \rightarrow \text{cok}(S(k) \rightarrow S_\text{ad}(k))$.

**Proof.** There is $z \in Z(G_\text{sc})$ s.t. $F(s_\text{sc}) = z_\text{sc} \cdot s_\text{sc}$, where $F$ denotes the Frobenius endomorphism. Hence $ws_\text{sc}w^{-1}s_\text{sc}^{-1} \in S_\text{sc}(k)$. The independence of the choice of $s_\text{sc}$ is immediate. If $s_\text{ad}$ is the image of $s \in S(k)$, then the image of $ws_\text{sc}w^{-1}s_\text{sc}^{-1}$ under $S_\text{sc}(k) \rightarrow S(k)$ equals $wsw^{-1}s^{-1}$, which lies in the kernel of $\theta$. Multiplicativity in $s_\text{ad}$ is obvious. Multiplicativity in $\Omega(S, G)(k)_\theta$ follows from $uws_\text{sc}w^{-1}u^{-1}s_\text{sc}^{-1} = u(ws_\text{sc}w^{-1}s_\text{sc}^{-1})u^{-1}$ and the fact that $u$ fixes $\theta$. 

11
The fact that \((w, s') \rightarrow \theta'(ws'w^{-1}s'^{-1})\) is well-defined is immediate. Given \(s' \in S'(k)\) let \(s_{ad} \in S_{ad}(k)\) be its image and choose a lift \(s_{sc} \in S_{sc}(k)\) of \(s_{ad}\). Then \(ws'w^{-1}s'^{-1}\) belongs to \(S(k)\) and equals the image of \(w_{sc}s_{sc}w^{-1}s_{sc}^{-1}\) in \(S(k)\).

To prove that the left kernel of (either) pairing is trivial, we observe that since \(\theta\) is non-singular, so is \(\theta'\), but then [DL76, Proposition 5.16] implies that \(\theta'\) is in general position. If \(w\) is such that \(\theta(w_{sc}w^{-1}s_{sc}^{-1}) = 1\) for all \(s_{ad} \in S_{ad}(k)\), then \(\theta'(ws'w^{-1}s'^{-1}) = 1\) for all \(s' \in S'(k)\), but then \(w = 1\).

**Corollary 2.1.2.** The group \(\Omega(S, G)(k)_{\theta}\) is abelian.

### 2.2 The Deligne-Lusztig packet as a torsor

We review here the main results of [Lus88] in our special case, but formulate them without reference to the dual group of \(G\). A crucial technical result is the following.

**Theorem 2.2.1** (Lusztig, [Lus88]). The representation \(R_{\theta}\) is multiplicity free.

The conjugation action of \(G_{ad}(k)\) on \(G(k)\) induces an action of \(\text{cok}(G(k) \rightarrow G_{ad}(k))\) on the set of isomorphism classes of irreducible representations of \(G(k)\). The embedding \(S \rightarrow G\) induces an isomorphism of groups \(\text{cok}(S(k) \rightarrow S_{ad}(k)) \rightarrow \text{cok}(G(k) \rightarrow G_{ad}(k))\), and since conjugation by \(S_{ad}(k)\) preserves the pair \((S, \theta)\), the action of \(\text{cok}(G(k) \rightarrow G_{ad}(k))\) preserves the set \([R_{\theta}]\).

The dual of (2.2) is a natural surjective map of abelian groups

\[
\text{cok}(G(k) \rightarrow G_{ad}(k)) = \text{cok}(S(k) \rightarrow S_{ad}(k)) \rightarrow \Omega(S, G)(k)^*_{\theta}. \tag{2.3}
\]

**Lemma 2.2.2.** The natural action of \(\text{cok}(G(k) \rightarrow G_{ad}(k))\) on the Deligne-Lusztig packet \([R_{\theta}]\) factors through a simply transitive action of \(\Omega(S, G)(k)^*_{\theta}\).

**Proof.** Consider again a regular embedding \(G \rightarrow G'\), let \(S' \subset G'\) be the maximal torus containing \(S\), and let \(\theta' : S'(k) \rightarrow \hat{Q}_{\theta}^{\times}\) be an extension of \(\theta\). Recall from Appendix D that there is a natural isomorphism \(R_{\theta} \rightarrow R_{\theta'}\) that intertwines the embedding \(G(k) \rightarrow G'(k)\). In other words, the representation \(R_{\theta}\) of \(G(k)\) is the restriction of the representation \(R_{\theta'}\) of \(G'(k)\). Since \(\theta'\) is in general position, the latter is irreducible. We can thus apply Clifford theory to the exact sequence

\[
1 \rightarrow G(k) \rightarrow G'(k) \rightarrow S'(k)/S(k) \rightarrow 1,
\]

bearing in mind Theorem 2.2.1. By Lemma A.11 the action of \(S'(k)/S(k)\) on \(G(k)\) induces a transitive action on the set of irreducible constituents of \(R_{\theta}\), whose kernel is the annihilator of the subgroup of \([S'(k)/S(k)]^*\) that stabilizes \(R_{\theta'}\). To see what the latter is, let \(\delta : S'(k)/S(k) \rightarrow \hat{Q}_{\theta'}^{\times}\). By Appendix D we have \(\delta \otimes R_{\theta'} = R_{\theta'}\). By [DL76, Proposition 5.26 and Corollary 6.3] this is isomorphic to \(R_{\theta'}\) if and only if the characters \(\theta'\) and \(\delta\theta'\) are conjugate under \(\Omega(S', G')(k)\). That is, if and only if there exists \(w \in \Omega(S', G')(k) = \Omega(S, G)(k)\) such that \(\delta = w\theta' \cdot \theta'^{-1}\). The stabilizer of \(R_{\theta'}\) in \([S'(k)/S(k)]^*\) is thus the image of \(\Omega(S, G)(k)_{\theta}\) under (2.2). Therefore the kernel of the action of \([S'(k)/S(k)]\) on \([R_{\theta}]\) is precisely the kernel of (2.3).

We thus see that the set \([R_{\theta}]\) is a torsor for the finite abelian group \(\Omega(S, G)(k)_{\theta}\). For our applications it will be important to reinterpret this torsor using the group \(N(S, G)(k)_{\theta}\).
Proposition 2.2.3. The character $\theta$ extends to the group $N(S, G)(k)_\theta$.

Proof. According to Lang’s theorem we have $\Omega(S, G)(k) = N(S_{sc}, G_{sc})(k)/S_{sc}(k)$ and this implies $N(S, G)(k) = N(S_{sc}, G_{sc})(k) \cdot S(k)$. It is thus enough to show that $\theta|_{S_{sc}(k)}$ extends to $N(S_{sc}, G_{sc})(k)_\theta$. Since the latter is a subgroup of the stabilizer of $\theta|_{S_{sc}(k)}$ in $N(S_{sc}, G_{sc})(k)$, we may as well assume that $G = G_{sc}$. Then $G$ is a product of $k$-simple factors and we may deal with each factor individually and then take the product of the extensions. Thus we may assume that $G$ is $k$-simple. Then $G = Res_{k'/k} G'$, with $G'$ an absolutely simple simply connected group defined over a finite extension $k'$ of $k$. Since $S$ is defined over $k$, it is of the form $S = Res_{k'/k} S'$ with $S' \subset G'$ an elliptic maximal torus defined over $k'$. If $G'$ is of type other than $D_{2n}^{(1)}$, the group $\Omega(S, G)(k)_\theta$ is cyclic, hence $H^2(\Omega(S, G)(k)_\theta, \mathbb{Q}_l^\times) = 1$ and the extendibility of $\theta$ follows from Lemma A.11. If $G'$ is of type $D_{2n}^{(1)}$ then $\Omega(S, G)(k)_\theta$ is one of $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$. In the first two cases the extendibility of $\theta$ follows by the same argument, while in the third case it follows from Lemma A.11 together with Lemma I.2. \(\square\)

We thus obtain a second torsor for the group $\Omega(S, G)(k)_\theta$, namely the set of extensions of $\theta$ to $N(S, G)(k)_\theta$. This set is thus in non-canonical bijection with $[R_{\theta}]$. In the following we shall discuss what it takes to specify such a bijection.

2.3 Natural intertwining operators

Recall that a specific representation of $G(k)$ in the isomorphism class of $R_{\theta}$ is obtained as follows. Let $F$ be the Frobenius endomorphism of $G$. Let $U \subset G$ be the unipotent radical of a Borel subgroup of $G$, defined over $k$, and containing $S$. The corresponding Deligne-Lusztig variety

$$Y_U = \{gU \in G/U | g^{-1} F(g) \in U \cdot FU\}$$

receives an action of $G(k)$ by left multiplication and of $S(k)$ by right multiplication. The $l$-adic cohomology group $H^*_c(Y_U, \mathbb{Q}_l)$ inherits both actions, and we may consider the $\theta$-isotypic component for the right action of $S(k)$

$$H^*_c(Y_U, \mathbb{Q}_l)_\theta = \{v \in H^*_c(Y_U, \mathbb{Q}_l) | vs = \theta(s)s \in S(k)\}.$$ 

According to [DL76, Corollary 9.9] and [He08] this component vanishes for all but one $i$, namely $i = d(U, FU)$, where $d(U, FU)$ denotes the number of root hyperplanes separating the Weyl chambers of $U$ and $FU$ respectively, and is also equal to the dimension of the variety $X_U = Y_U/S(k)$. We have $R_{\theta} = H^{d_U}_c(Y_U, \mathbb{Q}_l)_\theta$ for $d_U = d(U, FU)$.

Let $V$ be the unipotent radical of another Borel subgroup containing $S$. Then $H^{d_V}_c(Y_V, \mathbb{Q}_l)_\theta$ is another model for $R_{\theta}$. Thus there exists a $G(k)$-equivariant isomorphism $H^{d_U}_c(Y_U, \mathbb{Q}_l)_\theta \to H^{d_V}_c(Y_V, \mathbb{Q}_l)_\theta$. By Theorem 2.2.1, the set of all $G(k)$-equivariant morphisms $H^{d_U}_c(Y_U, \mathbb{Q}_l)_\theta \to H^{d_V}_c(Y_V, \mathbb{Q}_l)_\theta$ is a $\mathbb{Q}_l$-vector space of dimension $\# [R_{\theta}]$. Our first observation is that it has a distinguished line.

Let $G \to G'$ be a regular embedding, $S' \subset G'$ the maximal torus containing $S$, and $\theta' : S'(k) \to \mathbb{Q}_l^\times$ an extension of $\theta$. As recalled in Appendix D, we have a natural isomorphism $H^{d_U}_c(Y_U^G, \mathbb{Q}_l)_\theta \to H^{d_U}_c(Y_U^{G'}, \mathbb{Q}_l)_{\theta'}$ of $\mathbb{Q}_l$-vector spaces that intertwines the inclusion $G(k) \to G'(k)$. In this way, we obtain an action
of $G'(k)$ on $H_{c}^{d_{u}}(Y_{U}, \mathbb{Q}_{l})_{\theta}$ that extends the action of $G(k)$. The same is true for $H_{c}^{d_{v}}(Y_{V}, \mathbb{Q}_{l})_{\theta}$. Since $\theta'$ is in general position, these representations of $G'(k)$ are irreducible. Being isomorphic, the set of $G'(k)$-equivariant isomorphisms between them is a 1-dimensional $\mathbb{Q}_{l}$-vector space.

**Definition 2.3.1.** A natural intertwining operator $H_{c}^{d_{u}}(Y_{U}, \mathbb{Q}_{l})_{\theta} \rightarrow H_{c}^{d_{v}}(Y_{V}, \mathbb{Q}_{l})_{\theta}$ is $G'(k)$-equivariant linear map.

**Lemma 2.3.2.** This notion is independent of the choices of $G \rightarrow G'$ and $\theta'$.

**Proof.** Another choice of $\theta'$ is of the form $\theta' \cdot \delta$ for a character $\delta : S'(k)/S(k) \rightarrow \mathbb{Q}_{l}^{\times}$. Let $Y'_{U}$ be the Deligne-Lusztig variety for $G'$. As reviewed in Appendix D the action of $G'(k)$ on $H_{d_{v}}^{d_{u}}(Y_{U}, \mathbb{Q}_{l})_{\theta}$ obtained from $\theta' \cdot \delta$ is equal to the twist by $\delta$ of the action obtained from $\theta'$, where now $\delta$ is viewed as a character of $G'(k)/G(k)$. It is now clear that if a linear map $H_{c}^{d_{u}}(Y_{U}, \mathbb{Q}_{l})_{\theta} \rightarrow H_{c}^{d_{v}}(Y_{V}, \mathbb{Q}_{l})_{\theta}$ is equivariant for the $G'(k)$-actions obtained from $\theta'$, then it is also equivariant for the $G'(k)$-actions obtained from $\theta' \cdot \delta$. The independence from the choice of $G \rightarrow G'$ follows from [Lus88, Lemma 7.1].

**Corollary 2.3.3.** The composition of two natural intertwining operators is again a natural intertwining operator.

**Definition 2.3.4.** Assume given a subset $X$ of the set of unipotent radicals of Borel subgroups containing $S$.

1. A collection of natural intertwining operators on $X$ consist of a non-zero natural intertwining operator $\Phi_{V,U} : H_{c}^{d_{u}}(Y_{U}, \mathbb{Q}_{l})_{\theta} \rightarrow H_{c}^{d_{v}}(Y_{V}, \mathbb{Q}_{l})_{\theta}$ for any $U, V \in X$, and with the convention $\Phi_{U,U} = id$.

2. The collection $\Phi$ is called normalized if $\Phi_{U_{3},U_{2}} \circ \Phi_{U_{2},U_{1}} = \Phi_{U_{3},U_{1}}$ for all $U_{1}, U_{2}, U_{3} \in X$.

3. Let $\Gamma$ be a group acting on $G$ by automorphisms and preserving $S, \theta$, and $X$. The collection $\Phi$ is called $\Gamma$-equivariant if $\gamma \circ \Phi_{U,V} \circ \gamma^{-1} = \Phi_{\gamma(U),\gamma(V)}$ for all $\gamma \in \Gamma$.

4. The collection $\Phi$ is called coherent, if it is $\Gamma$-equivariant, its restriction to any $\Gamma$-orbit is normalized, and for all $U, V \in X$ and $\gamma \in \Gamma$ we have $\Phi_{\gamma(U),\gamma(V),U} = \Phi_{\gamma(V),V} \circ \Phi_{V,U}$.

We shall now investigate the question whether, for a given $\Gamma$, normalized $\Gamma$-equivariant collections exist. Corollary 2.3.3 allows us to measure the failure of a collection $\Phi$ to be normalized by the following function of $(U_{1}, U_{2}, U_{3}) \in X^{3}$

$$
\eta_{\Phi}(U_{1}, U_{2}, U_{3}) = \Phi_{U_{1},U_{2}} \circ \Phi_{U_{2},U_{3}} \circ \Phi_{U_{1},U_{3}}^{-1} \in \mathbb{Q}_{l}^{\times}.
$$

(2.4)

Any other collection is of the form $\epsilon \Phi$ for a function $\epsilon : X \times X \rightarrow \mathbb{Q}_{l}^{\times}$ with $\epsilon(U, U) = 1$. Here $[\epsilon \Phi]_{U_{1},U_{2}} = \epsilon(U_{1},U_{2})\Phi_{U_{1},U_{2}}$. The collection $\epsilon \Phi$ is normalized if and only if $\epsilon(U_{1},U_{2})\epsilon(U_{2},U_{3})\epsilon(U_{1},U_{3})^{-1} = \eta_{\Phi}(U_{1},U_{2},U_{3})^{-1}$. If $\Phi$ is $\Gamma$-equivariant, then $\epsilon \Phi$ is $\Gamma$-equivariant if and only if $\epsilon$ is $\Gamma$-invariant in the sense that $\epsilon(\gamma(U_{1}),\gamma(U_{2})) = \epsilon(U_{1},U_{2})$.

**Lemma 2.3.5.** A normalized $\Gamma$-equivariant collection exists if and only if a coherent collection exists.
Proof. A normalized $\Gamma$-equivariant collection is automatically coherent. Conversely, assume that $\Phi$ is coherent. Choose arbitrarily a set of representatives $U_0, \ldots, U_k$ for $X/\Gamma$. For any $V_1, V_2 \in X$ let $U_1, U_2$ be such that $V_i \in \Gamma \cdot U_i$ and define

$$\epsilon(V_1, V_2) := \Phi_{V_1, V_2} \Phi_{U_1, U_2} \Phi_{U_2, U_0} \Phi_{U_3, V_2} \in \bar{Q}_1 \times.$$

Then one checks easily that $\epsilon$ is $\Gamma$-invariant and satisfies

$$\eta_{\Phi}(V_1, V_2, V_3)^{-1} = \epsilon(V_1, V_2) \epsilon(V_2, V_3) \epsilon(V_1, V_3)^{-1},$$

so that $\epsilon\Phi$ is normalized and $\Gamma$-equivariant. \hfill $\Box$

It is thus enough to understand under what circumstances coherent collections exist. For this, we shall assume the existence of a $\Gamma$-equivariant collection and see under what conditions it can be modified to ensure coherence.

**Lemma 2.3.6.** Let $\Phi$ be a $\Gamma$-equivariant collection on $X$ and let $U \in X$. Assume that the stabilizer of $U$ is normal in $\Gamma$ and let $\bar{\Gamma}$ be the quotient.

1. Then the function

$$\eta_{\Phi,U}(a, b, c) = \eta_{\Phi}(aU, bU, cU) \in \bar{Q}_1 \times \tag{2.5}$$

is a homogeneous 2-cocycle of $\bar{\Gamma}$. It's class is independent of $\Phi$.

2. Given $\epsilon_U \in C^1(\bar{\Gamma}, \bar{Q}_1 \times)$ with $\partial \epsilon = \eta_{\Phi,U}^1$ define a collection $\epsilon_U \Phi$ on $Y = \Gamma \cdot U \subset X$ by $\epsilon_U \Phi|_{U \cdot V} = \epsilon_U(a, b) \Phi_{U \cdot V}$. This is a normalized $\Gamma$-equivariant collections on $Y$ and every such collection is given in this way.

Proof. Left to the reader. \hfill $\Box$

In order to treat all $\Gamma$-orbits in $X$ simultaneously we introduce for any two $U, V \in X$ the function

$$\beta_{\Phi,V,U}(a, b) := \Phi_u \Phi_{aU} \Phi_{bU} \Phi_{aU} \Phi_{bU} \Phi_{bU} \in \bar{Q}_1 \times.$$

It is an element of $C^1(\bar{\Gamma}, \bar{Q}_1 \times)$. From now on we assume that the stabilizers in $\Gamma$ of all $U \in X$ are equal. In particular, $\beta_{\Phi,V,U}$ is inflated from $C^1(\bar{\Gamma}, \bar{Q}_1 \times)$.

**Lemma 2.3.7.** Given $U, V, W \in X$ we have

1. $\beta_{\Phi,U,U} = 1$, $\beta_{\Phi,U,V} = \beta_{\Phi,V,U}^{-1}$, and $\beta_{\Phi,U,V} \cdot \beta_{\Phi,V,W} = \beta_{\Phi,U,W} \cdot \beta_{\Phi,V,U} = 1$.

2. $\partial \beta_{\Phi,V,U} = \eta_{\Phi,U} \cdot \eta_{\Phi,V}^{-1}$.

3. If $V = xU$ then $\beta_{\Phi,V,U} = \eta_{\Phi,U}(a, b) \eta_{\Phi,U}(ax, bx, b)^{-1}$.

Proof. Left to the reader. \hfill $\Box$

**Corollary 2.3.8.** The cohomology class $[\eta]$ of $\eta_{\Phi,U}$ depends neither on $\Phi$, nor on $U$.

**Definition 2.3.9.** A collection $\{\epsilon_U \in C^1(\bar{\Gamma}, \bar{Q}_1 \times)|U \in X\}$ is called a coherent splitting of $\{\eta_{\Phi,U}|U \in X\}$ if for any $U \in X$ we have $\partial \epsilon_U = \eta_{\Phi,U}^{-1}$ and for any $U, V \in X$ we have $\epsilon_V = \beta_{\Phi,V,U} \cdot \epsilon_U$. Given a coherent splitting $\epsilon = \{\epsilon_U\}$ we define the collection $\epsilon\Phi$ as follows: For each $\Gamma$-orbit $Y \subset X$ choose arbitrarily an element $U \in Y$ and define $\epsilon\Phi|_Y$ by $\epsilon\Phi|_Y$ as in Lemma 2.3.6. By Lemma 2.3.7 the resulting $\epsilon\Phi|_Y$ is independent of the choice of $U$. For any $U, V \in X$ belonging to different $\Gamma$-orbits, define $[\epsilon\Phi]_{U,V} = \Phi_{U,V}$. It is immediate to check that $\epsilon\Phi$ is a coherent collection.
Corollary 2.3.10. Assume that a $\Gamma$-equivariant collection on $X$ exists and that the stabilizers in $\Gamma$ of all $U \in X$ are equal. The following are equivalent.

1. The class $[\eta]$ is trivial.
2. A $\Gamma$-equivariant normalized collection of natural intertwining operators exists on some $\Gamma$-orbit of $X$.
3. A coherent splitting of $\{\eta_{U,V}|U \in X\}$ exists.
4. A coherent collection of natural intertwining operators on $X$ exists.
5. A normalized $\Gamma$-equivariant collection of natural intertwining operators on $X$ exists.

Proof. Left to the reader. \hfill \Box

Fact 2.3.11. If $\{\epsilon_U|U \in X\}$ is a coherent splitting of $[\eta]$, then any other such is of the form $\{\epsilon_U \cdot \delta|U \in X\}$ for $\delta \in Z^1(\Gamma, \mathbb{Q}_\ell^\times) = \text{Hom}(\Gamma, \mathbb{Q}_\ell^\times)$. In this way, the set of coherent splittings is a torsor for $\text{Hom}(\Gamma, \mathbb{Q}_\ell^\times)$.

2.4 Geometric intertwining operators

In order to apply Lemma 2.3.6 or Corollary 2.3.10, one needs to start with a $\Gamma$-equivariant collection of natural intertwining operators. In this subsection we shall review work of Bonnafé-Dat-Rouquier [BDR17] that provides a canonical such collection. More precisely, for any two $U,V$ the authors construct a non-zero intertwining operator $\Psi_{V,U} : H^0_c(Y_U, \mathbb{Q}_\ell)_\theta \rightarrow H^0_c(Y_V, \mathbb{Q}_\ell)_\theta$ that is naturally given, i.e. independent of any additional choices. It is very easy to see, and we shall do so below, that the resulting collection $\Psi$ is equivariant with respect to the full group of automorphisms of $G$ that preserve $S$ and $\theta$ and consists of natural intertwining operators in the sense of Definition 2.3.1.

The construction of $\Psi_{V,U}$ is as follows. Given $U,V$, Bonnafé-Dat-Rouquier define [BDR17, §6.A] the variety

$$Y_{U,V} = \{(g,h) \in G/U \times G/V | g^{-1}h \in U \cdot V, h^{-1}F(g) \in V \cdot F(U)\}$$

and the closed subvariety

$$Y_{U,V}^{(2)} = \{(g,h) \in Y_{U,V} | g^{-1}F(g) \in U \cdot F(U)\}.$$

Just like $Y_U$, these varieties are equipped with an action of $G(k)$ by left multiplication and of $S(k)$ by right multiplication. It is shown in [BDR17, Lemma 6.1] that the map $Y_{U,V}^{(2)} \rightarrow Y_U$ given by $(g,h) \rightarrow g$ is a smooth affine fiber bundle of dimension equal to the codimension of $Y_{U,V}^{(2)}$ in $Y_{U,V}$. This dimension is $d_{U,V} = \dim(U \cap F(U)) - \dim(U \cap V \cap F(U))$. Pulling back along the inclusion $Y_{U,V}^{(2)} \rightarrow Y_{U,V}$ gives a map $H^*_c(Y_{U,V}, \mathbb{Q}_\ell) \rightarrow H^*_c(Y_{U,V}^{(2)}, \mathbb{Q}_\ell)$. Pushing forward along the smooth morphism $Y_{U,V}^{(2)} \rightarrow Y_U$ gives an isomorphism $H^*_c(Y_{U,V}, \mathbb{Q}_\ell) \rightarrow H^*_c(Y_{U,V}^{(2)}, \mathbb{Q}_\ell)$. Both of these are equivariant for the actions of $G(k)$ and $S(k)$. One of the main results, [BDR17, Theorem 6.2], is that the composed map $H^*_c(Y_{U,V}, \mathbb{Q}_\ell)_\theta \rightarrow H^*_c(Y_{U,V}^{(2)}, \mathbb{Q}_\ell)_\theta$ is an isomorphism. This is combined with the observation that the map $Y_{U,V} \rightarrow Y_{V,FU}$ given by $(g,h) \mapsto (h,F(g))$ induces
an isomorphism of étale sites, and hence also an isomorphism $H^*_c(Y_{U,V}, \tilde{\mathcal{Q}}_I) \to H^*_c(Y_{V,FU}, \tilde{\mathcal{Q}}_I)$, again equivariant for $G(k)$ and $S(k)$. Composing the three isomorphisms $H^*_c(Y_{U,V}, \tilde{\mathcal{Q}}_I)_\theta \to H^*_{c,dv}^{(2-v)}(Y_{U}, \tilde{\mathcal{Q}}_I)_\theta, H^*_c(Y_{U,V}, \tilde{\mathcal{Q}}_I) \to H^*_c(Y_{V,FU}, \tilde{\mathcal{Q}}_I)$, and the inverse of $H^*_c(Y_{V,FU}, \tilde{\mathcal{Q}}_I)_\theta \to H^*_{c,dv}^{(2-v)}(Y_{V}, \tilde{\mathcal{Q}}_I)_\theta$, leads to the isomorphism

$$\Psi_{V,U} : H^*_{c}^{(2)}(Y_{U}, \tilde{\mathcal{Q}}_I)_\theta \to H^*_{c}^{(2)}(Y_{V}, \tilde{\mathcal{Q}}_I)_\theta.$$  \hspace{1cm} (2.6)

We find it convenient to remember the definition of this isomorphism in terms of the following symbolic diagram

$$Y_U \leftrightarrow Y_{U,V}^{(2)} \leftrightarrow Y_{U,V} \leftrightarrow Y_{V,FU} \leftrightarrow Y_{V,FU}^{(2)} \leftrightarrow Y_V.$$

Note that this isomorphism also depends on $G$ and $\theta$, so when there is a danger of confusion we will write $\Psi_{U,V}^\theta$, or $\Psi_{U,V}^\theta$, to record this dependence.

**Lemma 2.4.1.** Let $\tilde{G} \to G$ be a morphism with abelian kernel and cokernel. The diagram

$$H^*_{c}^{(2)}(Y_{U}, \tilde{\mathcal{Q}}_I)_\theta \xrightarrow{\phi_{U}^{\tilde{G}}} H^*_{c}^{(2)}(Y_{V}, \tilde{\mathcal{Q}}_I)_\theta \xrightarrow{\phi_{U}^{\tilde{G}}} H^*_{c}^{(2)}(Y_{V}, \tilde{\mathcal{Q}}_I)_\theta$$

commutes, where the vertical arrows are the isomorphisms reviewed in Appendix D.

**Proof.** This follows from the commutativity of the following diagram, in which all vertical arrows are induced by $\tilde{G} \to G$.

$$Y_{U}^{\tilde{G}} \xleftarrow{\alpha_{\tilde{G}}} Y_{U,V}^{(2)} \xrightarrow{\alpha_{\tilde{G}}} Y_{U,V} \xleftarrow{\alpha_{\tilde{G}}} Y_{V,FU} \xrightarrow{\alpha_{\tilde{G}}} Y_{V,FU}^{(2)} \xrightarrow{\alpha_{\tilde{G}}} Y_{V}$$

$$Y_{U}^{\tilde{G}} \xleftarrow{\alpha_{\tilde{G}}} Y_{U,V}^{(2)} \xrightarrow{\alpha_{\tilde{G}}} Y_{U,V} \xleftarrow{\alpha_{\tilde{G}}} Y_{V,FU} \xrightarrow{\alpha_{\tilde{G}}} Y_{V,FU}^{(2)} \xrightarrow{\alpha_{\tilde{G}}} Y_{V}$$

**Corollary 2.4.2.** The operator $\Psi_{V,U}$ is natural in the sense of Definition 2.3.1.

Let $\alpha$ be an automorphism of $G$ commuting with $F$ and preserving $S$. Then $\alpha : G \to G$ restricts to an isomorphism $\alpha : Y_U \to Y_{\alpha(U)}$ of varieties, and on cohomology, we obtain the isomorphism $\alpha : H^*_c(Y_U, \tilde{\mathcal{Q}}_I) \to H^*_c(Y_{\alpha(U)}, \tilde{\mathcal{Q}}_I)$ satisfying $\alpha(v)(\alpha(s) = \theta(s)\alpha(v)$ for $v \in H^*_c(Y_U, \tilde{\mathcal{Q}}_I)$ and $s \in S(k)$. Noting that $d_{\alpha(U)} = d_{U}$, we obtain $\alpha : H^*_{c}^{(2)}(Y_{U}, \tilde{\mathcal{Q}}_I)_\theta \to H^*_{c}^{(2)}(Y_{\alpha(U)}, \tilde{\mathcal{Q}}_I)_\theta \alpha^{-1}$.

**Lemma 2.4.3.** For any $U, V$ we have the equality

$$\alpha \circ \Psi_{V,U}^\theta \circ \alpha^{-1} = \Psi_{\alpha(V),\alpha(U)}^{\theta\alpha^{-1}}.$$

**Proof.** This follows from the commutativity of the following diagram

$$Y_U \xleftarrow{\alpha} Y_{U,V}^{(2)} \xrightarrow{(\alpha,\alpha)} Y_{U,V} \xleftarrow{(\alpha,\alpha)} Y_{V,FU} \xrightarrow{(\alpha,\alpha)} Y_{V,FU}^{(2)} \xrightarrow{(\alpha,\alpha)} Y_V$$

$$Y_{\alpha(U)} \xleftarrow{\alpha_{\alpha(U)}} Y_{\alpha(U),\alpha(V)} \xrightarrow{(\alpha,\alpha)} Y_{\alpha(U),\alpha(V)} \xleftarrow{\alpha_{\alpha(U),\alpha(V)}} Y_{\alpha(U),F\alpha(U)} \xrightarrow{(\alpha,\alpha)} Y_{\alpha(U),F\alpha(U)}^{(2)} \xrightarrow{(\alpha,\alpha)} Y_{\alpha(V)}$$

where $(\alpha, \alpha)$ sends $(g, h)$ to $(\alpha(g), \alpha(h))$. \hspace{1cm} \square
We consider the function $\eta_{\psi}(U_1, U_2, U_3)$ of (2.4). By Lemma 2.4.1 this function depends only on the simply connected cover of the derived subgroup of $G$. For the study of $\eta_{\psi}$ we may therefore assume that $G$ is simply connected. Then it is the product of $k$-simple factors.

**Lemma 2.4.4.** Let $G = G_1 \times G_2$, $U = U_1 \times U_2$, $V = V_1 \times V_2$, $S = S_1 \times S_2$, $\theta = \theta_1 \otimes \theta_2$. Then

1. The isomorphism $G_1 \times G_2 \to G$ restricts to an isomorphism $Y_{U_1}^{G_1} \times Y_{U_2}^{G_2} \to Y_U^G$.
2. This induces an isomorphism
   
   $$H^\epsilon_c(Y_{U_1}^{G_1}, \bar{Q}_l)_\theta \otimes H^\epsilon_c(Y_{U_2}^{G_2}, \bar{Q}_l)_\theta \to H^\epsilon_c(Y_U^G, \bar{Q}_l)_\theta.$$
3. The latter isomorphism identifies $\Psi_{V_1,U_1}^{G_1} \otimes \Psi_{V_2,U_2}^{G_2}$ with $\Psi_{V,U}^G$.

**Proof.** The first claim is immediate from the definitions. The second follows from the Künneth formula and the vanishing theorem [DL76, Cor 9.9]. The third follows from the fact that the analogous decomposition as for $Y_U^G$ also holds for $Y_{V,U}^G$ and $Y_{V,U}^{G,(2)}$, as well as the maps between them.

**Fact 2.4.5.** Let $\Gamma_1$ and $\Gamma_2$ be two groups and $M$ a trivial $\Gamma_1 \times \Gamma_2$-module. Let $z \in Z^2(\Gamma_1 \times \Gamma_2, M)$ have the property $z((\gamma_1, 1), (1, \gamma_2)) = 0$ and $z((1, \gamma_2), (\gamma_1, 1)) = 0$ for all $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$. Then

$$z((\gamma_1, \gamma_2), (\gamma_1', \gamma_2')) = z((\gamma_1, 1), (\gamma_1', 1)) + z((1, \gamma_2), (1, \gamma_2')).$$

In other words

$$z = \Inf_{\Gamma_1}^{\Gamma_1 \times \Gamma_2} \Res_{\Gamma_1}^{\Gamma_1 \times \Gamma_2} z + \Inf_{\Gamma_2}^{\Gamma_1 \times \Gamma_2} \Res_{\Gamma_2}^{\Gamma_1 \times \Gamma_2} z.$$

**Corollary 2.4.6.** Let $G = G_1 \times \cdots \times G_k$, $S = S_1 \times \cdots \times S_k$, $\Gamma = \Gamma_1 \times \cdots \times \Gamma_k$, $U = U_1 \times \cdots \times U_k$. Then

$$\eta_{\psi, U}^G = \sum_{i=1}^k \Inf_{\Gamma_i}^{\Gamma_1 \times \cdots \times \Gamma_k} \eta_{\psi, U_i}^{G_i}.$$

**Proof.** By induction we may assume $k = 2$. Applying above Fact it is enough to show that when $U = U_1 \times U_2$, $U' = V_1 \times U_2$, and $U'' = V_1 \times V_2$, we have $\eta_{\psi}(U, U', U'') = 1$. This follows from Lemma 2.4.4 by observing $\Psi_{V_1,V_1}^{G_1} = \id$ and $\Psi_{S_2,U_2}^{G_2} = \id$.

Let $\phi : G(k) \to \bar{Q}_l^\times$ be a character whose restriction to $G_{sc}(k)$ is trivial. Recall from [DL76, Corollary 1.27] that there exists a naturally given isomorphism of $G(k)$-representations $\phi \otimes H^\epsilon_c(Y_{U_1}, \bar{Q}_l)_\theta \to H^\epsilon_c(Y_U, \bar{Q}_l)_\theta$, as follows. The action of $G(k) \times S(k)$ on $Y_U$ by the formula $x \to gx$ extends to an action of $[(G \times S)/Z](k)$, where $Z$ is the center of $G$ embedded anti-diagonally in $G \times S$. Let $Y_U^{\sigma}$ denote the Deligne-Lusztig variety for $G_{sc}$. The natural map $G_{sc} \to G$ induces an étale map $Y_U^{\sigma} \to Y_U$ which realizes the $[(G \times S)/Z](k)$-representation $H^\epsilon_c(Y_U, \bar{Q}_l)$ as the induction of the $[(G_{sc} \times S_{sc})/Z_{sc}](k)$-representation $H^\epsilon_c(Y_U^{\sigma}, \bar{Q}_l)$. We now use the basic fact that if $\eta : A \to B$ is a homomorphism of finite groups,
\[ \phi : B \to \mathbb{Q}^\times \] is a character, and \((\rho, V)\) is a representation of \(A\), then multiplication by \(\phi^{-1}\) is an automorphism of the vector space \(\{ f : B \to V \mid f(\eta(a)b) = af(b) \}\) that is an isomorphism

\[ \text{Ind}^B_A((\phi \circ \eta) \otimes \rho) \to \phi \otimes \text{Ind}^A_B \rho. \]

Note that this automorphism is functorial in \(V\). We apply this basic fact to the induced representation \(H^i_c(Y_U, \mathbb{Q}_l)\) and the character \((g, s) \mapsto \phi(gs)\) of \([ (G \times S)/Z ](k)\). It gives us an automorphism of the \(\mathbb{Q}_l\)-vector space \(H^i_c(Y_U, \mathbb{Q}_l)\) whose restriction to each \(\theta\)-isotypic component realizes the isomorphism \(\phi \otimes H^i_c(Y_U, \mathbb{Q}_l)_\theta \to H^i_c(Y_U, \mathbb{Q}_l)_{\phi \theta}\).

**Lemma 2.4.7.** For any two \(U, V\) the following diagram commutes.

\[
\begin{array}{ccc}
\phi \otimes H^d_{U} (Y_U, \mathbb{Q}_l)_{\phi} & \xrightarrow{\psi^{d,U}_{\phi \theta}} & \phi \otimes H^d_{V} (Y_V, \mathbb{Q}_l)_{\theta} \\
\downarrow \multicolumn{2}{c}{=} & \downarrow \\
H^d_{U} (Y_U, \mathbb{Q}_l)_{\phi \theta} & \xrightarrow{\psi^{d,V}_{\phi \theta}} & H^d_{V} (Y_V, \mathbb{Q}_l)_{\phi \theta}
\end{array}
\]

Given an automorphism \(\alpha\) of \(G\) commuting with \(F\) and preserving \(S\) we obtain a morphism \(\alpha : Y_U \to Y_{\alpha(U)}\) and hence a linear map \(H^i_c(Y_U, \mathbb{Q}_l) \to H^i_c(Y_{\alpha(U)}, \mathbb{Q}_l)\), which we shall also denote by \(\alpha\). Then the following diagram of \(G(k)\)-representations commutes

\[
\begin{array}{ccc}
(\phi \otimes H^i_c(Y_U, \mathbb{Q}_l))_\theta \alpha & \xrightarrow{\alpha} & (\phi \otimes H^i_c(Y_{\alpha(U)}, \mathbb{Q}_l))_{\theta \alpha^{-1}} \\
\downarrow \multicolumn{2}{c}{=} & \downarrow \\
(H^i_c(Y_U, \mathbb{Q}_l))_{\phi \theta} \alpha & \xrightarrow{\alpha} & (H^i_c(Y_{\alpha(U)}, \mathbb{Q}_l))_{(\phi \theta) \alpha^{-1}}
\end{array}
\]

where for a \(G(k)\)-representation \(\pi\) we write \(\pi(\alpha^{-1}(g)) = \pi^{-1}(g)\)) is commutative.

**Proof.** Consider the first diagram. An argument as in the proof of [DL76, Proposition 1.25] shows that the \([ (G \times S)/Z ](k)\)-spaces \(Y_{U,V}\) and \(Y^{(2)}_{U,V}\) are induced from the \([ (G_{sc} \times S_{sc})/Z_{sc} ](k)\)-spaces \(Y_{U,V}^{sc}\) and \(Y_{U,V}^{sc,(2)}\). In addition, the maps 

\[ H^i_c(Y_U, \mathbb{Q}_l) \to H^i_c(Y_{U,V}, \mathbb{Q}_l) \]

as well as 

\[ H^i_c(Y_{U,V}, \mathbb{Q}_l) \to H^i_c(Y_{V,U}, \mathbb{Q}_l) \]

are induced from the corresponding maps for \(Y^{sc}\). The claim now follows from the functoriality in \(V\) of the automorphism \(f \mapsto \phi \cdot f\), in the abstract set-up explained above.

Now consider the second diagram. The two horizontal maps are induced from the corresponding maps for \(Y^{sc}\), due to the commutativity of

\[
\begin{array}{ccc}
Y^{sc}_{U} & \xrightarrow{\alpha} & Y^{sc}_{\alpha(U)} \\
\downarrow \multicolumn{2}{c}{=} & \downarrow \\
Y_{U} & \xrightarrow{\alpha} & Y_{\alpha(U)}
\end{array}
\]

and the claim follows again from the functoriality of \(f \mapsto \phi \cdot f\).

\[\square\]

**Corollary 2.4.8.** The 2-cocycle \(\eta_{\phi,U}\) does not change if we replace \(\theta\) by \(\phi \cdot \theta\).
2.5 A parameterization of Deligne-Lusztig packets

Let $U$ be the unipotent radical of a Borel subgroup containing $S$. We shall apply the foregoing discussion to the case of $\Gamma = N(S,G)(k)_\theta$, $\Gamma = \Omega(S,G)(k)_\theta$, $X$ the set of unipotent radicals of all Borel subgroups containing $S$. According to Lemma 2.4.3 and Corollary 2.4.2 the collection $\Psi$ is a $\Gamma$-equivariant collection of natural intertwining operators on $X$. Hence the class $[\eta]$ of Corollary 2.3.8 is defined. We shall see shortly that it is trivial.

For any $\Gamma$-equivariant collection $\Phi$ of natural intertwining operators on $Y = \Gamma \cdot U \subset X$ and $n \in N(S,G)(k)_\theta$ define a self-intertwining operator

$$R^\Phi_n : H^\ell_c(Y_U, \bar{Q}_l)_\theta \rightarrow H^\ell_c(Y_U, \bar{Q}_l)_\theta$$

as follows. The map $gU \mapsto gnU^{-1}$ is an isomorphism of varieties $Y_U \rightarrow Y_{nU}$, where we write $nU = nU^{-1}$, that commutes with the left $G(k)$-action and translates the action of $s \in S(k)$ to the action of $nsn^{-1} \in S(k)$. It induces a linear isomorphism $r_{n-1} : H^\ell_c(Y_U, \bar{Q}_l) \rightarrow H^\ell_c(Y_{nU}, \bar{Q}_l)$ that respects the $\theta$-isotypic components. We define $R^\Phi_n = \Phi_{U^nU} \circ r_{n-1}$.

If $l_n$, $r_n$, and $c_n$ denote the maps $g \mapsto ng$, $g \mapsto gn$, and $g \mapsto ngn^{-1}$, then we have $c_n = l_n r_{n-1}$. Using that $\Phi$ is a $\Gamma$-equivariant collection of $G(k)$-equivariant operators, we compute

$$R^\Phi_n \circ R^\Phi_m = \eta_{\Phi,U}(1, n, nm) \cdot R^\Phi_{nm}.$$

Lemma 2.5.1. The element $[\eta] \in H^2(\bar{\Gamma}, \bar{Q}_l^\times)$ of Corollary 2.3.8 is trivial.

Proof. Choose an irreducible constituent $\pi \in [R_\theta]$. According to Theorem 2.2.1 for each $n \in N(S,G)(k)_\theta$ the operator $R^\Phi_n$ preserves $\pi$ and hence acts on it by a scalar $c_n \in \bar{Q}_l^\times$. We have $c \in C^1(N(S,G)(k)_\theta, \bar{Q}_l^\times)$ and for $s \in S(k)$ we have $c_s = \theta(s)^{-1}$. By Proposition 2.2.3 there exists an extension $\hat{\theta}$ of $\theta$ to $N(S,G)(k)_\theta$. We set $\tilde{c}_n := \hat{\theta}(n) \cdot c_n$. Then $\tilde{c} \in C^1(\Omega(S,G)(k)_\theta, \bar{Q}_l^\times)$ and using inhomogenous notation we see that $\partial \tilde{\psi}(n,m) = \partial \tilde{\varepsilon}(n,m) = \eta_{\Phi,U}(n,m)$ holds for all $n, m \in N(S,G)(k)_\theta$, hence $\partial \tilde{c} = \eta_{\Phi,U}$ holds in $Z^2(\Omega(S,G)(k)_\theta, \bar{Q}_l^\times)$. \qed

Corollary 2.3.10 now implies that a coherent splitting $\{\epsilon_U\}$ of $\{\eta_{\Phi,U}\}$ exists. We fix one such and obtain the coherent collection $\Phi = \epsilon \Psi$. Let us write $R^\Phi_n$ for the operator $R^\Phi_n$ obtained in this way. The resulting map $n \mapsto R^\Phi_n$ gives an action of $N(S,G)(k)_\theta$ on $H^d_U(Y_U, \bar{Q}_l)_\theta$ that commutes with the action of $G(k)$ and extends the action of $S(k)$ on this vector space obtained by inverting the action coming from right multiplication, i.e. the action given by $s \mapsto (s)^{-1}$. In other words, we now have an action of $G(k) \times N(S,G)(k)_\theta$ on $H^d_U(Y_U, \bar{Q}_l)_\theta$.

Remark 2.5.2. Note that the cocycles $\{\eta_{\Phi,U}|U \in X\}$ are naturally given. Studying them would hopefully lead to a natural choice of $\{\epsilon_U\}$. We shall come to this in a forthcoming paper.

The coherence of the collection $\Phi$ immediately implies the following:

Fact 2.5.3. Given $U, V \in X$ and $n \in N(S,G)(F)_\theta$ we have

$$\Psi_{V,U} \circ R^\Phi_n \circ \Psi_{V,U}^{-1} = R^\Phi_n \circ \Psi_{V,U}.$$ 

Theorem 2.5.4. 1. The isotypic constants for the action of $\{1\} \times N(S,G)(k)_\theta$ are precisely the irreducible factors for the action of $G(k) \times \{1\}$ on $H^d_U(Y_U, \bar{Q}_l)_\theta$. 

20
2. We obtain a bijection
\[
\{ \bar{\theta} : N(S, G)_{\bar{\theta}} \to \bar{\Omega}^{\times}_l | \bar{\theta}|_{S(k)} = \theta \} \to [R_{\theta}]
\]
that assigns to \( \bar{\theta}^{-1} \) the isotypic component
\[
R^\theta_{\bar{\theta}} := R_{\bar{\theta}} := \text{Hom}_{N(S, G)_{\bar{\theta}}}((\bar{\theta}^{-1}, H^1_{\bar{\theta}}(Y_U, \bar{\Omega}_l)_{\bar{\theta}})).
\]

3. This bijection is equivariant with respect to the action of \( \Omega(S, G)(k)_{\bar{\theta}} \) on the left hand side by multiplication and the action on the right hand side given by Lemma 2.2.2.

4. This bijection is independent of the choice of \( U \).

5. The dependence of this bijection on the coherent splitting is given by \( R^\delta_{\bar{\theta}} = R^\epsilon_{\bar{\theta}} \rho_{\delta} \) for \( \delta \in H^1(\Omega(S, G)(k)_{\bar{\theta}}, \bar{\Omega}^{\times}_l) = \Omega(S, G)(k)_{\bar{\theta}} \) as in Fact 2.3.11.

6. If \( \phi : G(k) \to \bar{\Omega}^{\times}_l \) has trivial restriction to \( G_{\text{ad}}(k) \), then the natural isomorphism \( \phi \otimes R_{\theta} \to R_{\phi, \theta} \) identifies \( \phi \otimes R^\epsilon_{\bar{\theta}} \) with \( R^\delta_{\bar{\theta}} \).

Proof. The actions of \( N(S, G)(k)_{\theta} \) and \( G(k) \) on \( H^1_{\bar{\theta}}(Y_U, \bar{\Omega}_l)_{\theta} \) commute with each other, so for every \( \bar{\theta} \) the isotypic component \( R_{\bar{\theta}} \) is a \( G(k) \)-subrepresentation. Choose one \( \bar{\theta} \) for which \( R_{\bar{\theta}} \) is non-zero. We claim that for any \( \bar{s} \in S_{\text{ad}}(k) \) the \( G(k) \)-representations \( R_{\bar{\theta}} \circ \text{Ad}(\bar{s})^{-1} \) and \( R_{\bar{\theta} \circ \delta_{\bar{s}}} \) are isomorphic, where \( \delta_{\bar{s}} \) is the image of \( s \) under the map \( S_{\text{ad}}(k) \to \Omega(S, G)(k)_{\bar{\theta}} \) given by (2.3). Granting this claim, we see that for all characters \( \bar{\theta} \) the isotypic component \( R_{\bar{\theta}} \) is non-zero. By Theorem 2.2.1 and Lemma A.11 the number of irreducible \( G(k) \)-subrepresentations of \( R_{\theta} \) is equal to the number of characters \( \bar{\theta} \) extending \( \theta \), namely \( |\Omega(S, G)(k)_{\theta}| \). This shows that each isotypic component \( R_{\bar{\theta}} \) is irreducible, and the first three parts of the theorem follow.

To prove the claim we consider the isomorphism \( \text{Ad}(\bar{s}) : Y_U \to Y_U \). It induces a vector space isomorphism
\[
H^1_{\bar{\theta}}(Y_U, \bar{\Omega}_l)_{\theta} \to H^1_{\bar{\theta}}(Y_U, \bar{\Omega}_l)_{\theta}
\]
that translates the action of \( g \in G(k) \) on its source to the action of \( \text{Ad}(\bar{s})g \) on its target. We claim that it also translates the action of \( R^\epsilon_{\bar{\theta}} \) on its source to the action of \( R^\bar{\epsilon}_{\text{Ad}(\bar{s})n} \) on its target. This would follow if we knew that the collection \( \Psi \) is \( S_{\text{ad}}(k) \)-equivariant. By Lemma 2.3.6 and Corollary 2.4.2 we have \( \Phi = \epsilon \Psi \), where \( \Psi \) is the collection of geometric intertwining operators and \( \epsilon \) is a coherent splitting in the sense of Definition 2.3.9 for \( \Gamma = \Omega(S, G)(k)_{\theta} \). By Lemma 2.4.3 the collection \( \Psi \) is \( S_{\text{ad}}(k) \)-equivariant, and then so is the collection \( \Phi \). Therefore the isomorphism \( \text{Ad}(\bar{s}) \) identifies the representations \( R_{\bar{\theta}} \) and \( R_{\bar{\theta} \circ \text{Ad}(\bar{s})^{-1}} \circ \text{Ad}(\bar{s}) \), or equivalently \( R_{\bar{\theta}} \circ \text{Ad}(\bar{s})^{-1} \) and \( R_{\bar{\theta}} \circ \text{Ad}(\bar{s})^{-1} \). Clearly \( \bar{\theta} \circ \text{Ad}(\bar{s})^{-1} \) extends \( \theta \), so it is given by \( \bar{\theta} \cdot \delta \) for a uniquely determined \( \delta \in \Omega(S, G)(k)_{\theta} \), which we can then evaluate at \( n \in N(S, G)(k)_{\theta} \) by the formula \( \delta(n) = \bar{\theta}((\text{Ad}(\bar{s})^{-1}n) \cdot n^{-1}) = \bar{\theta}(n \cdot n^{-1} \cdot n^{-1}) \) where \( \bar{s} \in S_{\text{ad}}(k) \) is any lift of \( \bar{s} \). A look at Lemma 2.1.1 reveals that \( \delta = \delta_{\bar{s}} \).

We now discuss independence of \( U \). Let \( V \) be the unipotent radical of another Borel subgroup containing \( S \). Then \( H^1_{\bar{\theta}}(Y_V, \bar{\Omega}_l)_{\theta} \) is another realization of the representation \( R_{\theta} \). We have the action of \( N(S, G)(k)_{\theta} \) on this realization given by \( n \mapsto R^\Phi_{n} \). Our claim is that the \( \bar{\theta}^{-1} \)-isotypic component of the action of \( N(S, G)(k)_{\theta} \) on \( H^1_{\bar{\theta}}(Y_V, \bar{\Omega}_l)_{\theta} \) via \( R^\Phi_{\bar{\theta}} \) and the \( \bar{\theta}^{-1} \)-isotypic component
of the action of $N(S, G)(k)_\theta$ on $H^d_{c}(Y_U, \mathbb{Q}_l)_\theta$ via $R^{\Phi, U}$ are isomorphic $G(k)$-representations. For this we consider the isomorphism $\Psi_{V, U} : H^d_{c}(Y_U, \mathbb{Q}_l)_\theta \to H^d_{c}(Y_V, \mathbb{Q}_l)_\theta$. Both sides being of multiplicity 1, this isomorphism restricts to an isomorphism between the individual irreducible constituents, and the claim follows from Fact 2.5.3.

The dependence on the coherent splitting is a direct computation. The compatibility with character twists follows from Lemma 2.4.7 applied to $\alpha = c_n$.  

3 Non-singular Deligne-Lusztig packets over local fields

Let $G$ be a connected reductive group defined over a non-archimedean local field $F$, $S \subset G$ an elliptic maximally unramified maximal torus, $\theta : S(F) \to \mathbb{C}^\times$ a character of depth zero. Let $S' \subset S$ be the maximal unramified subtorus. We consider the set $R_{\text{res}}(S', G)$ of restrictions to $S'$ of the absolute roots $R(S, G)$. Since $S'$ is a maximally split torus of $G$ over $F_{\text{ur}}$, this is in fact a (possibly non-reduced) root system.

Let $x \in B(G, F)$ be the point associated to $S$. It is a vertex [Kal19, Lemma 3.4.3]. Let $G^x_\alpha$ be the reductive quotient of the connected parahoric group scheme of $G$ associated to the vertex $x$. Let $S^x$ be special fiber of the connected Neron model of $S'$, or equivalently the reductive quotient of the special fiber of the connected Neron model of $S$. Then $S^x \subset G^x_\alpha$ is an elliptic maximal torus. The restriction of $\theta$ to $S'(F)_0$ factors through a character $\overline{\theta} : S^x(k_F) \to \mathbb{C}^\times$.

Definition 3.0.1. Let $F'/F$ be an unramified extension splitting $S'$. The character $\theta$ will be called

1. $k_F$-non-singular, if for every $\alpha \in R(S^x, G^x_\alpha)$ the character
   $$\overline{\theta} \circ N_{k_F'/k_F} \circ \alpha_{\text{res}} : (k_F')^\times \to \mathbb{C}^\times$$
   is non-trivial;
2. $F$-non-singular, if for every $\alpha_{\text{res}} \in R_{\text{res}}(S', G)$ the character
   $$\theta \circ N_{F'/F} \circ \alpha_{\text{res}} : F'^\times \to \mathbb{C}^\times$$
   has non-trivial restriction to $O^x_{F'}$.

Remark 3.0.2. Note that the choice of $F'$ is irrelevant, because for any finite unramified extension $F'/F'$ the norm map $N_{F'/F''} : O^x_{F'} \to O^x_{F''}$ is surjective. Note further that by [Kal19, Lemma 3.4.14] the character $\theta$ is $k_F$-non-singular if and only if $\overline{\theta}$ is non-singular with respect to $G^x_\alpha$ in the sense of [DL76, Definition 5.15].

Remark 3.0.3. If $(S_1, \theta_1)$ and $(S_2, \theta_2)$ are stably conjugate pairs, then $\theta_1$ is $F$-non-singular if and only if $\theta_2$ is. On the other hand, the $k_F$-non-singularity of $\theta_1$ does not a-priori imply anything about the $k_F$-non-singularity of $\theta_2$.  

Fact 3.0.4. 1. If $\theta$ is $F$-non-singular then it is $k_F$-non-singular.
2. If the vertex $x$ is absolutely special and $\theta$ is $k_F$-non-singular, then it is $F$-non-singular.
3. If $\theta$ is regular in the sense of [Kal19, Definition 3.4.16], then $\theta$ is $F$-non-singular.
Proof. The root system $R(S^0, G^0)$ is a subset of $R_{\res}(S', G)$ and according to [Kal19, Lemma 3.4.14] $F$-non-singularity implies $k_F$-non-singularity.

Assume now that $x$ is absolutely special and that $\theta$ is $k_F$-non-singular. Then $R(S^0, G^0)$ is the set of reduced roots in $R_{\res}(S', G)$. Thus given $\alpha_{\res} \in R(S', G)$ either $\beta_{\res} = \alpha_{\res}$ or $\beta_{\res} = \frac{1}{2} \alpha_{\res}$ lies in $R(S^0, G^0)$. By assumption $\theta \circ N_{F'/F} \circ \beta_{\res}$ has non-trivial restriction to $O^\circ_{F'}$. Since $\beta_{\res} = \alpha_{\res}$ or $\beta_{\res} = 2\alpha_{\res}$ this implies that $\theta \circ N_{F'/F} \circ \alpha_{\res}$ has non-trivial restriction to $O^\circ_{F'}$. 

Assume now that $\theta$ is not $F$-non-singular. We want to show that it cannot be regular. The torus $S$ transfers to the quasi-split inner form of $G$ [Kal19, Lemma 3.2.2], so we may assume that $G$ is quasi-split. We may further replace $S$ by a stable conjugate to ensure that the vertex $x$ is absolutely special, according to [Kal19, Lemma 3.4.12]. By the previous point, $\theta$ is not $k_F$-non-singular. According to Remark 3.0.2 and [DL76, Corollary 5.18] there exists $w \in \Omega(S^0, G^0)(k_F)$ stabilizing $\theta$. Now [Kal19, Lemma 3.4.10] precludes the regularity of $\theta$. 

Let now $\theta$ be $k_F$-non-singular. By Remark 3.0.2 we thus have the cuspidal representation $R_{\bar{\theta}}$ of $G^0_x(k_F)$ studied in the previous Section. We inflate this representation to $G(F)_{x,0}$ and denote this inflation by $G_{x,0}$, the same notation used in [Kal19, §3.4.3], but we bear in mind that this representation is now reducible. Next we extend $\kappa_{(S, \bar{\theta})}$ to a representation $\kappa_{(S, \theta)}$ of $S(F) \cdot G(F)_{x,0}$, following the argument of [Kal19, §3.4.4]. Let us briefly recall the construction of this extension. Choose a Borel subgroup of $G^0_x$ containing $S^0$ and defined over $\bar{k}_F$ and let $U$ be its unipotent radical. As discussed in the previous Section, $R_{\bar{\theta}}$ is realized in $H^0_c(Y_0, \bar{Q}_l)_{\bar{\theta}}$. Let $S^0_{\ad}$ be the image of $S^0$ in the adjoint quotient of $G^0_x$. There is an action of $S^0_{\ad}(k_F)$ on $Y_0$ by conjugation. On the other hand, there is a natural map $S(F) \to S^0_{\ad}(k_F)$, whose construction we shall recall below. Via this map we define an action of $S(F)$ on $R_{\bar{\theta}}$ by $s \cdot v := \theta(s) \Ad(s) v$, and this gives the extension of $R_{\bar{\theta}}$ to $S(F) \cdot G(F)_{x,0}$.

If $V$ is the unipotent radical of another Borel subgroup containing $S$, the geometric intertwining operator $\Psi_{V, U} : H^0_c(Y_U, \bar{Q}_l)_{\bar{\theta}} \to H^0_c(Y_V, \bar{Q}_l)_{\bar{\theta}}$ intertwines both the action of $G(F)_{x,0}$ on the left, and the action of $S(F)$ by conjugation, the latter according to Lemma 2.4.3, thus $\kappa_{(S, \theta)}$ is independent of the choice of $U$.

Both $\kappa_{(S, \bar{\theta})}$ and $\kappa_{(S, \theta)}$ can fail to be irreducible. Furthermore, the restriction of an irreducible constituent of $\kappa_{(S, \theta)}$ to $G(F)_{x,0}$ need not remain irreducible.

The group $S(F)G(F)_{x,0}$ is normal and of finite index in the full stabilizer $G(F)_x$ of $x$. The main technical goal in this section is to describe the irreducible constituents of $\Ind_{G(F)_x}^{G(F)_{x,0}} S(F)G(F)_{x,0} \kappa_{(S, \theta)}$. Via the results of Moy and Prasad [MP96] this leads rather directly to the description of all irreducible supercuspidal representations of $G(F)$ that stem from the non-singular Deligne-Lusztig character $R_{\bar{\theta}}$. The situation will turn out to be rather more complicated than that over the finite field $k_F$. In particular, the analogs of Theorem 2.2.1 and Proposition 2.2.3 fail; the irreducible constituents of $\Ind_{G(F)_x}^{G(F)_{x,0}} S(F)G(F)_{x,0} \kappa_{(S, \theta)}$ may well occur with higher multiplicities. We shall need to compute these multiplicities as part of the description. In order to resolve these difficulties, we will make essential use of the geometric intertwining operators introduced in the previous section.
3.1 The basic bicharacter again

Inside of the Weyl group \( \Omega(S, G)(F) \) we have the stabilizer \( \Omega(S, G)(F)_\theta \) of the character \( \theta \), as well as the stabilizer \( \Omega(S, G)(F)_{\bar{\theta}} \) of the restriction \( \bar{\theta} \) of \( \theta \) to \( S(F)_0 \). In the unramified case we have \( S(F) = Z_G(F) \cdot S(F)_0 \), see e.g. [Kal11, Lemma 7.1.1], so \( \Omega(S, G)(F)_\theta = \Omega(S, G)(F)_{\bar{\theta}} \). In the ramified case the equality \( S(F) = Z_G(F) \cdot S(F)_0 \) fails, see [Kal19, §3.4.3], so the two groups could a-priori be different, even though we do not know of an example. Their difference can be measured by the following bicharacter, closely related to the one in Lemma 2.1.1.

**Lemma 3.1.1.** 1. The map

\[
\frac{\Omega(S, G)(F)_{\bar{\theta}}}{\Omega(S, G)(F)_\theta} \times \frac{S(F)}{S(F)_0} \to \mathbb{C}^\times, \quad (w, s) \mapsto \bar{\theta}(ws^{-1}s^{-1})
\]  
(3.1)

is well-defined, bi-multiplicative, and has a trivial left kernel.

2. If \( w \in [N(S, G(F),_0)]_{\bar{\theta}} \), then the value at \( (w, s) \) is equal to the value of the bicharacter of Lemma 2.1.1 at \( (w, \bar{s}) \), where \( \bar{s} \in S^{\text{ad}}_G(k_F) \) is the image of \( s \).

**Proof.** Consider \( s \in S(F) \) and \( w \in \Omega(S, G)(F)_{\bar{\theta}} \). Then \( ws^{-1}s^{-1} \in S(F) \) lies in the image of \( S_{\text{sc}}(F) \to S(F) \). Since \( S_{\text{sc}}(F) \) is maximally unramified and anisotropic, \( S_{\text{sc}}(F) = S_{\text{sc}}(F)_0 \), so \( ws^{-1}s^{-1} \in S(F)_0 \). This allows us to consider \( \bar{\theta}(ws^{-1}s^{-1}) \). If \( s \in S(F)_0 \), then also \( \bar{\theta}(ws^{-1}s^{-1}) = 1 \). For any \( s, w \) we have \( \bar{\theta}(ws^{-1}s^{-1}) = \bar{\theta}(w sns^{-1}) \). This shows that \( \bar{\theta}(ws^{-1}s^{-1}) \) is trivial for \( w, s \in \Omega(S, G)(F)_\theta \), and is multiplicative in \( s \). For multiplicity in \( \Omega(S, G)(F)_{\bar{\theta}} \) the argument is as in the proof of Lemma 2.1.1 – we have for \( u, v \in \Omega(S, G)(F)_{\bar{\theta}} \)

\[
usv^{-1}s^{-1}u^{-1} = (usv^{-1}s^{-1})u^{-1}(usv^{-1}s^{-1})
\]

and the two terms in parenthesis belong to \( S(F)_0 \). Since \( u \) fixes \( \bar{\theta} \) the desired multiplicativity follows. Finally, for a fixed \( w \) we have \( \bar{\theta}(ws^{-1}s^{-1}) = 1 \) for all \( s \) precisely when \( w \in \Omega(S, G)(F)_{\bar{\theta}} \).

For the second point, let \( \bar{s} \in S'(F) \) be a preimage of \( \bar{s} \). Then \( \bar{s}^{-1} \in S'(F) \)

centralizes \( G_\bar{s} \), so \( ws^{-1}s^{-1} = ws^{-1}s^{-1} \).

**Corollary 3.1.2.** The bicharacter (3.1) induces group homomorphisms

\[
\frac{\Omega(S, G)(F)_{\bar{\theta}}}{\Omega(S, G)(F)_\theta} \to \left( \frac{S(F)}{S(F)_0} \right)^\times \quad \text{and} \quad \frac{S(F)}{S(F)_0} \to \left( \frac{\Omega(S, G)(F)_{\bar{\theta}}}{\Omega(S, G)(F)_\theta} \right)^\times
\]

the first of which is injective, and the second surjective.

**Remark 3.1.3.** Note that the injective homomorphism describes \( \Omega(S, G)(F)_{\bar{\theta}} \) as the kernel of the map \( \Omega(S, G)(F)_{\bar{\theta}} \to (S(F)/S(F)_0)^\times \). Since this map involves only \( \bar{\theta} \), we see that \( \Omega(S, G)(F)_{\bar{\theta}} \) depends on \( \theta \) only through its restriction \( \bar{\theta} \).

**Lemma 3.1.4.** The group \( \Omega(S, G)(F)_{\bar{\theta}} \) is abelian. If \( G \) is absolutely simple and simply connected, then \( \Omega(S, G)(F)_{\bar{\theta}} \) is cyclic except when \( G \) is split of type \( D_{2n} \), in which case the possibilities are \( \{1\} \), \( \mathbb{Z}/2\mathbb{Z} \), and \( (\mathbb{Z}/2\mathbb{Z})^2 \).

**Proof.** According to [Kal19, Lemma 3.4.12] we may assume that \( G \) is quasi-split and the point \( x \in B(G, F) \) associated to \( S \) is absolutely special. Then \( \Omega(S, G)(F)_{\bar{\theta}} = \Omega(S^\circ, G^\circ_x)(k_F)_{\bar{\theta}} \), the latter being abelian by Corollary 2.1.2.
Assume now that $G$ is absolutely simple and simply connected. Then $G_2^\circ$ is absolutely simple and semi-simple. By Corollary 2.1.2 we have $\Omega(S^0, G_2^\circ)_{(k_F)\bar{\theta}} \subset \text{cok}(S^0(k_F) \to S_{ad}^0(k_F)) = H^1(k_F, Z(G_2^\circ))$. This group is cyclic as soon as $Z(G_2^\circ)$ is, which is always the case except possibly when $G_2^\circ$ is of type $D_{2n}$. This happens if and only if $G$ itself is unramified of type $D_{2n}$. In that case $G_2^\circ$ is simply connected and its center is $\mu_2 \times \mu_2$. If $G_2^\circ$ is not split, then $H^1(k_F, Z(G_2^\circ)) = \mathbb{Z}/2\mathbb{Z}$, so we may assume that $G_2^\circ$ is split, which is the case if and only if $G$ is split. In that case, take $G_2^\circ$ to be Lusztig’s dual group of $G_2^\circ$ and $S^* \subset S^G_2$ the semi-simple element of the dual torus of $S^0$ corresponding to $\theta$. Then $\Omega(S^0, G_2^\circ)_{(k_F)\bar{\theta}} = \Omega(S^0, G_2^\circ)_{(k_F)\bar{s}}$. The latter can be either of $\{1\}$, $\mathbb{Z}/2\mathbb{Z}$, or $(\mathbb{Z}/2\mathbb{Z})^2$, and all possibilities are realized, as one observes using [Ree10, Proposition 2.1].

\section{3.2 A parameterization of $\kappa_{(S,\theta)}$}

In Subsection 2.5 we had extended the action of $G_2^\circ(k_F)$ on $H^{d_0}_{\nu_0}(Y_{\bar{\theta}}, \bar{Q}_{\bar{l}})_{\bar{\theta}}$ to an action of $G_2^\circ(k_F) \times N(S^0, G_2^\circ)_{(k_F)\bar{\theta}}$. The action of $\{1\} \times S^0(k_F)$ is via the character $\bar{\theta}^{-1}$. We then proved that the isotypic components of $\{1\} \times N(S^0, G_2^\circ)_{(k_F)\bar{\theta}}$ are exactly the irreducible components of $G_2^\circ(k_F) \times \{1\}$. Via the surjective homomorphism $N(S, G(F)_{x,0})_{\bar{\theta}} \to N(S^0, G_2^\circ)_{(k_F)\bar{\theta}}$ we inflate the action of $\{1\} \times N(S^0, G_2^\circ)_{(k_F)\bar{\theta}}$. The isotypic components of course remain the same, but they are not subrepresentations of $\kappa_{(S,\theta)}$, because the resulting action of $N(S, G(F)_{x,0})_{\bar{\theta}}$ commutes with the action of $G(F)_{x,0}$, but not with the action of $S(F)$.

\textbf{Lemma 3.2.1.} The restriction to $N(S, G(F)_{x,0})_{\bar{\theta}}$ of the action of $N(S, G(F)_{x,0})_{\bar{\theta}}$ commutes with $S(F)$.

\textbf{Proof.} A simple computation shows that for $s \in S(F)$, $n \in N(S, G(F)_{x,0})_{\bar{\theta}}$, and $v \in H^{d_0}_{\nu_0}(Y_{\bar{\theta}}, \bar{Q}_{\bar{l}})_{\bar{\theta}}$, we have $R_n s v = s R_{n^{-1} s n^{-1}} R_n v$. By Lemma 3.1.1 we have $s^{-1} n s^{-1} \in S(F)_{0}$, so we get $R_n s v = \bar{\theta}(n s^{-1} n^{-1} s) s R_n v$, and $\bar{\theta}(n s^{-1} n^{-1} s)$ is trivial according to Lemma 3.1.1 provided $n \in N(S, G(F)_{x,0})(F)_{\bar{\theta}}$. \hfill \Box

We have thus extended the $\kappa_{(S,\theta)}$ from a representation of $S(F)G(F)_{x,0}$ to a representation of $S(F)G(F)_{x,0} \times N(S, G(F)_{x,0})_{(F)\bar{\theta}}$.

\textbf{Lemma 3.2.2.} The isotypic components for $\{1\} \times N(S, G(F)_{x,0})_{\bar{\theta}}$ are precisely the irreducible constituents for $S(F)G(F)_{x,0} \times \{1\}$. Assigning to an extension $\bar{\theta}$ of $\theta$ to $N(S, G(F)_{x,0})_{\bar{\theta}}$ the $\bar{\theta}^{-1}$-isotypic component gives a bijection

$$\{ \bar{\theta} : N(S, G(F)_{x,0})_{\bar{\theta}} \to C^\times | \bar{\theta}|_{S(F)_{/0}} = \bar{\theta} \} \rightarrow [\kappa_{(S,\theta)}].$$

\textbf{Proof.} Let $\bar{\tilde{\theta}}$ be an extension of $\bar{\theta}$ to $N(S, G(F)_{x,0})_{\bar{\theta}}$. Let $s \in S(F) \subset S(F)G(F)_{x,0}$ and let $\bar{s} \in S_{ad}^0(k_F)$ be its image. By definition $sv = \theta(s) \text{Ad}(\bar{s})v$. The collection $\Phi$ of natural intertwining operators is $S_{ad}^0(k_F)$-invariant and this implies that $v \mapsto sv$ sends the $\theta$-constituent to the $\theta \circ \text{Ad}(s)^{-1}$-constituent. Thus, two $G(F)_{x,0}$-irreducible constituents belong to the same $S(F)G(F)_{x,0}$-irreducible constituent if and only if they are isotypic components for the $N(S, G(F)_{x,0})_{\bar{\theta}}$-action that correspond to characters that are conjugate under $S(F)$. In other words, under the correspondence between $G(F)_{x,0}$-irreducible constituents and $N(S, G(F)_{x,0})_{\bar{\theta}}$-isotypic components, the $S(F)G(F)_{x,0}$-irreducible constituents
correspond to orbits of the action of $S(F)$ by conjugation on the set of characters of $N(S,G(F)_{x,0})_\theta$ that extend $\bar{\theta}$. We claim that such an orbit is precisely the set of characters with a fixed restriction to $N(S,G(F)_{x,0})_\theta$. In other words, we claim that $S(F)$ acts transitively on those fibers of the restriction map $N(S,G(F)_{x,0})^*_\theta \to N(S,G(F)_{x,0})_\theta$ that contain extensions of $\bar{\theta}$. Indeed, if $\tilde{\theta}$ is an extension of $\bar{\theta}$ to $N(S,G(F)_{x,0})_\theta$, then for $n \in N(S,G(F)_{x,0})_\theta$ and $s \in S(F)$ we have

$$\tilde{\theta}(s n s^{-1}) \cdot \tilde{\theta}(n)^{-1} = \tilde{\theta}(s n s^{-1} n^{-1}) = \theta(s) \theta(n s n^{-1})^{-1},$$

which vanishes if $n \in N(S,G(F)_{x,0})_\theta$, showing that $\tilde{\theta}$ and $\tilde{\theta} \circ \text{Ad}(s)$ have the same restriction to $N(S,G(F)_{x,0})_\theta$. Conversely, if $\tilde{\theta}'$ and $\tilde{\theta}$ both extend $\bar{\theta}$ and have the same restriction to $N(S,G(F)_{x,0})_\theta$, then $\tilde{\theta}' = \tilde{\theta} \cdot \delta^{-1}$ with $\delta \in (\Omega_{x,\bar{\theta}}/\Omega_{x,\theta})^*$, where $\Omega_x = N(S,G(F)_{x,0})/S(F)_0$. By Corollary 3.2.2 there is $s \in S(F)$ s.t. $\delta(w) = \bar{\theta}(w s w^{-1} s^{-1})$ for all $w \in \Omega_{x,\bar{\theta}}$ and we conclude $\tilde{\theta}'(n) = \tilde{\theta}(s n s^{-1})$, as claimed. 

The action of $\{1\} \times N(S,G(F)_{x,0})_\theta$ can be extended to an action of $\{1\} \times N(S,S(G(F)_{x,0})_\theta = \{1\} \times S(F)N(S,G(F)_{x,0})_\theta$, with $S(F)$ acting by the character $\theta^{-1}$.

**Corollary 3.2.3.** Let $\epsilon$ be a coherent splitting for the $N(S,G_{x,0},(G^e_{x,0})(k_F)_{\theta})$-equivariant collection of geometric intertwining operators $\Psi$,

1. The isotypic components for the action of $\{1\} \times N(S,S(G(F)_{x,0})_\theta$ are precisely the irreducible constituents for $S(F)G(F)_{x,0}$.

2. Assigning to an extension $\tilde{\theta}$ of $\theta$ to $N(S,S(G(F)_{x,0})_\theta$ the $\tilde{\theta}^{-1}$-isotypic component $\kappa^\epsilon_{(S,\tilde{\theta})}$ gives a bijection

$$\{\tilde{\theta} : N(S,S(G(F)_{x,0})_\theta \to \mathbb{C}^\times | \tilde{\theta}|_{S(F)} = \theta \} \to [\kappa_{(S,\theta)}].$$

3. The irreducible factor $\kappa^\epsilon_{(S,\tilde{\theta})}$ of $\kappa_{(S,\theta)}$ is independent of the choice of $U$.

4. For $\delta \in \Omega(S,G(F)_{x,0})^*_\theta$ we have $\kappa^\epsilon_{(S,\tilde{\theta})} = \kappa^\delta_{(S,\delta,\theta)}$. Notice that this depends only on the image of $\delta$ in $\Omega(S,G(F)_{x,0})^*_\theta$.

5. Let $G'_\ell$ be the image in $G_x$ of $S(F^\ell)G(F^\ell)_{x,0}$. For a character $\phi : G'_\ell(k_F) \to \mathbb{C}^\times$ trivial on $G^e_{x,0}(k_F)$ we have $\phi \otimes \kappa^\epsilon_{(S,\theta)} = \kappa^\phi_{(S,\phi,\theta)}$.

**Proof.** We have $N(S,S(G(F)_{x,0})_\theta = S(F)N(S,G(F)_{x,0})_\theta$. Hence restriction along the inclusion $N(S,G(F)_{x,0})_\theta \to N(S,S(G(F)_{x,0})_\theta$ provides a bijection between the set of extensions of $\bar{\theta}$ to $N(S,G(F)_{x,0})_\theta$ and the set of extensions of $\theta$ to $N(S,S(G(F)_{x,0})_\theta$. Since the group $S(F)$ acts on the right by $\theta^{-1}$, the first two statements follow from Lemma 3.2.2. The other statements follow as in the proof of Theorem 2.5.4. 

Before we proceed, note that the quotient $G(F)_{x}/G(F)_{x,0}$ is abelian. Thus every subgroup of $G(F)_{x}$ that contains $G(F)_{x,0}$ is automatically normal. This holds in particular for $S(F)G(F)_{x,0}$.

**Lemma 3.2.4.** 1. Let $g \in G(F)_{x}$. Then $\kappa_{(S,\theta)} \circ \text{Ad}(g)^{-1} \simeq \kappa_{(S,G,\theta)}$.
2. Let $S_1$ and $S_2$ be two maximally unramified maximal tori of $G$ with vertex $x$, and let $\theta : S_1(F) \to \mathbb{C}^\times$ be $k_F$-non-singular. If the restrictions of $\kappa_{(S,\theta)}$ to $S_1(F)G(F)_{x,0} \cap S_2(F)G(F)_{x,0}$ are not disjoint, then there exists $g \in G(F)_{x,0}$ with $Ad(g)(S_1, \theta_1) = (S_2, \theta_2)$. Hence $\kappa_{(S_1,\theta_1)} = \kappa_{(S_2,\theta_2)}$ as representations of $S_1(F)G(F)_{x,0} = S_2(F)G(F)_{x,0}$.

3. Let $g \in G(F)_{x,0}$. The representations $\kappa_{(S,\theta)}$ and $\kappa_{(S,G)}$ are either equal or disjoint. They are equal if and only if $g \in N(S,G)_{\theta,F} \cdot G(F)_{x,0}$.

Proof. Let $V = g Ug^{-1}$, $S_1 = gSg^{-1}$. Then $Ad(g) : Y_0 \to Y_V$ is an isomorphism of varieties that translates left multiplication by $g^{-1}hg \in G(F)_{x,0}$ to left multiplication by $h \in G(F)_{x,0}$, right multiplication by $s \in S(F)_{\theta}$ to right multiplication by $gs^{-1} \in S(F)_{\theta}$, and conjugation by $s \in S(F)_{\theta}$ to conjugation by $gsg^{-1} \in S_1(F)$, and hence produces an isomorphism $\kappa_{(S,\theta)} \circ Ad(g)^{-1} \cong \kappa_{(S,G)}$.

Assume that $\kappa_{(S_1,\theta_1)}$ and $\kappa_{(S_2,\theta_2)}$ have a common constituent upon restriction to $S_1(F)G(F)_{x,0} \cap S_2(F)G(F)_{x,0}$. Then the representations $\kappa_{(S_1,\theta_1)}$ and $\kappa_{(S_2,\theta_2)}$ of $G(F)_{x,0}$ have a common constituent. By [DL76, Theorem 6.8] there exists $g \in G(F)_{x,0}$ s.t. $Ad(g)(S_1, \theta_1) = (S_2, \theta_2)$. By [Kal19, Lemma 3.45] there exists $l \in G(F)_{x,0}$ s.t. $Ad(lg)(S_1) = S_2$. In particular $S_1(F)G(F)_{x,0} = S_2(F)G(F)_{x,0}$. Conjugating $\kappa_{(S_1,\theta_1)}$ by $lg$ we may assume $S_1 = S_2 = S$ and $\theta_1 = \theta = \theta_2$. Then $\theta_2 \theta_1^{-1}$ is a character of $S(F)/S(F)_0 = [S(F)G(F)_{x,0}]/G(F)_{x,0}$ and $\kappa_{(S,\theta)} = (\theta_2 \theta_1^{-1}) \otimes \kappa_{(S,\theta)}$. There thus exist two irreducible constituents $\rho_1$ and $\rho_2$ of $\kappa_{(S,\theta)}$ s.t. $\rho_2 = (\theta_2 \theta_1^{-1}) \otimes \rho_1$. By Theorem 2.2.1 each irreducible constituent of $\kappa_{(S,\theta)}$ is uniquely determined by its decomposition into irreducible $G(F)_{x,0}$-representations. Restricting the relation $\rho_2 = (\theta_2 \theta_1^{-1}) \otimes \rho_1$ to $G(F)_{x,0}$ we obtain $\rho_2 = \rho_1$. We thus have an irreducible constituent $\rho$ of $\kappa_{(S,\theta)}$ s.t. $\rho = (\theta_2 \theta_1^{-1}) \otimes \rho$. Theorem 2.2.1, [BH06, Lemma 2.7], and Lemma A.4 applied to the exact sequence

$$1 \to G(F)_{x,0} \to S(F)G(F)_{x,0} \to S(F)/S(F)_0 \to 1$$

imply that $\theta_2 \theta_1^{-1}$ annihilates the kernel of the action of $S(F)/S(F)_0$ on the irreducible constituents of $\kappa_{(S,\theta)}$. According to Theorem 2.5.4 the kernel of the action of $S_\text{ad}(k_F)$ on this set is equal to the kernel of the map $S_\text{ad}(k_F) \to \Omega(S_\text{ad},G_\text{ad})(k_F)_\theta^\circ$ given by (2.3). The action of $S(F)$ factors through the map $S(F) \to S_\text{ad}(k_F)$. We can apply Lemma 3.1.1 and conclude that the composition $S(F) \to S_\text{ad}(k_F) \to \Omega(S_\text{ad},G_\text{ad})(k_F)_\theta^\circ$ is given by $(s,w) \mapsto \theta(ws^{-1}w^{-1}s^{-1})$, and its image is the subgroup $(\Omega_{x,\theta}/\Omega_{x,\theta})^\circ$ of $\Omega_{x,\theta}$, where we have written $\Omega_x = \Omega(S_\text{ad},G_\text{ad})(k_F)$. It follows that there exists an element $w \in \Omega_{x,\theta}$ s.t. $\theta_2(s)\theta_1(s)^{-1} = \theta(ws^{-1}w^{-1}s^{-1})$, i.e. $\theta_2(s) = \theta_1(ws^{-1}).$

The third point is an immediate consequence of the second.

3.3 On the existence of normalized intertwining operators

Our next goal would be to understand the structure of the induction of $\kappa_{(S,\theta)}$ from $S(F)G(F)_{x,0}$ to $G(F)_{x,0}$. The essential tool for this will be a conjugation action of $N(S,G)(F)_\theta(g)$ on $H^0_F(Y_0,\mathcal{Q}_1)_\theta$ that extends the conjugation action of $S(F)$ that was used in the construction of $\kappa_{(S,\theta)}$. The construction of this action will in turn rely on the existence of a coherent family of intertwining operators on the set $X$ of unipotent radicals of Borel subgroups containing $S$, in the sense of §2.3. Such a family was already used in §2.5 with $\Gamma = N(S,G)(S)_\theta$ and
its existence was established in Lemma 2.5.1. Here we shall need to use \( \Gamma = N(S, G)(F)_{\theta} \). The proof of Lemma 2.5.1 relied on the fact that \( \Gamma \) has a natural action on \( Y_{\theta} \) on the right, i.e., commuting with the left \( G_{\mathbb{E}}^o(k_E) \)-action. This is no longer true for the group \( \Gamma \) considered here, so this argument no longer applies.

Since \( N(S, G)(F) \) normalizes \( S \), it fixes the point \( x \in B(G, F) \), and hence acts on \( G_{\mathbb{E}}^o \) by automorphisms that preserve \( S \). The action of \( N(S, G)(F)_{\theta} \) on the set \( \mathcal{S} \) of Borel subgroups of \( G_{\mathbb{E}}^o \) containing \( S \) factors through the quotient modulo \( S(F) \), which is a subgroup of \( \Omega(S, G)(F)_{\theta} \), and hence abelian by Lemma 3.1.4. It follows that the stabilizers in \( \Gamma \) of all Borel subgroups containing \( S \) are equal.

By Corollary 2.4.2 and Lemma 2.4.3 the geometric intertwining operators \( \Psi \) of Bonnafé-Dat-Rouquier provide a \( \Gamma \)-equivariant collection on \( X \). The class \( [\eta] \in H^2(\bar{\Gamma}, \bar{\mathbb{Q}}^\times_l) \) of Corollary 2.3.8 is thus defined.

**Proposition 3.3.1.** The class \([\eta]\) is trivial.

**Proof.** First note that \( Z(F) \subset \Gamma \) acts trivially, so within this proof we can replace \( \Gamma \) by \( N(S, G)(F)_{\theta}/Z(F) \), as this does not affect \( \bar{\Gamma} \). Next we notice that we are free to enlarge \( \bar{\Gamma} \) if we like. It will be convenient to take for \( \bar{\Gamma} \) the stabilizer of \( \theta_{sc} = \theta|_{\mathbb{Q}^0(F)} \) in \( N(S, G_{ad})(F) \). Then \( \bar{\Gamma} \) becomes the quotient of \( [N(S, G_{ad})(F)_{\theta_{sc}}/S_{ad}(F)] \) by the stabilizer of \( U \). Since \( \Omega(S, G)(F)_{\theta_{sc}} \) is abelian by Lemma 3.1.4 applied to \( G_{sc} \), this quotient is a group.

Let \( G_{sc}^o \) be the reductive quotient of the parahoric subgroup of \( G_{sc} \) associated to the vertex \( x \). Then \( G_{sc}^o \to G_{sc}^o \) is a morphism with abelian kernel and cokernel. By Lemma 2.4.1 the class in \( H^2(\bar{\Gamma}, \bar{\mathbb{Q}}^\times_l) \) is unchanged if we replace \( G \) by \( G_{sc} \), so we may assume from now on that \( G \) is simply connected. Note that now \( \theta = \bar{\theta} \).

By Corollary 2.4.6 we may further assume that \( G \) is \( F \)-simple. Thus \( G = \text{Res}_{E/F} H \) for some absolutely simple simply connected group \( H \) defined over a finite extension \( E \) of \( F \). We have \( B(G, F) = B(H, E) \). Moreover, \( G_{x}^o = \text{Res}_{k_E/F} H_{x}^o \) by Appendix F. Let us represent \( G_{x}^o \) as the product \( H_{x}^o \times \cdots \times H_{x}^o \) of \( k = [k_E : k_F] \)-many factors, with Frobenius acting by \( F(h_1, \ldots, h_k) \to (F^{k_{h_k}} h_k, \ldots, h_{k-1}) \). Then \( S^o = T' \times \cdots \times T' \) for an elliptic maximal torus \( T' \subset H_{x}^o \). By Lemma 2.3.6 the class we are considering is independent of the choice of \( U \), so we may take \( U \) to have the form \( V \times \cdots \times V \) for some unipotent radical \( V \) of a Borel subgroup of \( H_{x}^o \) containing \( T' \). The diagonal embedding \( H_{x}^o \to G_{x}^o \) provides an isomorphism of varieties \( Y_{V} \to Y_{U} \). Under this isomorphism the geometric intertwining operators \( \Psi \) for \( G_{x}^o \) provide a collection of \( \Gamma \)-equivariant operators for \( H_{x}^o \). We may therefore compute the class in \( H^2(\bar{\Gamma}, \bar{\mathbb{Q}}^\times_l) \) using \( H_{x}^o \).

We are thus looking at an absolutely simple simply connected group \( H \), a maximally unramified anisotropic maximal torus \( T \subset H \) with vertex \( x \in B(H, E) \), and a \( k_E \)-non-singular character \( \theta : T(E) \to \mathbb{C}^\times \). We have \( \Gamma = N(T, H_{ad})(E) \) and \( \bar{\Gamma} \) is a subquotient of \( \Omega(T, H)(E) \). This latter group is cyclic for all possible \( H \) except for \( H \) being of split type \( D_{2n} \), in which case it can be one of \( \{1\}, \mathbb{Z}/2\mathbb{Z}, \) or \( (\mathbb{Z}/2\mathbb{Z})^2 \) by Lemma 3.1.4. When \( \bar{\Gamma} \) is cyclic \( H^2(\bar{\Gamma}, \bar{\mathbb{Q}}^\times_l) \) vanishes, so we are left do deal with the case when \( H \) is of split type \( D_{2n} \), and \( \Omega(T, H)(E) \) equals \( (\mathbb{Z}/2\mathbb{Z})^2 \).

If \( x \) is hyperspecial, then \( \Omega(T, H)(E) = \Omega(T', H_{x}^o)(k_E) \) and we can apply Lemma 2.5.1 to conclude the triviality of the class in \( H^2(\bar{\Gamma}, \bar{\mathbb{Q}}^\times_l) \). Assume now that \( x \) is not hyperspecial. The root system of \( H_{x}^o \) is the product of two systems of type \( D \). If an element of \( \Omega(T, H)(E) \) swaps the two copies, then we
can choose the Borel subgroup containing $T$ to be invariant under this element, forcing the image of this element in $\tilde{\Gamma}$ to be trivial. Then $\tilde{\Gamma}$ is cyclic and again $H^2(\tilde{\Gamma}, \mathbb{Q}_\ell^\times)$ is trivial. We can thus assume that $\Omega(T, H)(E)_\theta$ preserves each of the two irreducible factors of the root system of $H^n$. We apply Lemma 2.4.1 to replace $H^n$ with its simply connected cover and write it as a product of its two absolutely irreducible factors, and then apply Corollary 2.4.6 to reduce to studying each factor separately.

There exist choices $\Delta$ and $\Delta'$ of simple roots for $R(T, H)$ with the following properties. There is a basis $e_1, \ldots, e_{2n}$ of $X^*(T)_\mathbb{R}$ such that $\Delta = \{ e_1 - e_2, \ldots, e_{2n-1} - e_{2n}, e_2 - e_{2n+1} \}$ and the group $\Omega(T, H)(E)_\theta$ is given by $\langle w_1, w_2 \rangle$, where $w_1 = e_1 e_{2n}$, with $e_j(e_j) = (-1)^{\delta_{i,j}}$, and $w_2 = (-1)^m$, where $m(e_i) = e_{2n+i}$. There exists a second basis $e'_1, \ldots, e'_{2n}$ of $X^*(T)_\mathbb{R}$ such that $\Delta' = \{ e'_1 - e'_2, \ldots, e'_{2n-1} - e'_{2n}, e'_{2n} - e'_{2n+1} \}$ and the set $\{ e'_1 - e'_2, e'_1 - e'_2, e'_2 - e'_3, \ldots, e'_{k-1} - e'_k \} \cup \{ e'_k + e'_{k+2}, \ldots, e'_{2n-1} - e'_2, e'_{2n-1} - e'_{2n} \}$ is a set of simple roots for $R(T, H^n)$. Let $w \in \Omega(T, H)$ be the unique element that sends $\Delta$ to $\Delta'$. It is a signed permutation of $\{ e_i \}$ and thus sends $e_i$ to $(-1)^{s} e_{\sigma(i)}$. Then $w_1 = e'_i e'_j$ where $1 = \sigma(i)$ and $2 = \sigma(j)$. The transposition swapping $e'_i$ and $e'_j$ is a factor of the permutation part of $w_2$. Since $w_2$ preserves both irreducible factors of the root system $R(T, H^n)$, we must have that either $i, j \leq k$ or $i, j > k$.

We thus see that $w_1$ acts trivially on one of the factors, and belongs to the Weyl group of the other factor. Consider now $w_2$. Its permutation part is a product of $n$ disjoint transpositions. Since it preserves both factors, these factors are of type $D_{2a}$ and $D_{2b}$ respectively for some $a + b = n$. The restriction of $w_2$ to $D_{2a}$ has a permutation part given by a product of $a$ disjoint transpositions, and the restriction of $w_2$ to $D_{2b}$ has a permutation part given by a product of $b$ disjoint transpositions. We conclude that $w_2$ acts on each factor by a Weyl element.

We can now apply Lemma 2.5.1 again to each of the factors to conclude the triviality of the class $[\theta]$.  

\section*{3.4 A parameterization of $\text{Ind}^{G}_{S, \theta}$}

In this subsection we consider the induction of $\kappa(S, \theta)$ from $S(F)G(F)[\mathfrak{X}, 0]$ to $G(F)[\mathfrak{X}]$. An irreducible constituent of $\kappa(S, \theta)$ may induce to a reducible representation of $G(F)[\mathfrak{X}]$, and the irreducible constituents of that induction may occur with multiplicity greater than 1. Moreover, two irreducible constituents of $\kappa(S, \theta)$ may have isomorphic inductions. We will describe these phenomena by finding an exact mirror of this situation in a simpler setting, namely the group $N(S, G)(F)_\theta$ and the character $\theta$ in place of the group $G(F)[\mathfrak{X}]$ and the representation $\kappa(S, \theta)$. This simpler setting will eventually be related to the dual group of $G$, and is moreover readily computable in examples shall the need arise.

According to Proposition 3.3.1 and Corollary 2.3.10 there exists a coherent splitting $\{ U \}$ of $\{ \eta_{\Psi, \theta} \}$, and we remind ourselves that now $\Gamma = N(S, G)(F)_\theta$. Thus we have the coherent collection $\cdot \Psi$. We use this collection to define a conjugation action of $N(S, G)(F)_\theta$ on $H^d_c(Y_U, \mathbb{Q}_\ell)_\theta$ in a way very similar to the action on the right defined in \S 2.5. Namely, if $c_n$ denotes the action of $n \in N(S, G)(F)$ on $G^n_\theta$ by conjugation, then we have an isomorphism of varieties $c_n : Y_U \to Y^n_{\mu}$, and we define the linear operator

\[ C^\mu_n : H^d_c(Y_U, \mathbb{Q}_\ell) \to H^d_c(Y_U, \mathbb{Q}_\ell)_\theta \]

for $n \in N(S, G)(F)_\theta$ as the composition $C^\mu_n = [\cdot \Psi]_{[\mu]} \circ c_n$.  

\[ C^\mu_n : H^d_c(Y_U, \mathbb{Q}_\ell) \to H^d_c(Y_U, \mathbb{Q}_\ell)_\theta \]

for $n \in N(S, G)(F)_\theta$ as the composition $C^\mu_n = [\cdot \Psi]_{[\mu]} \circ c_n$.  

\[ C^\mu_n : H^d_c(Y_U, \mathbb{Q}_\ell) \to H^d_c(Y_U, \mathbb{Q}_\ell)_\theta \]
Unlike the analogous operator $R_n^U$ of §2.5, this operator does not commute with the left action of $G^x_k(k_F)$, but instead translates it as $C_n^{\epsilon,U} \circ l_g = l_{gmn^{-1}} \circ C_n^{\epsilon,U}$. For $m \in N(S,G(F)_{x,0})$ and $n \in N(S,G)(F)$ the two operators $R_m^U$ and $C_n^{\epsilon,U}$ satisfy the relation $C_n^{\epsilon,U}P_m^U = P_{mn^{-1}}^U C_n^{\epsilon,U}$, as one sees immediately using the equivariance of the collection $\epsilon \Psi$. Furthermore, Fact 2.5.3 holds with $C_n^{\epsilon,U}$ in place of $R_n^U$.

The map $N(S,G)(F)_{\theta} \to \text{Aut}_{\mathcal{O}}(H^d_{\mu}(Y_0, \tilde{Q}_{\tilde{L}}))$ given by $n \mapsto C_n^{\epsilon,U}$ is a group homomorphism and defines the conjugation action that we sought. This action extends the conjugation action of $S(F)$ on $H^d_{\mu}(Y_0, \tilde{Q}_{\tilde{L}})$ that gave rise to the extension $\kappa_{(S,\theta)}$ of $\kappa_{(S,\tilde{b})}$. For $n \in N(S,G)(F)_{\theta} \subset N(S,G)(F)_{\theta}$ and $s \in S(F)$ we have $C_n^{\epsilon,U}(sv) = (nssn^{-1})C_n^{\epsilon,U}(v)$.

We have thus extended the $S(F)G(F)_{x,0} \times N(S,G)(F)_{\theta}$. This extension depends on the choice of $\epsilon$, so we may denote it by $\kappa_{(S,\theta)}^\epsilon$. Let $I_{(S,\theta)}^\epsilon$ denote its induction to $G(F)_{x} \times N(S,G)(F)_{\theta}$. Using the fact that $N(S,G)(F)_{\theta} \subset G(F)_{x}$ we can now form the representation $I_{(S,\theta)}^\epsilon(g,n) := J_{(S,\theta)}^\epsilon(gn^{-1}, n)$ of $G(F)_{x} \times N(S,G)(F)_{\theta}$.

Note that $I_{(S,\theta)}^\epsilon \triangleright\triangleleft G(F)_{x} = \text{Ind}_{S(F)G(F)_{x,0}}^{G(F)_{x}} \kappa_{(S,\theta)}^\epsilon$. By definition we have $I_{(S,\theta)}^\epsilon(1 \times s) = \theta^{-1}(s)$ for $s \in S(F)$. Therefore we can decompose $\text{Ind}_{S(F)G(F)_{x,0}}^{G(F)_{x}} \kappa_{(S,\theta)}^\epsilon$ under the action of $\{1 \times N(S,G)(F)_{\theta}\} \times N(S,G)(F)_{\theta}$ and the pieces will be isotypic for elements of $\text{Irr}(N(S,G)(F)_{\theta}, \theta^{-1})$, i.e. duals of elements of $\text{Irr}(N(S,G)(F)_{\theta}, \theta)$. Given $\rho \in \text{Irr}(N(S,G)(F)_{\theta}, \theta)$, consider the subrepresentation of $\text{Ind}_{S(F)G(F)_{x,0}}^{G(F)_{x}} \kappa_{(S,\theta)}^\epsilon$ defined as

$$\kappa_{(S,\theta),\rho}^\epsilon = \text{Hom}_{\{1 \times N(S,G)(F)_{\theta}\}}(\rho^\vee, I_{(S,\theta)}^\epsilon).$$

**Proposition 3.4.1.**

1. For each $\rho \in \text{Irr}(N(S,G)(F)_{\theta}, \theta)$ the $\rho^\vee$-isotypic constituent $\kappa_{(S,\theta),\rho}^\epsilon$ is irreducible.

2. We obtain a bijection

$$[\text{Ind}\kappa_{(S,\theta)}] \longleftrightarrow \text{Irr}(N(S,G)(F)_{\theta}, \theta), \quad \kappa_{(S,\theta),\rho}^\epsilon \leftrightarrow \rho.$$

3. Under this bijection, the multiplicity of $\kappa_{(S,\theta),\rho}$ in $\text{Ind}\kappa_{(S,\theta)}$ equals $\text{dim}(\rho)$.

4. For $\delta \in (N(S,G)(F)_{\theta}/S(F))^\ast$ we have $\kappa_{(S,\theta),\rho}^{\delta} = \kappa_{(S,\theta),\delta \otimes \rho}^\epsilon$.

5. For $\phi : G_{x}(k_F) \to \mathbb{C}^\times$ trivial on $G_{x,m}(k_F)$ we have $\phi \otimes \kappa_{(S,\theta),\rho}^\epsilon = \kappa_{(S,\theta),\rho,\phi \otimes \rho}^\epsilon$.

**Proof.** We apply Proposition B.3 to the representation $I'$. The first assumption of this Proposition is satisfied by Corollary 3.2.3 and Lemma B.1 applied to the representation $\kappa_{(S,\theta)}$ of $S(F)G(F)_{x,0} \times N(S,F,G)(F)_{x,0}$. The second assumption of the Proposition is satisfied by Lemma 3.2.4. Now Lemma B.1 applied to the representation $I$ gives the first three points.

For the behavior under $\epsilon \mapsto \delta \cdot \epsilon$ let us write $\kappa_{(S,\theta)}^\epsilon$ for the extension of $\kappa_{(S,\theta)}$ to a representation of $S(F)G(F)_{x,0} \times N(S,G)(F)_{\theta}$. Then $\kappa_{(S,\theta)}^{\delta \epsilon} = \kappa_{(S,\theta)}^\epsilon \otimes \delta$, where now $\delta$ is inflated to a character of $S(F)G(F)_{x,0} \times N(S,G)(F)_{\theta}$ trivial on $S(F)G(F)_{x,0}$. Therefore $(I')^{\delta \epsilon} = (I')^{\epsilon} \otimes \delta$ and $I^{\delta \epsilon} = I^{\epsilon} \otimes \delta$, and the claim follows.

For the behavior under $\rho \mapsto \phi \cdot \rho$ we apply Lemma 2.4.7 to $\alpha = v_n$ to conclude that the isomorphism $\phi \otimes \kappa_{(S,\theta)} \to \kappa_{(S,\theta),\rho}$ of $S(F)G(F)_{x,0}$-representations
of Corollary 3.2.3 is also equivariant with respect to the operators $C_n$ and therefore is an isomorphism of representations of $S(F)G(F)_{x,0} \rtimes N(S, G)(F)_\theta$. Note that there are two equivalent ways to think of $\phi \otimes \kappa_{(S, \theta)}$ as a representation of $S(F)G(F)_{x,0} \rtimes N(S, G)(F)_\theta$—either as the vector space underlying $\kappa_{(S, \theta)}$, on which the action of $S(F)G(F)_{x,0}$ is twisted by $\phi$, and the action of $N(S, G)(F)_\theta$ is unaltered, or as the representation $\kappa_{(S, \theta)}$ of $S(F)G(F)_{x,0} \rtimes N(S, G)(F)_\theta$ twisted by $\phi$, where $\phi$ is now viewed as a character of the group $S(F)G(F)_{x,0} \rtimes N(S, G)(F)_\theta$ that is trivial on $N(S, G)(F)_\theta$. After induction we obtain the isomorphism $\phi \otimes J^*_{(S, \theta)} \to J^*_{(S, \theta, \phi)}$ of $G(F)_x \rtimes N(S, G)(F)_\theta$ representations, and thus the isomorphism $(\phi \otimes \phi^{-1}) \otimes J^*_{(S, \theta)} \to J^*_{(S, \theta, \phi)}$ of $G(F)_x \times N(S, G)(F)_\theta$. □

### 3.5 A parameterization of the depth-zero Deligne-Lusztig packet

Consider a tuple $(S, \theta, \rho, \epsilon)$, where $S \subset G$ is a maximally unramified maximal torus, $\theta : S(F) \to \mathbb{C}^\times$ is a $k_F$-non-singular character in the sense of Definition 3.0.1, $\rho$ is an irreducible representation of $N(S, G)(F)_\theta$ whose restriction to $S(F)$ contains $\theta$, and $\epsilon$ is a coherent splitting of the family of 2-cocycle $\{\eta_{S, \theta}\}$ as in §2.3, where $U$ ranges over the unipotent radicals of the Borel subgroups of $G^\circ$ containing $S$. We shall consider two such tuples $(S_i, \theta_i, \rho_i, \epsilon_i)$ for $i = 1, 2$ equivalent if there exist $g \in G(F)$ and $\delta \in (N(S_1, G)(F)_{\theta_i}/S_i(F))^*$ such that $(S_2, \theta_2, \rho_2, \epsilon_2) = \text{Ad}(g)(S_1, \theta_1, \rho_1 \otimes \delta, \delta^{-1} \epsilon_1).

**Lemma 3.5.1.** Fix a vertex $x \in \mathcal{B}(G, F)$. The map $(S, \theta, \rho, \epsilon) \mapsto \kappa^\epsilon_{(S, \theta, \rho)}$ is a bijection between the set of equivalence classes of tuples $(S, \theta, \rho, \epsilon)$ s.t. $x$ is the vertex for $S$, and the set of irreducible representations of $G(F)_x \rtimes G(F)_{x,0}$ whose restriction to $G(F)_{x,0}$ contains a non-singular cuspidal representation of $G^\circ_x(k_F)$.

**Proof.** By Proposition 3.4.1 the map is well-defined.

For injectivity, assume that $\kappa^\epsilon_{(S_1, \theta_1, \rho_1)} \cong \kappa^\epsilon_{(S_2, \theta_2, \rho_2)}$. This means that the inductions to $G(F)_x$ of $\kappa_{(S_1, \theta_1, \rho_1)}$ and $\kappa_{(S_2, \theta_2, \rho_2)}$ share a common constituent, so there exists $g \in G(F)_x$ s.t. $\kappa^g_{(S_1, \theta_1)}$ and $\kappa^g_{(S_2, \theta_2)}$ share a common constituent. Lemma 3.2.4 implies that $(S_1, \theta_1)$ and $(S_2, \theta_2)$ are conjugate under $G(F)_x$. We may therefore assume that $S_1 = S_2 = S$ and $\theta_1 = \theta_2 = \theta$. Furthermore we may arrange that $\epsilon_1 = \epsilon_2 = \epsilon$. Now we have $\kappa^\epsilon_{(S, \theta, \rho_1)} \cong \kappa^\epsilon_{(S, \theta, \rho_2)}$, which by Proposition 3.4.1 implies $\rho_1 \cong \rho_2$.

For surjectivity let $\tau$ be an irreducible representation of $G(F)_x \rtimes G(F)_{x,0}$ whose restriction to $G(F)_{x,0}/G(F)_{x,0}$ contains a non-singular cuspidal representation. It is enough to find a maximally unramified maximal torus $S \subset G$ with vertex $x$ and a $k_F$-non-singular character $\theta$ s.t.

$$0 \neq \Hom_{G(F)_x}(\tau, \text{Ind}_{S(F)G(F)_{x,0}}^{G(F)_x} \kappa_{(S, \theta)}) = \Hom_{S(F)G(F)_{x,0}}(\tau, \kappa_{(S, \theta)}).$$

By assumption there exists an elliptic maximal torus $S \subset G^\circ_x$ and a non-singular character $\theta : S(k_F) \to \mathbb{C}^\times$ s.t. $\Hom_{G^\circ_x(k_F)}(\tau, R_{S, \theta}) \neq 0$. By [Kal19, Lemma 3.4.4] there exists a maximally unramified maximal torus $S \subset G$ with vertex $x$ whose image in $G^\circ_x$ is $S$. Let $\theta$ be any extension of $\theta$ to $S(F)$. The representation $\kappa_{(S, \theta)}$ is an extension of $\kappa_{(S, \theta)}$ to $S(F)G(F)_{x,0}$ of the inflation to $G(F)_{x,0}$ of $R_{S, \theta})$. Trivially we still have $\Hom_{G(F)_{x,0}}(\tau, \kappa_{(S, \theta)}) \neq 0$. By Corollary 3.2.3 the irreducible constituents of $\kappa_{(S, \theta)}$ are given by $\kappa^\epsilon_{(S, \theta)}$, for extensions $\hat{\theta}$ of $\theta$ to $N(S, S(F)G(F)_{x,0})\theta$. Thus for one such $\hat{\theta}$ we have $\Hom_{G(F)_{x,0}}(\tau, \kappa^\epsilon_{(S, \theta)}) \neq 0$.
The restriction $\tau \mid_{S(F)G(F)_{x,0}}$ is semi-simple by [BH06, Prop. 2.7(1)]. Let $\tau^0$ be an irreducible factor of $\tau \mid_{S(F)G(F)_{x,0}}$ s.t. $\text{Hom}_{G(F)_{x,0}}(\tau^0, \kappa_{(S,\theta)}) \neq 0$. Apply Lemma A.4 to the exact sequence

$$1 \to G(F)_{x,0} \to S(F)G(F)_{x,0} \to S(F)/S(F)_{0} \to 1$$

to find a character $\phi : S(F)/S(F)_{0} \to \mathbb{C}^\times$ s.t. $\tau^0 = \phi \otimes \kappa^{r}_{(S,\phi \theta)}$, the latter being equal by Corollary 3.2.3 to $\kappa^{r}_{(S,\phi \theta)}$. Replacing $\theta$ by $\phi \theta$ we now have $\text{Hom}_{S(F)G(F)_{x,0}}(\tau_0, \kappa_{(S,\theta)}) \neq 0$. 

\[\square\]

**Proposition 3.5.2.** The representation

$$\pi^{r}_{(S,\theta,\rho)} = \text{c-Ind}^{G(F)}_{S(F)G(F)_{x,0}} \kappa^{r}_{(S,\theta,\rho)}$$

is irreducible and supercuspidal. Two tuples lead to isomorphic representations if and only if they are equivalent. If $\phi : G(F) \to \mathbb{C}^\times$ is a character of depth zero and trivial on $G_{\mathfrak{u}}(F)$, then $\chi \otimes \pi^{r}_{(S,\theta,\rho)} = \pi^{r}_{(S,\chi \otimes \chi \otimes \rho)}$.

**Proof.** That $\pi^{r}_{(S,\theta,\rho)}$ is irreducible and supercuspidal follows from [MP96, Proposition 6.6]. It is clear that conjugating a triple doesn’t change the representation. That replacing $(S, \theta, \rho, \epsilon)$ by $(S, \theta, \rho \otimes \delta, \delta^{-1} \epsilon)$ also doesn’t follow from Proposition 3.4.1. Assume conversely that two tuples $(S_i, \theta_i, \rho_i, \epsilon_i)$ for $i = 1, 2$ lead to isomorphic representations. By [MP96, Theorem 3.5] there exists $g \in G(F)$ s.t. if $x_1$ and $x_2$ are the vertices for $S_1$ and $S_2$, then $gx_1 = x_2$ and $\text{Ad}(g)\kappa^{r}_{(S_1, \theta_1, \rho_1)} = \kappa^{r}_{(S_2, \theta_2, \rho_2)}$. We may conjugate $(S_1, \theta_1, \rho_1, \epsilon_1)$ by $g$ to assume $x_1 = x_2 = x$ and $\kappa^{r}_{(S_1, \theta_1, \rho_1)} = \kappa^{r}_{(S_2, \theta_2, \rho_2)}$, after which Lemma 3.5.1 completes the proof. \[\square\]

Define further

$$\pi(S, \theta) := \text{c-Ind}^{G(F)}_{S(F)G(F)_{x,0}} \kappa(S, \theta). \quad (3.2)$$

This is a supercuspidal representation of $G(F)$. When $\theta$ is regular, $\pi(S, \theta)$ is irreducible, and was the subject of study in [Kal19]. When $\theta$ is $k_F$-non-singular, but not necessarily regular, $\pi(S, \theta)$ may be reducible. We shall write $[\pi(S, \theta)]$ for the set of irreducible constituents of $\pi(S, \theta)$ and refer to this set as the non-singular Deligne-Lusztig packet associated to $(S, \theta)$. From Proposition 3.4.1 and Proposition 3.5.2 we immediately obtain:

**Corollary 3.5.3.** Let $\epsilon$ be a coherent splitting for $\{\eta_{\Psi, 0}\}$.

1. The irreducible constituents of $\pi(S, \theta)$ are precisely the representations $\pi^{r}_{(S,\theta,\rho)}$, for irreducible smooth representations $\rho$ of $N(S, G)(F)_{\theta}$ whose restriction to $S(F)$ contains $\theta$.

2. Two such representations $\pi^{r}_{(S,\theta,\rho_1)}$ and $\pi^{r}_{(S,\theta,\rho_2)}$ are isomorphic if and only if $\rho_1 \equiv \rho_2$.

3. The multiplicity of $\pi^{r}_{(S,\theta,\rho)}$ in $\pi(S, \theta)$ equals $\text{dim} \rho$.

4. The sets $[\pi(S, \theta_i)]$ for two pairs $(S_i, \theta_i)$ are either equal or disjoint. They are equal if and only if the pairs are $G(F)$-conjugate.
3.6 General depth

Let $G$ be a connected reductive group defined over $F$ and split over a tame extension of $F$. We assume that the residual characteristic $p$ of $F$ is odd, is not a bad prime for $G$ in the sense of [SS70, §4.3], and does not divide the order of the fundamental group of $G_{der}$. If $M \subset G$ is a Levi subgroup, then $p$ satisfies the same assumptions relative to $M$.

**Definition 3.6.1.** Let $S \subset G$ be a maximal torus and $\theta : S(F) \to \mathbb{C}^\times$ a character. We shall call the pair $(S, \theta)$ tame $k_F$-non-singular elliptic (resp. tame $F$-non-singular elliptic) if

1. $S$ is elliptic and its splitting extension $E/F$ is tame;
2. Inside the connected reductive subgroup $G^0 \subset G$ with maximal torus $S$ and root system
   \[ R_{0+} = \{ \alpha \in R(S, G) | \theta(N_{E/F}(E^\times_0)) = 1 \}, \]
   the torus $S$ is maximally unramified.
3. The character $\theta$ is $k_F$-non-singular (resp. $F$-non-singular) with respect to $G^0$ in the sense of Definition 3.0.1.

**Remark 3.6.2.** A tame regular elliptic pair, in the sense of [Kal19, Definition 3.7.5], is a special case of a tame non-singular elliptic pair, due to Fact 3.0.4.

**Remark 3.6.3.** The subgroup $G^0$ is a tame twisted Levi subgroup, according to [Kal19, Lemma 3.6.1].

**Remark 3.6.4.** When $G = \text{GL}_N$, all vertices are special, in fact absolutely special, and Fact 3.0.4 shows that the notions of $F$-non-singular and $k_F$-non-singular coincide. The argument in the proof of [Kal19, Lemma 3.7.8] shows furthermore that when $p \nmid N$ a tame non-singular elliptic pair is admissible in the sense of Howe. Thus for $G = \text{GL}_N$, $p \nmid N$, the notions of non-singular, regular, extra regular, and admissible, are all equivalent.

Given a tame $k_F$-non-singular elliptic pair $(S, \theta)$ we apply [Kal19, Proposition 3.6.7] and obtain a Howe factorization $(\phi_{-1}, \ldots, \phi_d)$ for $(S, \theta)$. Then $S \subset G^0$ is a maximally unramified maximal torus. Moreover [Kal19, Fact 3.6.4] tells us that $\theta|_{\mathbb{C}_l(F)} = \phi_{-1}|_{\mathbb{C}_l(F)}$, from which we see that $\phi_{-1}$ is a $k_F$-non-singular character of $S(F)$ with respect to $G^0$. The failure of $\phi_{-1}$ to be regular mirrors the failure of $\theta$ to be regular, in the following sense.

**Lemma 3.6.5.** The natural inclusion $\Omega(S, G^0)(F) \to \Omega(S, G)(F)$ gives the identifications
\[ \Omega(S, G^0)(F)_{\phi_{-1}} = \Omega(S, G^0)(F)_{\theta} = \Omega(S, G)(F)_{\theta}. \]

The natural inclusion $N(S, G^0)(F) \to N(S, G)(F)$ gives the identifications
\[ N(S, G^0)(F)_{\phi_{-1}} = N(S, G^0)(F)_{\theta} = N(S, G)(F)_{\theta}. \]

**Proof.** This follows from [Kal19, Lemma 3.6.5].
Associated to the tame $k_F$-non-singular elliptic pair $(S, \phi_{-1})$ of $G^0$ we have the family of 2-cocycles $\eta_\emptyset = \{\eta_{\emptyset,1}\}$ of Lemma 2.3.6. This family depends only on $(S, \theta)$, but not on the Howe factorization. Indeed, by [Kal19, Lemma 3.6.6] any two factorizations differ by a refactorization. So if $(\phi_{-1}, \ldots, \phi_d)$ is another Howe factorization, then $\phi_{-1} = \phi_{-1} \cdot \chi_0$, where $\chi_0$ is a character of $G^0(F)$ of depth zero, and trivial on $G^0_{sc}(F)$, and Corollary 2.4.8 implies the claim.

Consider a tuple $(S, \theta, \rho, \epsilon)$, where $(S, \theta)$ is a tame $k_F$-non-singular elliptic pair, $\rho$ is an irreducible smooth representation of $N(S, G)(F)_\theta$ whose restriction to $S(F)$ contains $\theta$, and $\epsilon$ is a coherent splitting for the family of 2-cocycles $\eta_\emptyset$. We shall consider two such tuples $(S_i, \theta_i, \rho_i, \epsilon_i)$, $i = 1, 2$, equivalent if there exists $g \in G(F)$ and $\delta \in [N(S, G)(F)_\theta/S(F)]^*$ s.t. $(S_2, \theta_2, \rho_2, \epsilon_2) = \text{Ad}(g)(S_1, \theta_1, \rho_1 \otimes \delta, \delta^{-1} \cdot \epsilon_1)$.

Put $\delta_0 := \prod_{i=0}^d \phi_i^{-1} : G^0(F) \to \mathbb{C}^\times$, so that $\phi_{-1} = \delta_0 \theta$. Then $\rho \mapsto \delta_0 \otimes \rho =: \rho_{-1}$ is a bijection between the smooth irreducible representations of $N(S, G)(F)_\theta = N(S, G^0)(F)_{\phi_{-1}}$ whose restriction to $S(F)$ contains $\theta$, and those whose restriction contains $\phi_{-1}$. For such $\rho$, we have the irreducible depth-zero supercuspidal representation $\pi^{\epsilon'}_{(G^0, S, \phi_{-1}, \rho_{-1})}$ of $G^0(F)$ obtained in Proposition 3.5.2, and

\[(\langle G^0 \subset G^1 \cdots \subset G^d \rangle, \pi^{\epsilon'}_{(G^0, S, \phi_{-1}, \rho_{-1})}, (\phi_0, \ldots, \phi_d)),\]

is a normalized reduced generic cuspidal $G$-datum in the sense of [Kal19, Definition 3.7.1], leading to a supercuspidal representation of $G(F)$, which we shall denote by $\pi^{\epsilon'}_{(S, \theta, \rho)}$.

**Proposition 3.6.6.** 1. The representation $\pi^{\epsilon'}_{(S, \theta, \rho)}$ depends only on $(S, \theta, \rho, \epsilon)$, and is independent of the choice of Howe factorization.

2. Two tuples $(S_i, \theta_i, \rho_i, \epsilon_i)$ produce isomorphic representations if and only they are equivalent.

3. If $\phi : G(F) \to \mathbb{C}^\times$ is a character trivial on $G^0_{sc}(F)$, then $\chi \otimes \pi^{\epsilon'}_{(S, \theta, \rho)} = \pi^{\epsilon'}_{(S, \chi \theta, \chi \rho)}$.

**Proof.** Another factorization $(\phi_{-1}, \ldots, \phi_d)$ is a refactorization of $(\phi_{-1}, \ldots, \phi_d)$ according to [Kal19, Lemma 3.6.6]. Using the notation of that Lemma, write $\chi_i = \prod_{j=0}^d \phi_j^{-1} \phi_j^{-1} \phi_j^{-1} \phi_j$, and in particular $\phi_{-1} = \phi_{-1} \chi_0$. We have $\hat{\delta}_0 = \hat{\chi_0} \delta_0$ and hence $\hat{\rho}_{-1} = \hat{\chi_0} \rho_{-1}$. By Proposition 3.5.2 we obtain the equality $\pi^{\epsilon'}_{(G^0, S, \phi_{-1}, \hat{\phi_{-1}})} = \chi_0 \otimes \pi^{\epsilon'}_{(G^0, S, \phi_{-1}, \hat{\phi_{-1}})}$. With this, [Kal19, Corollary 3.5.5], which is a mild strengthening of [HM08, Theorem 6.6], imply that the normalized generic cuspidal $G$-data for the two Howe factorizations produce the same representation of $G(F)$.

It is clear that conjugate tuples produce the same representation. Replacing $(S, \theta, \rho, \epsilon)$ by $(S, \theta, \rho \otimes \delta, \delta^{-1} \epsilon)$ replaces the depth-zero tuple $(S, \phi_{-1}, \rho_{-1}, \epsilon)$ by $(S, \phi_{-1}, \rho_{-1} \otimes \delta, \delta^{-1} \epsilon)$. By Proposition 3.5.2 the corresponding representation of $G^0(F)$ is unchanged, and then so is $\pi$ itself.

Now assume that two tuples $(S_i, \theta_i, \rho_i, \epsilon_i)$, $i = 1, 2$, produce isomorphic representations. Let $(S_i, \theta_i)$ be the tuples consisting of twisted Levi tower and Howe factorization of $\theta_i$, respectively. Then [HM08, Theorem 6.6] implies the existence of $g \in G(F)$ s.t. $\text{Ad}(g)(G_i, Ad(g)(\phi_{d,0} \ldots \phi_{d, d}))$ is a refactorization of $(\phi_{1,0} \ldots \phi_{1, d})$, and $\text{Ad}(g)(\pi^{\epsilon'}_{(G^0, S, \phi_{d,0} \ldots \phi_{d, d})} \otimes \delta_0^{-1}) = [\pi^{\epsilon'}_{(G^0, S, \phi_{d,0} \cdots \phi_{d,1})} \otimes \delta_1^{-1}]$.
where as before $\delta_{i,0}^{-1}$ is the product of the restrictions to $G^0_i(F)$ of $\phi_{i,0} \ldots \phi_{i,d}$. We conjugate $(S_i, \theta_i, \rho_i, \epsilon_i)$ by $g$ to assume $g = 1$ and then have $\pi_{(G^0, S, \theta_2, \ldots, \theta_{d-1}, \rho_2, \ldots, \rho_{d-1})} \otimes \delta_{2,0}^{-1} \delta_{1,0} = \pi_{(G^0, S, \theta_1, \rho_1, \epsilon_1)}$. By [Kal19, Lemma 3.4.28] the depth of $\delta_{2,0}^{-1} \delta_{1,0}$ is zero, so we may apply Proposition 3.5.2 and see that the two depth-zero tuples for $G^0$ given by $(S_2, \theta_2 \delta_{1,0}, \rho_2, \delta_{1,0}, \epsilon_2)$ and $(S_1, \theta_1 \delta_{1,0}, \rho_1, \delta_{1,0}, \epsilon_1)$ are equivalent. But then so are the original tuples for $G$.

Finally, $(\phi_{-1}, \phi_0, \ldots, \phi_{d-1}, \chi \cdot \phi_d)$ is a Howe factorization of $\chi \cdot \theta$. \hfill $\Box$

We define, just as in the depth-zero case, also

$$\pi_{(S, \theta)}$$

(3.3)

to be the supercuspidal representation produced by Yu’s construction applied to the datum $((G^0 \subset G^1 \subset \ldots \subset G^d), \pi_{(G^0, S, \theta, \epsilon)}((\phi_0, \ldots, \phi_d)))$, where $\pi_{(G^0, S, \theta, \epsilon)}$ is the representation (3.2) for the group $G^0$ and the pair $(S, \phi_{-1})$. Again this representation may be reducible. We shall refer to this set of irreducible constituents as the non-singular Deligne-Lusztig packet associated to $(S, \theta)$ and write $[\pi_{(S, \theta)}]$ for it. From Proposition 3.6.6 and Corollary 3.5.3 we obtain the following:

**Corollary 3.6.7.**

1. The representation $\pi_{(S, \theta)}$ depends only on $(S, \theta)$, but not on the choice of Howe factorization.

2. The irreducible constituents of $\pi_{(S, \theta)}$ are $\pi_{(S, \theta, \rho)}$ for varying smooth irreducible representations $\rho$ of $N(S, G)(F) \theta$ whose restriction to $S(F)$ is $\theta$-isotypic.

3. The multiplicity of $\pi_{(S, \theta, \rho)}$ in $\pi_{(S, \theta)}$ is equal to the dimension of $\rho$.

4. The sets $[\pi_{(S_i, \theta_i)}]$ for two pairs $(S_i, \theta_i)$ are either equal or disjoint. They are equal if and only if the pairs are $G(F)$-conjugate.

5. The assignment $\delta_{0}^{-1} \otimes \pi_{(S, \theta, \rho)} \mapsto \pi_{(S, \theta, \rho)}$ is a bijection $[\pi_{(G^0, S, \theta)}] \to [\pi_{(G, S, \theta)}]$ independent of any choices.

### 3.7 Remarks on the character formula

The representation $\pi_{(S, \theta)}$ constructed in the previous subsection is a direct sum of finitely many supercuspidal representations. Despite the fact that it is not irreducible, the material in [Kal19, §4] applies to it. In particular we have the formula of [Kal19, Corollary 4.10.1] for the character values of $\pi_{(S, \theta)}$ at shallow elements of $S(F)$. We shall give here a slight reformulation of this formula that will be better suited to address the issues arising from the fact that $\theta$ is not regular.

We first recall the character formula as given in [Kal19, Corollary 4.10.1]. Let $R(S, G)$ be the absolute root system of the maximal torus $S$. Let $\Lambda : F \to C^\times$ be a non-trivial character. For each symmetric $\alpha \in R(S, G)$ consider the characters $\Lambda \circ \text{tr}_{F_a/F} : F_a \to C^\times$ and $\theta \circ N_{F_a/F} \circ \alpha^\vee : F_a^\times \to C^\times$ and let $r_{\Lambda, \alpha}$ and $r_{\theta, \alpha}$ denote their depths. Let $r_{\alpha} = r_{\Lambda, \alpha} - r_{\theta, \alpha}$ and define $\bar{a}_{\alpha} \in [F_a]_{r_{\alpha}}/[F_a]_{r_{\alpha}+}$ by the formula

$$\theta \circ N_{F_a/F} \circ \alpha^\vee (X + 1) = \Lambda \circ \text{tr}_{F_a/F}(\bar{a}_{\alpha} X),$$

where $X$ is a variable in $[F_a]_{r_{\theta, \alpha}}/[F_a]_{r_{\theta, \alpha}+}$. Then $(r_{\alpha}, \bar{a}_{\alpha})_\alpha$ is a set of mod-$\alpha$-data in the sense of [Kal19, Definition 4.6.8]. Specify a character $\chi'_{\alpha} : F_a^\times \to C^\times$
that extends the sign character \( \kappa_a : F_{\pm a}/N_{F_{\pm a}}(F_{\pm a}^\times) \to \{\pm 1\} \) by taking \( \chi'_a \) to be unramified when \( F_{\pm a}/F_{\pm a}^\times \) is unramified, and otherwise taking \( \chi'_a \) to be the unique tamely ramified character that satisfies
\[
\chi'_a(2\alpha a) = \lambda_{F_{\pm a}}(\Delta \circ \text{tr}_{F_{\pm a}/F}).
\]
Then \( (\chi'_a)_\alpha \) is a set of minimally ramified \( \chi \)-data in the sense of [Kal19, Definition 4.6.1]. We have the function
\[
\Delta_{II}^{\text{abs}}[\bar{a}, \chi'] : S(F) \to \mathbb{C}^\times,
\]
\[
\gamma \mapsto \prod_{\alpha \in R(S,G)/T \atop \alpha(\gamma) \neq 1} \chi_{\alpha} \left( \frac{\alpha(\gamma) - 1}{a_\alpha} \right)
\]
defined in [Kal19, Definition 4.6.2]. Then if \( \gamma \in S(F) \) is regular and shallow, then the character \( \theta' \) at \( \gamma \) is given by
\[
e(G) D_G(\gamma)^{-\frac{1}{2}} \epsilon_L(X^*(T_0)_{\mathbb{C}} - X^*(S)_{\mathbb{C}}, \Lambda) \sum_{w \in N(S,G)(F)/S(F)} \Delta_{II}^{\text{abs}}[\bar{a}, \chi'](\gamma^w) \theta(\gamma^w),
\]
where \( \epsilon(G) \) is the Kottwitz sign of \( G \), \( D_G(\gamma) \) is the Weyl discriminant, \( T_0 \) is the minimal Levi subgroup of the quasi-split inner form of \( G \), \( \epsilon_L \) is the Langlands normalization of the local \( \epsilon \)-factor. The character \( \theta' \) is obtained from \( \theta \) by \( \theta'(\gamma) = \epsilon_{f, \text{ram}}(\gamma) \epsilon_{\text{ram}}(\gamma) \theta(\gamma) \), where \( \epsilon_{f, \text{ram}} \) is defined in [Kal19, Definition 4.7.3] and \( \epsilon_{\text{ram}} \) in [Kal19, (4.3.3)].

For the construction of \( L \)-packets in [Kal19, §5] it was essential that the regularity of \( \theta \) implies that of \( \theta' \). The problem that we now have to face is that, even if \( \theta \) is \( F \)-non-singular with respect to \( G^0 \), \( \theta' \) might fail to be even \( k_F \)-non-singular with respect to \( G^0 \). This is due to the properties of \( \epsilon_{\text{ram}} \). There will be a parallel phenomenon occurring on the Galois side. Our task here will be to reformulate (3.4) in such a way that the parallell between the representation-theoretic and Galois-theoretic phenomena can be recognized.

Recall the twisted Levi subgroup \( G^0 \subset G \) from Definition 3.6.1. Let \( S^0 = Z(G^0)^e \) and let \( R(S^0, G) \) be the set of weights for the adjoint action of \( S^0 \) on \( g \). It is the image of \( R(S,G) \) under the restriction map \( X^*(T) \to X^*(S^0) \). Write \( \alpha_0 \in R(S^0, G) \) for the image of \( \alpha \in R(S,G) \). Then \( \alpha_0 \neq 0 \) if and only if \( \alpha \notin R(S,G^0) \). Let \( \chi_0 = (\chi_{\alpha_0})_{\alpha_0} \) be a set of minimally ramified \( \chi \)-data for \( R(S^0, G) \).

In [Kala, §5.3] the notation \( \inf\chi_0 \) was introduced for the \( \chi \)-data \( (\chi''_\alpha)_{\alpha} \), where \( \chi''_\alpha = \chi_{\alpha_0} \circ N_{F_{\pm a}/F_{\pm a}} \) for all \( \alpha \in R(S,G) \setminus R(S,G^0) \), and \( \chi''_\alpha \) is the unramified quadratic character for all symmetric (automatically unramified) \( \alpha \in R(S,G^0) \). It was shown in [Kala, §5.3] that \( (\chi''_\alpha)_{\alpha} \) is a set of \( \chi \)-data for \( R(S,G) \). Note however that this set is not minimally ramified – it is possible that \( \chi''_\alpha \) is ramified even if \( \alpha \) is unramified, and it is possible that \( \chi''_\alpha \) is non-trivial even if \( \alpha \) is asymmetric. Nonetheless, \( (\chi''_\alpha)_{\alpha} \) is tamely ramified, and hence has a canonical minimal \( \chi \)-data \( m_{\chi''} \) associated to it as in [Kala, §5.4].

Define \( \zeta_{\alpha} = \chi''_{\alpha} \cdot (\chi''_\alpha)^{-1} \). Then \( (\zeta_{\alpha})_{\alpha} \) is a set of \( \zeta \)-data in the sense of [Kal19, Definition 4.6.4] and [Kal19, Lemma 4.6.6] implies
\[
\Delta_{II}^{\text{abs}}[\bar{a}, \chi''](\gamma) = \Delta_{II}^{\text{abs}}[\bar{a}, \chi'](\gamma) \cdot \zeta_S(\gamma),
\]
where \( \zeta_S : S(F) \to \mathbb{C}^\times \) is the character of [Kal19, Definition 4.6.5]. Define now \( \theta''(\gamma) = \zeta_S(\gamma) \epsilon_{f, \text{ram}}(\gamma) \epsilon_{\text{ram}}(\gamma) \theta(\gamma) \). Then we can write (3.4) as
\[
e(G) D_G(\gamma)^{-\frac{1}{2}} \epsilon_L(X^*(T_0)_{\mathbb{C}} - X^*(S)_{\mathbb{C}}, \Lambda) \sum_{w \in N(S,G)(F)/S(F)} \Delta_{II}^{\text{abs}}[\bar{a}, \chi''](\gamma^w) \theta''(\gamma^w),
\]
(3.5)
The point of rewriting the character formula in this way is the following.

**Lemma 3.7.1.** If \( \theta \) is \( k_F \)-non-singular with respect to \( G^0 \) then so is \( \theta'^0 \).

**Proof.** It is shown in [FKS] that the product \( \zeta_S(\gamma)\epsilon_{f,\text{ram}}(\gamma)\epsilon_{\text{ram}}(\gamma) \), i.e. the difference \( \theta'^0/\theta \), extends from \( S(F) \) to \( G^0(F)_x \), hence the lemma. \( \square \)

## 4 Supercuspidal \( L \)-packets

### 4.1 Factorization of parameters

Let \( G \) be a quasi-split connected reductive group defined over \( F \) and split over a tame extension of \( F \), \( \hat{G} \) its complex dual group, \( L \) its \( L \)-group. We assume that the residual characteristic \( p \) of \( F \) is odd, is not a bad prime for \( G \) in the sense of [SS70, §4.3], and does not divide the order of the fundamental group of \( G_{\text{sc}} \). If \( M \subset G \) is a Levi subgroup then \( p \) is not a bad prime for \( M \) and does not divide the order of the fundamental group of \( M_{\text{der}} \). Then the same properties hold for \( \hat{G} \) in place of \( G \).

**Definition 4.1.1.** A supercuspidal Langlands parameter for \( G \) is a discrete Langlands parameter \( W_F \rightarrow \hat{L} \).

In other words, it is a discrete parameter \( W_F \times \mathrm{SL}_2(\mathbb{C}) \rightarrow \hat{L} \) whose restriction to \( \mathrm{SL}_2(\mathbb{C}) \) is trivial. It is expected that these parameters correspond precisely to those discrete series \( L \)-packets of \( G \) that consists entirely of supercuspidal representations. This expectation was formulated in [DR09, §3.5] and is in hindsight a special case of a more precise conjecture [AMS].

**Definition 4.1.2.** A supercuspidal parameter is called **torally wild** if \( \varphi(P_F) \) is contained in a maximal torus of \( \hat{G} \).

**Lemma 4.1.3.** Let \( \varphi : W_F \rightarrow \hat{L} \) be a supercuspidal parameter.

1. If \( p \) does not divide the order of the Weyl group of \( G \), then \( \varphi \) is torally wild.
2. If \( \varphi \) is torally wild, then \( \text{Cent}(\varphi(I_F), \hat{G})^0 \) is a torus.

**Proof.** The image \( \varphi(P_F) \subset \hat{G} \) is a finite \( p \)-group, hence nilpotent, hence supersolvable. As a supersolvable group of semi-simple automorphisms of \( \hat{G} \) it normalizes a maximal torus \( \hat{T} \subset \hat{G} \), by [SS70, §II,Theorem 5.16]. By assumption \( p \nmid \Omega(\hat{T}, \hat{G}) \), so the image of \( \varphi(P_F) \) in \( \Omega(\hat{T}, \hat{G}) \) is trivial, so \( \varphi(P_F) \subset \hat{T} \).

Let \( \hat{M} \) be the centralizer of \( \varphi(P_F) \), a Levi subgroup of \( \hat{G} \) by [Kal19, Lemma 5.2.2]. Let \( \hat{L} \subset \hat{M} \) be the connected centralizer of \( \varphi(I_F) \). It is a connected reductive group and is a torus if and only if \( \hat{L}/\text{Z}(\hat{G})^0 \) is a torus. We may thus replace \( \hat{G} \) by \( \hat{G}/\text{Z}(\hat{G})^0 \) and assume that \( \hat{G} \) is semi-simple. This has the effect that \( \hat{L}\epsilon(\text{Frob}) \subset \text{Cent}(\varphi(W_F), \hat{G}) \) is finite, where \( \text{Frob} \in W_F \) is any Frobenius element. Now \( \varphi(\text{Frob}) \) is a semi-simple automorphism of \( \hat{L} \), which we decompose as a product \( \text{Ad}(l)\theta \) of an automorphism \( \theta \) of \( \hat{L} \) that preserves a pinning \( (\hat{T}, \hat{B}, \{X_\alpha\}) \) of \( \hat{L} \) and an inner automorphism by an element \( l \) of \( \hat{L} \). The map \( \hat{L} \rightarrow \hat{L}, \quad x \mapsto x^{-1}l\theta(x) \)
gives an isomorphism from the coset space \( \hat{L}/\text{Cent}(l_\theta, \hat{L}) \) to the \( \theta \)-twisted conjugacy class of \( l \). This \( \theta \)-twisted conjugacy class is an irreducible closed subvariety of \( \hat{L} \) and the finiteness of \( \text{Cent}(l_\theta, \hat{L}) = \hat{L}^{\varphi(F_{\text{Frob}})} \) implies that its dimension is equal to that of \( \hat{L} \). Since \( \hat{L} \) is an irreducible variety this means that the \( \theta \)-twisted conjugacy class of \( l \) is equal to \( \hat{L} \). In other words, \( \hat{L} \) is a single \( \theta \)-twisted conjugacy class. This is only possible of \( \hat{L} \) is a torus, for otherwise there is a 1-1 correspondence between \( \theta \)-twisted conjugacy classes in \( \hat{L} \) and \( \Omega(\hat{T}, \hat{L})^0 \)-orbits in the group \( \hat{T}_0 \) of \( \theta \)-coinvariants in \( \hat{T} \), the latter being a non-trivial algebraic torus, see [KS99, Lemma 3.2.A].

We will now introduce the concept of torally wild \( L \)-packet data and show that there is a natural 1-1 correspondence between \( G \)-conjugacy classes of torally wild Langlands parameters \( \hat{\theta} \) and equivalence classes of such data. The data is closely related to the regular supercuspidal \( L \)-packet data from [Kal19, §5.2], with one subtle difference.

**Definition 4.1.4.** A torally wild supercuspidal \( L \)-packet datum is a tuple \((S, \hat{j}, \chi_0, \theta)\), where

1. \( S \) is a torus of dimension equal to the absolute rank of \( G \), defined over \( F \) and split over a tame extension of \( F \);
2. \( \hat{j} : \hat{S} \to \hat{G} \) is an embedding of complex reductive groups whose \( \hat{G} \)-conjugacy class is \( \Gamma \)-stable;
3. \( \chi_0 = (\chi_{\alpha_0})_{\alpha_0} \) is minimally ramified \( \chi \)-data for \( R(S^0, G) \), as explained below;
4. and \( \theta : S(F) \to \mathbb{C}^\times \) is a character.

subject to the condition that \((S, \theta)\) is a tame \( F \)-non-singular elliptic pair in the sense of Definition 3.6.1.

We need to explain the notation in the third point. As discussed in [Kal19, §5.1] we obtain from \( \hat{j} \) a \( \Gamma \)-invariant root system \( R(S, G) \subset X^\times(S) \). We can then define the subsystem \( R_{0+} \subset R(S, G) \) as in Definition 3.6.1. Let \( S^0 \subset S \) be the connected component of the intersection of the kernels of all elements of \( R_{0+} \) and \( R(S^0, G) \) be the image of \( R(S, G) \) under the restriction map \( X^\times(S) \to X^\times(S^0) \). The notion of minimally ramified \( \chi \)-data was introduced in [Kal19, Definition 4.6.1] in the case of root systems, but it makes sense in the more general context here too and means that \( \chi_{\alpha_0} = 1 \) if \( \alpha_0 \) is not symmetric, \( \chi_{\alpha_0} \) is the unramified quadratic character if \( \alpha_0 \) is unramified symmetric, and \( \chi_{\alpha_0} \) is one of the two possible tamely ramified characters of \( F_{\alpha_0}^\times \), that restrict to the non-trivial character of \( F_{\pm \alpha_0}/N(F_{\alpha_0}) \) if \( \alpha_0 \) is ramified symmetric.

The difference between this definition and [Kal19, Definition 5.2.4] is, besides requiring that \((S, \theta)\) be non-singular rather than regular, is the usage of \( \chi \)-data for \( R(S^0, G) \) in place of \( R(S, G) \). This is related to the issue discussed in §3.7.

**Definition 4.1.5.** A morphism \((S, \hat{j}, \chi_0, \theta) \to (S', \hat{j}', \chi_0', \theta')\) of torally wild supercuspidal \( L \)-packet data is a triple \((\iota, g, \zeta_0)\), where

1. \( \iota : S \to S' \) is an isomorphism of \( F \)-tori;
2. \( g \in \hat{G} \)

3. and \( \zeta_0 = (\zeta_{0_i})_{i=1}^k \) is a set of \( \zeta \)-data for \( R(S^0, G) \) in the sense of [Kal19, Definition 4.6.4].

We require that \( \hat{j} \circ \hat{\iota} = \text{Ad}(g) \circ j' \), that \( \chi_{\alpha_i} \circ \iota = \chi_{\alpha_i}' \cdot \zeta_{0_i} \), and that \( \zeta_{0_i}^{-1} \cdot \theta' \circ \iota = \theta \).

Here we take \( \zeta = \inf_{\alpha_i} \) and take \( \zeta' \) to be the character of \( S'(F) \) corresponding to \( \zeta \) as in [Kal19, Definition 4.6.5]. Composition of morphisms is defined in the obvious way.

**Remark 4.1.6.** Every morphism is an isomorphism. When \( \theta \) is regular [Kal19, Lemma 5.2.6] shows that \( \chi \) is regular in the sense of Remark 4.1.6. obvious way.

**Proposition 4.1.7.** There is a natural 1-1 correspondence between \( \hat{G} \)-conjugacy classes of torally wild Langlands parameters for \( G \) and isomorphism classes of torally wild \( L \)-packet data.

The rest of this subsection is devoted to the proof of this proposition. The arguments are an amplification of those in the proof of [Kal19, Proposition 5.2.7]. We will present them here in an abbreviated form and a slightly different structure, in the hope that this will help shed a better light on them.

First, we give the two inverse constructions. Starting with a torally wild \( L \)-packet datum \((S, j, \chi_0, \theta)\) we extend \( j \) to an \( L \)-embedding \( \hat{j} : L S \to \hat{L} G \) using the \( \chi \)-data \( \chi = \inf_{\chi_0} \) recalled in §3.7 and let \( \phi = \hat{j} \circ \varphi_S \), where \( \varphi_S : W_F \to L S \) is the Langlands parameter for the character \( \theta \). In this way we obtain from the tuple \((S, j, \chi_0, \theta)\) a Langlands parameter \( \phi \).

Conversely, given a torally wild parameter \( \varphi : W_F \to \hat{L} G \) we apply [Kal19, Lemma 5.2.2] and Lemma 4.1.3 to obtain the Levi subgroup \( \hat{M} \subset \hat{G} \) and a maximal torus \( \hat{T} \subset \hat{M} \), both normalized by \( \varphi(W_F) \). Conjugating \( \varphi \) in \( \hat{G} \) if necessary we may arrange that \( \hat{T} \) is part of a \( \Gamma \)-invariant Borel pair of \( \hat{G} \). Then \( \varphi : W_F \to N(\hat{T}, \hat{G}) \rtimes W_F \). The action of \( W_F \) on \( \hat{T} \) via \( \text{Ad}(\varphi(\cdot)) \) extends to \( \Gamma_F \).

We denote by \( \hat{S} \) the corresponding \( \Gamma_F \)-module structure on \( \hat{T} \), and by \( \hat{i} : \hat{S} \to \hat{G} \) the tautological embedding \( \hat{T} \to \hat{G} \). Let \( S \) be the algebraic torus defined over \( F \) and dual to \( \hat{S} \). Write \( R_{0+} = R(\hat{S}, \hat{M}) \) and let \( S^0 \subset S \) be defined with respect to this \( R_{0+} \). Choose minimal \( \chi \)-data \( \chi_0 \) for \( R(S^0, G) \) and use \( \chi = \inf_{\chi_0} \) to extend \( \hat{j} \) to an \( L \)-embedding \( \hat{j}_{\chi_0} : L S \to \hat{L} G \). The parameter \( \varphi \) factors through this embedding as \( \varphi = \hat{j}_{\chi_0} \circ \varphi_{S, \chi_0} \) for \( \varphi_{S, \chi_0} : W_F \to L S \). We let \( \theta_{\chi_0} : \hat{S}(F) \to \mathbb{C}^\times \) be the corresponding character. In this way we obtain from \( \varphi \) the tuple \((S, j, \chi_0, \theta_{\chi_0})\).

This concludes the description of the two constructions. The proofs that the isomorphism class of the tuple \((S, j, \chi_0, \theta_{\chi_0})\) produced from \( \varphi \) depends only on the \( \hat{G} \)-conjugacy class of \( \varphi \), and conversely that the \( \hat{G} \)-conjugacy class of the parameter \( \varphi \) produced from a tuple \((S, j, \chi_0, \theta)\), depends only on the isomorphism class of that tuple, are very similar to the ones given in the proof of [Kal19, Proposition 5.2.7]. They are routine and we will not repeat them. Moreover, the fact that the two constructions are inverse to each other is clear.
What remains to be checked is that the tuple \((S, \tilde{j}, \chi_0, \theta_{\chi_0})\) produced from a torally wild \(\varphi\) is a torally wild \(L\)-packet datum, and conversely that the parameter \(\varphi\) produced from a torally wild \(L\)-packet datum \((S, \tilde{j}, \chi_0, \theta)\) is torally wild.

We begin by noting that, given a torally wild parameter \(\varphi\), the definition of \(\hat{M}\) implies that \(\varphi(P_F) \subset Z(\hat{M}) \subset \hat{T}\), so the \(\Gamma\)-module \(\hat{S}\) is tame. Moreover, [Kal19, Lemma 5.2.2] implies that \(\varphi(I_F)\) preserves a Borel subgroup of \(\hat{M}\) containing \(\hat{T}\), so the action of \(I_F\) on \(R(\hat{S}, \hat{M})\) preserves a positive chamber.

**Lemma 4.1.8.** Under the identification \(R(\hat{S}, \hat{G}) = R^\psi(S, G)\) the root system \(R(\hat{S}, \hat{M})\) is identified with the coroot system of the root system \(R_{0+}\) of Definition 3.6.1.

**Proof.** We have \(R(\hat{S}, \hat{M}) = \{\hat{\alpha} \in R(\hat{S}, \hat{G})|\hat{\alpha}(\varphi(P_F)) = 1\}\). For any \(\hat{\alpha} \in R(\hat{S}, \hat{G})\) let \(\alpha^\vee \in R^\psi(S, G)\) be the corresponding cocharacter. Letting \(E/F\) be the tame Galois extension splitting \(S\), the parameter of the character \(\theta \circ N_{E/F} \circ \alpha^\vee\) is equal to the restriction to \(E_F\) of \(\hat{\alpha} \circ \varphi_S\). The tameness of the \(\chi\)-data implies that \(\varphi|_{P_E} = \varphi|_{P_E}\). Since \(P_E = P_F\) we see using [Yu09, Theorem 7.10] that \(R(\hat{S}, \hat{M})\) is the subset of \(R^\psi(S, G)\) consisting of those \(\alpha^\vee\) for which \(\theta \circ N_{E/F} \circ \alpha^\vee\) restricts trivially to \(E_F\), as claimed.

**Lemma 4.1.9.** Let \((S, \tilde{j}, \chi_0, \theta)\) be a tuple as in Definition 4.1.4, but without assuming that \((S, \theta)\) is a tame non-singular elliptic pair. Instead we only assume that \(S\) is tame and maximally unramified in \(G^0\). Let \(\varphi = L^\chi_0 \circ \varphi_S\). Then \(\theta\) is \(F\)-non-singular with respect to \(G^0\) if and only if \(\hat{M}\varphi(I_F)^o\) is a torus.

Granting this lemma, we complete the proof of Proposition 4.1.7 as follows.

Since \(\varphi\) is torally wild Lemma 4.1.3 implies that \(\hat{M}\varphi(I_F)^o = \text{Cent}(\varphi(I_F), \hat{G})^o\) is a torus, so \(\theta\) is \(F\)-non-singular with respect to \(G^0\) by Lemma 4.1.9. Furthermore, \(\tilde{j}\) identifies \(\hat{S}^\Gamma,^o\) with \(\hat{T}^\varphi(W_F),^o\subset \hat{M}(\varphi(W_F)),^o = \text{Cent}(\varphi(W_F), \hat{G})^o\). The discreteness of \(\varphi\) implies that \(\hat{S}^\Gamma/Z(\hat{G})^\Gamma\) is finite, thus \(S/Z(G)\) is anisotropic.

Conversely, starting with a torally wild \(L\)-packet datum \((S, \tilde{j}, \chi_0, \theta)\) we have \(\text{Cent}(\varphi(I_F), \hat{G})^o = \hat{M}(\varphi(I_F)),^o\), which is a torus by Lemma 4.1.9. By [Kal19, Lemma 5.2.2] it normalizes \(\hat{T}\), so by rigidity of tori it centralizes \(\hat{T}\), but since \(\hat{T}\) is maximal it must then lie inside of \(\hat{T}\). Thus \(\text{Cent}(\varphi(W_F), \hat{G})^o \subset \hat{T}^\varphi(W_F),^o = \hat{T}^\varphi,^o\). But \(S/Z(G)\) is anisotropic, so \(\hat{S}^\Gamma/Z(\hat{G})^\Gamma\) is finite, so \(\varphi\) is discrete.

The proof of Proposition 4.1.7 is thus reduced to the proof of Lemma 4.1.9. As a preparation for that, we take a closer look at the construction of the \(L\)-embedding \(L^\chi_0 : L^S \to L^G\) of [LS87, §2.5,§2.6]. Recall that its \(\tilde{G}\)-conjugacy class is uniquely determined by the \(\chi\)-data. Let \((\hat{T}, \hat{\theta}, \{X_\alpha\}_{\alpha \in \Delta^\vee})\) be a \(\Gamma\)-invariant pinning of \(\hat{G}\). Assume that \(\varphi(W_F)\) normalizes \(\hat{T}\), as [Kal19, Lemma 5.2.2] allows. The equation \(\varphi = L^\chi_0 \circ \varphi_{S,\chi_0}\) specifies \(L^\chi_0\) further up to \(\hat{T}\)-conjugacy.

When \(G = G^0\) then the datum \(\chi_0\) is empty and \(\chi = \inf \chi_0\) is the unique minimally ramified datum, in which \(\chi_0\) is the unramified quadratic character when \(\alpha\) is symmetric, and trivial when \(\alpha\) is asymmetric. We shall denote by \(L^\chi_0\) the corresponding \(L\)-embedding.
**Lemma 4.1.10.** Assume that $G = G^0$. There exists a representative of the $\tilde{T}$-conjugacy class of $L_{j_0}$ so that for all $x \in I_F$

$$L_{j_0}(1 \times x) = 1 \times x.$$ 

**Proof.** In order to obtain a representative of the $\tilde{T}$-conjugacy class we follow [LS87, §2.5] and make the following choices: First we choose one representative $\hat{\alpha} \in R(\hat{S}, \hat{G})_{\text{sym}}$ within each $\Gamma$-orbit. We make sure $\hat{\alpha} > 0$. For each such chosen $\hat{\alpha}$ we choose a set of representatives $w_1, \ldots, w_n \in W_F$ for the quotient $\Gamma_{\hat{\alpha}} \setminus \Gamma$, making sure that $w_i^{-1}_i \hat{\alpha} > 0$. Choose also $u_1 \in W_{\pm \hat{\alpha}} \setminus W_{\hat{\alpha}}$. Set $v_0 = 1$. Note that these choices make the gauge $p$ of [LS87, §2.5] be given by $p(\beta) = 1 \Leftrightarrow \beta > 0$.

We now obtain the representative $L_{j_0}(s \times w) = \hat{\gamma}(s) \cdot r_p(\sigma) \cdot n_w \times w$. Here $n_w \in N(\hat{T}, \hat{G})$ is the Tits lift of $\omega_w \in \Omega(\hat{T}, \hat{G})$, and $\omega_w \times \sigma$ is the action on $\hat{T}$ by $\text{Ad}(\varphi(w))$. Furthermore, $r_p : W_F \to \hat{T}$ is defined by

$$r_p(w) = \prod_{\hat{\alpha}} \prod_{i=1}^n \hat{\alpha}^\vee(\chi_{\hat{G}}(v_0(u_i(w)))).$$ 

Here $u_i(w) \in W_{\pm \hat{\alpha}}$ is defined by $w_i w = u_i(w) w_i$, with $i' \in \{1, \ldots, n\}$ the unique possible index, and $v_0(u) \in W_{\hat{\alpha}}$ for $u \in W_{\pm \hat{\alpha}}$ by $u = v_0(u) v_i'$, where $i' \in \{0, 1\}$ is again the unique possible index.

As is shown in [LS87, §2.5] making different choices changes the 1-cochain $r_p$ up to 1-coboundaries, and hence $L_{j_0}$ up to $\tilde{T}$-conjugacy. We will now show how to make the choices so as to obtain $r_p(x) = 1$ for $x \in I_F$. The assumption $G = G^0$ implies $\omega_x = 1$ and hence $n_x = 1$, so the lemma will be proved.

What we want is for the contribution of the unramified symmetric roots to vanish when $w = x$. Thus let $\hat{\alpha}$ be an unramified symmetric root. Let $I_{\hat{\alpha}}$ and $I_{\pm \hat{\alpha}}$ be the intersections of $I_F$ with $\Gamma_{\hat{\alpha}}$ and $\Gamma_{\pm \hat{\alpha}}$ respectively. We have $I_{\hat{\alpha}} = I_{\pm \hat{\alpha}}$ as $\hat{\alpha}$ is symmetric unramified. We choose representatives $\tau_i \in W_F$ for the coset space $I_F \cdot \Gamma_{\pm \hat{\alpha}} \setminus \Gamma$, again maintaining $\tau_i^{-1}_i \hat{\alpha} > 0$, as well as representatives $\delta_j \in I_F$ of the coset space $I_{\pm \hat{\alpha}} \setminus I_F$. Then $\{\delta_j \tau_i\}$ is a set of representatives for $\Gamma_{\pm \hat{\alpha}} \setminus \Gamma$.

We claim that $(\delta_j \tau_i)^{-1}_i \hat{\alpha} > 0$. Indeed, this equals $(\tau_i^{-1}_i \delta_j^{-1} \tau_i) \tau_i^{-1} \hat{\alpha}$. By construction $\tau_i^{-1} \hat{\alpha} > 0$. Moreover, $\tau_i^{-1} \delta_j^{-1} \tau_i \in I_F$ and the assumption $G = G^0$ means that the action of $I_F$ on $R(\hat{S}, \hat{G})$ preserves the set of positive roots.

The claim we just proved means that we can take $\{\delta_j \tau_i\}$ as the set $w_1, \ldots, w_n$ of representatives above. For $w = x \in I_F$ we then have $u_{ij}(x) = \delta_j \tau_i x \tau_i^{-1}_i \tau_i^{-1}$. One now observes that $i' = i$, so $u_{ij}(x)$ is an element of $\Gamma_{\pm \hat{\alpha}} \cap I_F = I_{\pm \hat{\alpha}} = I_{\hat{\alpha}}$. Hence $v_0(u_{ij}(x)) = u_{ij}(x) \in I_{\hat{\alpha}}$. But $\chi_{\hat{\alpha}}$ is unramified, so $\chi_{\hat{\alpha}}(v_0(u_{ij}(x))) = 1$.

**Remark 4.1.11.** It may be tempting to drop the assumption $G = G^0$ in the above lemma and assert that one can arrange the choices so that the cochain $r_p$ only receives contributions from the ramified symmetric roots. That is, there is a representative of the $\tilde{T}$-conjugacy class of $L_{j_0}$ so that for all $x \in I_F$

$$L_{j_0}(1 \times x) = \left( \prod_{\hat{\alpha} \in R(\hat{S}, \hat{G})_{\text{sym}, \text{ram}} \setminus \Gamma} \hat{\alpha}^\vee(z_i(x)) \right) n_x \times x,$$
where \( z_i(x) \) are complex numbers and \( n_x \) is the Tits lift of \( \omega_x \) and \( \omega_x \times x \) is the action of \( \text{Ad}(\varphi(x)) \) on \( \hat{T} \). Note however that the proof will not go through, because there is no guarantee that \( (\tau_i^{-1} \delta_j^{-1} \tau_i) \) will send the positive root \( \tau_i^{-1} \alpha \) to a positive root. And indeed, this generalization is false. This is the Galois-theoretic expression of the fact mentioned in §3.7 that the character \( \theta' \) need not be non-singular even if \( \theta \) is.

Proof of Lemma 4.1.9. We first consider the special case that \( \hat{G} = \hat{M} \). We have \( \varphi : W_F \to N(T, \hat{G}) \times W_F \). The action of \( \text{Ad}(\varphi(I_F)) \) preserves a Borel subgroup of \( \hat{G} \) containing \( \hat{T} \). Upon further conjugating \( \varphi \) we may arrange that this Borel subgroup is the chosen one \( \hat{B} \). This implies \( \varphi(I_F) \subset \hat{T} \times I_F \). In particular, all symmetric roots in \( R(S, \hat{G}) \) are unramified and our \( \chi \)-data consists of unramified quadratic characters.

According to Lemma 4.1.10 we can arrange that for all \( x \in I_F \) we have \( Lj_\chi(1 \times x) = 1 \times x \). This means that if \( \varphi_S(x) = s \times x \) then \( \varphi(x) = \tilde{j}(x) \times x \). We now specify \( x \in I_F \) to be a lift of the topological generator of \( I_F / P_F \) and let \( t \in \hat{T} \) be determined by \( \varphi(x) = t \times x \). Thus \( \varphi_S(x) = s \times x \) and \( t = \tilde{j}(s) \).

Write again \( \hat{L} = \text{Cent}(\varphi(I_F), \hat{G})^\circ \). Then \( \hat{L} \) is a connected reductive group with maximal torus \( \hat{T}^{x, o} \). To determine its root system, we following [KS99, §1.3] and consider the relative root system \( R_\text{res}(\hat{T}^{x, o}, \hat{G}) \). We subdivide its elements into types \( R1 / R2 / R3 \) as follows: \( \hat{\alpha}_{\text{res}} \in R_\text{res}(\hat{T}^{x, o}, \hat{G}) \) is of type \( R1 \) if it is neither divisible nor multipliable, of type \( R2 \) if it is multipliable, and of type \( R3 \) if it is divisible. Types \( R2 \) and \( R3 \) occur only if \( \hat{G} \) has a component of Dynkin type \( A_{2n} \) and a power of \( \varphi(x) \) preserves and acts non-trivially on this component. In that case, they occur together: the restriction of \( \hat{\alpha} \in R(\hat{T}, \hat{G}) \) to \( \hat{T}^{x, o} \) is of type \( R2 \) if and only if the smallest \( l \in \mathbb{N} \) such that \( x^l \hat{\alpha} = \hat{\alpha} \) is even and \( \hat{\beta} = \hat{\alpha} + x^{1/2} \hat{\alpha} \) is also a root. Then the restriction of \( \hat{\beta} \) to \( T^{x, o} \) is of type \( R3 \), and every relative root of type \( R3 \) occurs this way.

An element \( \hat{\alpha}_{\text{res}} \in R_\text{res}(\hat{T}^{x, o}, \hat{G}) \) belongs to the root system \( R(\hat{T}^{x, o}, \hat{L}) \) if and only if either \( \hat{\alpha}_{\text{res}} \) is of type \( R1 \) or \( R2 \) and \( N\hat{\alpha}(t) = 1 \), or \( \hat{\alpha}_{\text{res}} \) is of type \( R3 \) and \( N\hat{\alpha}(t) = -1 \). Here \( \hat{\alpha} \in R(\hat{T}, \hat{G}) \) is any root restricting to \( \hat{\alpha}_{\text{res}} \) and \( N\hat{\alpha} \) is the sum of the members of the \( x \)-orbit of \( \hat{\alpha} \).

We now consider dually \( R(S, \hat{G}) \) and the relative root system \( R_\text{res}(S', \hat{G}) \). The bijection \( R(S, \hat{G}) \to R(\hat{S}, \hat{G}) \) given by \( \alpha \to \alpha^\vee = \hat{\alpha} \) induces a type-preserving bijection \( R_\text{res}(S', \hat{G}) \to R_\text{res}(\hat{S}^{\text{I, o}}, \hat{G}) \). If \( \hat{\alpha}_{\text{res}} \) is of type \( R1 \) or \( R3 \), the coroot of \( \alpha_{\text{res}} = (\hat{\alpha}^\vee)_{\text{res}} \) is \( N\alpha^\vee = N\hat{\alpha} \). And if \( \hat{\alpha}_{\text{res}} \) is of type \( R2 \) the corresponding coroot is \( 2N\alpha^\vee = 2N\hat{\alpha} \).

We now relate this to Definition 3.0.1. Let \( F'/F \) be an unramified extension splitting \( S' \). The Langlands parameter of the character \( \theta \circ N_{F'/F} \) of \( S'(F') \) is the composition

\[
W_{F'} \longrightarrow W_F \xrightarrow{\varphi_S} \hat{S} \times W_F \longrightarrow \hat{S}_{I_F} \times W_F
\]

where the first map is the natural inclusion and the last map is the natural projection. For \( \alpha_{\text{res}} \in R_\text{res}(S', \hat{G}) \) the dual of the \( F' \)-rational homomorphism \( \alpha_{\text{res}} : \mathbb{G}_m \to S' \) is the homomorphism \( \hat{S}_{I_F} \times W_{F'} \to \hat{C}^\times \) that is trivial on \( W_{F'} \), and given by the factorization of \( kN\hat{\alpha} : \hat{S} \to \hat{C}^\times \) to \( \hat{S}_{I_F} \), where \( k = 1 \) if \( \alpha_{\text{res}} \) is
of type R1 or R3, and \( k = 2 \) if \( \alpha_{\text{res}} \) is of type R2. Thus the Langlands parameter of \( \theta \circ N_{F'/F} \circ \alpha_{\text{res}} \) is \( (kN\tilde{\alpha}) \circ \varphi_{S}|_{W_{F'}} \) and the character \( \theta \circ N_{F'/F} \circ \alpha_{\text{res}} \) is trivial on \( O_{F'}^\times \), if and only if its parameter is has trivial restriction to \( I_{F'} = I_F \).

Let \( \hat{\alpha}_{\text{res}} \) be of type R1. Then \( \hat{\alpha}_{\text{res}} \) occurs in the root system of \( \hat{L} \) if and only if \( N\tilde{\alpha}(t) = 1 \), which is equivalent to the triviality of \( (N\tilde{\alpha}) \circ \varphi_{S}|_{I_F} \).

Let \( \hat{\alpha}_{\text{res}} \) be of type R2. Choose a lift \( \tilde{\alpha} \in R(\tilde{S}, \tilde{G}) \) and let \( \tilde{\beta} = \tilde{\alpha} + x/2 \tilde{\beta} \) be as above, so that \( \beta_{\text{res}} \) is of type R3. Note that \( N\tilde{\beta} = \tilde{N}\tilde{\beta} \). Then \( \hat{\alpha}_{\text{res}} \) occurs in the root system of \( \hat{L} \) if and only if \( N\tilde{\alpha}(t) = 1 \) and \( \beta_{\text{res}} \) occurs in that root system if and only if \( N\tilde{\beta}(t) = -1 \). Now \( \alpha_{\text{res}} = 2N\tilde{\alpha} \) and \( \beta_{\text{res}} = \tilde{N}\tilde{\beta} \). Thus \( \hat{\alpha}_{\text{res}} \) occurs in the root system of \( \hat{L} \) if and only if \( \theta \circ N_{F'/F} \circ \beta_{\text{res}} \) has trivial restriction to \( O_{F'}^\times \), while \( \beta_{\text{res}} \) occurs in that root system if and only if \( \theta \circ N_{F'/F} \circ \beta_{\text{res}} \) has trivial restriction to \( O_{F'}^\times \). This completes the proof in the case \( \hat{M} = \hat{G} \).

We now turn to the general case. By construction \( \varphi(W_F) \) normalizes \( \hat{M}, \hat{T} \), and in addition \( \varphi(I_F) \) normalizes a Borel subgroup of \( \hat{M} \) containing \( \hat{T} \), which we can arrange to be \( \hat{B} \cap \hat{M} \). The pinning of \( \hat{M} \) inherited from the chosen pinning of \( \hat{G} \) gives a section \( \text{Out}(\hat{M}) \to \text{Aut}(\hat{M}) \). We compose \( \varphi : W_F \to N(\hat{M}, \hat{G}) \to \text{Aut}(\hat{M}) \to \text{Out}(\hat{M}) \) with this section and obtain a new homomorphism \( W_F \to \text{Aut}(\hat{M}) \). It extends to \( \Gamma_F \) and induces an action of \( \Gamma_F \) on \( \hat{M} \) preserving the pinning. Let \( M \) be the quasi-split \( F \)-group whose dual group is \( \hat{M} \) with this pinned \( \Gamma \)-action.

The \( \chi \)-data for \( R(S^0, G) \) leads to an embedding \( L_{jM} : L\mathcal{M} \to L\mathcal{G} \) by the construction of [Kala, §6.1]. The image of this \( L \)-embedding contains the image of \( \varphi \) which leads to a factorization \( \varphi = L_{jM} \circ \varphi_M \) for a parameter \( \varphi_M : W_F \to L\mathcal{M} \).

The natural inclusion restricts to an isomorphism \( \text{Cent}(\varphi_M, \hat{M}) \to \text{Cent}(\varphi, \hat{G}) \).

We conclude that \( \varphi_M \) is a torally wild supercuspidal parameter. We apply the established special case \( \hat{M} = \hat{G} \) to the parameter \( \varphi_M \). Thus we have the \( L \)-embedding \( L_{jS,M} : L_{S,M} \to L\mathcal{M} \), obtained from unramified \( \chi \)-data, since \( S \) is maximally unramified with respect to \( M \), and the factorization \( \varphi_M = L_{jS,M} \circ \varphi_{S,M} \) for a parameter \( \varphi_{S,M} : W_F \to L\mathcal{S} \). The character \( \theta_M : S(F) \to \mathbb{C}^\times \) corresponding to \( \varphi_{S,M} \) is \( F \)-non-singular by the previously handled case.

According to [Kala, §6.2], the composition \( L_{jS,M} \circ L_{jM} \) is equal to the \( L \)-embedding \( L_{jS,G} : L\mathcal{S} \to L\mathcal{G} \) obtained by making \( (\chi_{\alpha^0}) \) into \( \chi \)-data for \( R(S, G) \) \( \times R(S, G^0) \) and complementing it with unramified \( \chi \)-data for \( R(S, G^0) \). Therefore the parameters \( \varphi_S \) and \( \varphi_{S,M} \) are \( \hat{S} \)-conjugate, so \( \theta = \theta_M \).

\( \square \)

4.2 Construction of the \( L \)-packet

Let \((S, \tilde{j}, \chi_0, \theta)\) be a torally wild \( L \)-packet datum. We write down a formula for a function \( \Theta : S(F)_{\text{reg}} \to \mathbb{C} \) just as in [Kal19, (5.3.1)]: We choose a non-trivial character \( \Lambda : F \to \mathbb{C}^\times \) and for each \( \alpha \in R(S, G) \) we define \( \tilde{a}_\alpha \in [F_\alpha]_{(r_\Lambda, a, r_{\theta, a})}/[F_\alpha]_{(r_\Lambda, a, -r_{\theta, a})}^+ \) by the formula

\[
\theta \circ N_{F_\alpha/F} \circ \alpha^\vee(X + 1) = \Lambda \circ \text{Tr}_{F_\alpha/F}(\tilde{a}_\alpha X),
\]

where \( r_{\Lambda, a} \) and \( r_{\theta, a} \) are the depths of the characters \( \Lambda \circ \text{Tr}_{F_\alpha/F} : F_\alpha \to \mathbb{C}^\times \) and \( \theta \circ N_{F_\alpha/F} \circ \alpha^\vee : F_\alpha^\times \to \mathbb{C}^\times \) respectively, and \( X \) is a variable in \([F_\alpha]_{r_{\theta, a}}/[F_\alpha]_{r_{\theta, a}}^+ \).
We will now put together the individual admissible embeddings \( j \) of the singular Deligne-Lusztig packets \([\text{mined by the isomorphism class of}] \) (same as in [Kal19, §5.3.1], this function depends only on the isomorphism class of \((S, j, \chi, \theta)\).

As explained in [Kal19, §5.1], the map \( \hat{j} \) gives rise to a unique stable conjugacy class of embeddings \( S \to G \), called admissible, which identify \( S \) with a maximal torus of \( G \) defined over \( F \). For every inner twist \( \xi : G \to G' \) we obtain by composition a stable conjugacy class of embeddings \( S \to G' \) with the same property, also called admissible.

To each admissible embedding \( j : S \to G' \) we shall now assign a non-singular Deligne-Lusztig packet \([\pi_{(j, S, \theta)}]\) by following the guiding principle that the formula for Harish-Chandra character of the (possibly reducible) supercuspidal representation \( \pi_{(j, S, \theta)} \) defined by (3.3) is given on shallow elements \( \gamma' \in jS(F) \) by the function

\[
e(G')|D_{G'}(\gamma')|^{-\frac{1}{2}} \sum_{w \in \Omega(jS(F), G')(F)} \Theta(j^{-1}(\gamma'^w)). \tag{4.2}
\]

This can be done explicitly as follows. Let \((\chi'_a)_{\alpha}\) be the minimal \( \chi \)-data for \( R(S, G) \) computed in terms of \( \theta \) as reviewed in §3.7. Let \( \chi = \inf_{\alpha} \chi_0 \) and let \( \zeta_\alpha = \chi_\alpha(\chi'_a)^{-1} \) as in §3.7. Set \( \theta_j = j_* \zeta_S \cdot e_{f, \text{ram}} \cdot e_{\text{ram}, j_*} \theta \). By Lemma 3.7.1 the pair \((jS, \theta_j)\) is \( k_F \)-non-singular and hence leads to the non-singular supercuspidal representation \( \pi_{(jS, \theta)} \). Furthermore, since \( \theta_j/\theta \) is trivial on \( S(F)_{0+} \) and the computation of \((a_{\alpha})_{\alpha}\) and \((\chi'_a)_{\alpha}\) depends only on \( \theta|_{S(F)_{0+}} \), we obtain the same data from \( \theta \) and \( \theta_j \). Therefore the character function of \( \pi_{(jS, \theta)} \) evaluated on shallow elements of \( jS(F) \) is given by (4.2).

The resulting non-singular Deligne-Lusztig packet \([\pi_{(j, S, \theta)}]\) is uniquely determined by the isomorphism class of \((S, \hat{j}, \chi, \theta)\) and the admissible embedding \( j : S \to G' \). We define the \( L \)-packet \( \Pi_{\varphi}(G') \) as the disjoint union of the non-singular Deligne-Lusztig packets \([\pi_{(j, S, \theta)}]\) for all \( G'(F) \)-conjugacy classes of admissible embeddings \( j \).

We will now put together the individual \( L \)-packets \( \Pi_{\varphi}(G') \) into a compound \( L \)-packet \( \Pi_{\varphi} \) encompassing all rigid inner forms of \( G \). The construction is the same as in [Kal19, §5.3]. We introduce the notion of a non-singular Deligne-Lusztig packet datum. It is a tuple \((S, \hat{j}, \chi, \theta, G', \xi, z, j)\), where \((S, \hat{j}, \chi, \theta)\) is a torally wild \( L \)-packet datum, \((G', \xi, z)\) is a rigid inner twist of \( G \) in the sense of [Kal16b, §5.1], and \( j : S \to G' \) is an admissible embedding defined over \( F \). We organize these data into a category, where a morphism

\[
(S_1, j_1, \chi_1, \theta_1, (G'_1, \xi_1, z_1), j_1) \to (S_2, j_2, \chi_2, \theta_2, (G'_2, \xi_2, z_2), j_2)
\]

is given by \((\iota, g, \zeta, f)\), where \((\iota, g, \zeta)\) is an isomorphism of the underlying regular torally wild \( L \)-packet data, \( f : (G'_1, \xi_1, z_1) \to (G'_2, \xi_2, z_2) \) is an isomorphism of rigid inner twists, and \( j_2 \circ \iota = f \circ j_1 \). There is an obvious forgetful functor from the category of non-singular Deligne-Lusztig packet data to the category of torally wild \( L \)-packet data. If we fix a torally wild \( L \)-packet datum \((S, \hat{j}, \chi, \theta)\), the set of isomorphism classes of non-singular Deligne-Lusztig packet data mapping to it is a torsor under \( H^1(u \to W, Z(G) \to S) \). This torsor is given by the relation

\[
x \cdot (G'_1, \xi_1, z_1, j_1) = (G'_2, \xi_2, z_2, j_2) \iff x = \text{inv}(j_1, j_2), \tag{4.3}
\]
We expect that the case of positive depth can be reduced to the Remark 4.2.2.

To each non-singular Deligne-Lusztig packet datum \((S, \hat{j}, \chi, \theta, (G', \xi, z), j)\) we associate the corresponding non-singular Deligne-Lusztig packet \([\pi_{(j, S, \theta)}]\) on the group \(G'(F)\). The compound packet \(\Pi_\varphi\) is then defined as the union of the sets \(\{(G', \xi, z, \pi) | \pi \in [\pi_{(j, S, \theta)}]\}\), as \((S, \hat{j}, \chi, \theta, (G', \xi, z), j)\) runs over the isomorphism classes of non-singular Deligne-Lusztig packet data that map to the isomorphism class of the torally wild packet datum \((S, \hat{j}, \chi, \theta)\).

It is possible that two non-isomorphic non-singular Deligne-Lusztig packet data \((S, \hat{j}, \chi, \theta, (G', \xi, z), j_i), i = 1, 2\) give the same non-singular Deligne-Lusztig packet. By Corollary 3.6.7 this happens if and only if there is \(w \in \Omega(S, G)(F)\) such that \((S, \hat{j}, \chi, \theta, (G', \xi, z), j_2 \circ w)\) is isomorphic to \((S, \hat{j}, \chi, \theta, (G', \xi, z), j_1)\).

**Lemma 4.2.1.** Assume \(\varphi\) has depth zero. For any given Whittaker datum \(\psi\) of \(G\) there exists a unique \(\psi\)-generic member of \(\Pi_\varphi\).

**Proof.** By Lemma H.1, \(\psi\) determines a absolutely special vertex \(x \in B(G, F)\), unique up to \((G, F)\)-conjugacy, s.t. \(\psi\) has depth zero at \(x\) for some \((B, \psi) \in \psi\). According to Proposition H.2 a depth-zero supercuspidal representation is \(G\)-generic if and only if it is induced from an irreducible representation of \(G(F)_x\) containing a \(\psi_x\)-generic cuspidal representation of \(G(F)_x\). By [Kal19, Lemmas 3.4.12] there exists precisely one admissible embedding \(j : S \rightarrow G\) up to \((G, F)\)-conjugacy such that the vertex \(x\) corresponds to the maximal torus \(j(S) \subset G\). A \(\psi\)-generic member of \(\Pi_\varphi\) can thus only come from the non-singular Deligne-Lusztig packet \([\pi_{(j, S, \theta)}]\). By [DLM92, Proposition 3.10], there exists a unique \(\psi_x\)-generic irreducible component of the Deligne-Lusztig character \(\kappa_{(j, S, \theta)}\), and hence a unique irreducible representation of \(G(F)_x\) containing it upon restriction. Its compact induction to \(G(F)\) is then the unique \(\psi\)-generic element of \([\pi_{(j, S, \theta)}]\). \(\square\)

**Remark 4.2.2.** We expect that the case of positive depth can be reduced to the case of depth zero using the local character expansions of [Spi17].

### 4.3 Study of the centralizer \(S_\varphi\)

Before we can construct a bijection between the compound \(L\)-packet \(\Pi_\varphi\) constructed in §4.2 and \(\text{Irr}(\pi_0(S^+_\varphi))\), we need to have a better understanding of the group \(S_\varphi\) and its cover \(S^+_\varphi\). Contrary to the case of regular supercuspidal parameters, where \(S_\varphi\) is always abelian and in fact canonically isomorphic to \(\hat{S}^\Gamma\), this is no longer true for arbitrary supercuspidal parameters, even for classical Dynkin types, as the following example shows.

**Example 4.3.1.** We consider the split group \(G = \text{Spin}_9\), which is of type \(B_4\). Its dual group is \(\hat{G} = \text{PSp}_8(\mathbb{C})\). We shall produce a discrete parameter of depth zero \(\varphi : W_F/P_F \rightarrow \hat{G}\) such that

\[
S_\varphi \cong [(\mathbb{Z}_4^\times)^2/\mathbb{Z}_2] \rtimes (\mathbb{Z}/2\mathbb{Z}),
\]

where \((\mathbb{Z}_4)^2\) is the subgroup of \(\mu_4^2\) consisting of those \((t_1, t_2, t_3, t_4)\) satisfying \(t_1^4 = t_2^4 = t_3^4 = t_4^4\), \(\mu_2\) is embedded diagonally in that subgroup, and \(\mathbb{Z}/2\mathbb{Z}\) acts on it by sending \((t_1, t_2, t_3, t_4)\) to \((t_4, t_3, t_2, t_1)\). For this, we assume that
\[ q = |k_F| > 8, \text{ choose a primitive } (q + 1)\text{-root of unity } \zeta \in \mathbb{C}^\times, \text{ and consider the matrices} \]

\[
\begin{pmatrix}
1 & 0 & 1 \\
0 & -1 & 1 \\
-1 & 0 & 1
\end{pmatrix}
\]

which we call \( j \) and \( n \), respectively, and let \( t \) be the diagonal matrix with diagonal entries \((\zeta, \zeta^2, -\zeta, -\zeta, -\zeta^{-1}, -\zeta^{-2}, \zeta^2, \zeta^{-1})\). We realize \( \text{Sp}_8 \) as the subgroup of \( \text{GL}_8 \) that preserves the symplectic form given by \( j \), i.e. the subgroup of matrices \( g \) satisfying \( j^{-1} \cdot g^T \cdot j = g^{-1} \). Then the matrices \( j, n, t \) all belong to \( \text{Sp}_4(\mathbb{C}) \). The element \( t \) is regular semi-simple. Its image in \( \hat{G} \) is not strongly regular, because it commutes with \( n \). In fact, \( n \) generates the stabilizer of \( t \) in the Weyl group of the diagonal maximal torus in \( \hat{G} \). In \( \hat{G} \) we have \( j \cdot t \cdot j^{-1} = t^n, \ jn = nj, nt = tn, \) and \( n^2 = 1 \). As before we let \( x \in I_F/P_F \) be a topological generator, and choose a Frobenius element \( y \in W_F/P_F \). We define \( \varphi(x) = t \) and \( \varphi(y) = j \). Then \( \hat{S} = \hat{T}^j \) is the 2-torsion subgroup of \( \hat{T} \) and is thus canonically isomorphic to \( (\mu_2)^3/\mu_2 \). The element \( n \) also belongs to \( S_\varphi \). It projects onto a generator of \( \Omega(S, G)(F)_\theta \cong \mathbb{Z}/2\mathbb{Z} \) and acts on \( \hat{S} \) as stated.

We consider the following functors from the category of torally wild \( L \)-packet data to the category of groups:

1. \( (S, \hat{j}, \chi, \theta) \rightarrow S_\varphi \), where \( \varphi := L_j \circ \varphi_S, L_j : L_S \rightarrow L_G \) is the extension of \( \hat{j} \) given by \( \chi \), well-defined up to conjugation by \( \hat{T} \), and \( \varphi_S : W_F \rightarrow L_S \) is the parameter of \( \theta \). It sends the morphism \((i, g, \zeta) : (S_1, \hat{j}_1, \chi_1, \theta_1) \rightarrow (S_2, \hat{j}_2, \chi_2, \theta_2)\) to the morphism \( \text{Ad}(g)_\varphi : S_\varphi \rightarrow S_{\text{Ad}(g)_\varphi} \).

2. \( (S, \hat{j}, \chi, \theta) \rightarrow \hat{S}^1 \). It sends the morphism \((i, g, \zeta) \) to the morphism \( \hat{t}^{-1} : \hat{S}^1 \rightarrow \hat{S}^2 \).

3. \( (S, \hat{j}, \chi, \theta) \rightarrow \Omega(S, G)(F)_\theta \). It sends the morphism \((i, g, \zeta) \) to the morphism \( \Omega(S_1, G)(F)_{\theta_1} \rightarrow \Omega(S_2, G)(F)_{\theta_2} \) induced by \( i \).

**Proposition 4.3.2.** There is a functorial exact sequence

\[ 1 \rightarrow \hat{S}^1 \rightarrow S_\varphi \rightarrow \Omega(S, G)(F)_\theta \rightarrow 1 \quad (4.4) \]

We begin with a preparatory lemma, considering a more general situation where \( j : S \rightarrow G \) is an embedding defined over \( F \) of a torus \( S \) into \( G \) as a maximal torus, \( \theta : S(F) \rightarrow \mathbb{C}^\times \) is a character, \( \varphi_S : W_F \rightarrow L_S \) its parameter, \( \chi \) a set of \( \chi \)-data for \( R(S, G) \), \( L_j : L_S \rightarrow L_G \) the corresponding \( L \)-embedding, with image \( \hat{T} := L_j(\hat{S}) \), and \( \varphi = L_j \circ \varphi \). We have the exact sequence \( 1 \rightarrow \hat{S} \rightarrow N(\hat{T}, \hat{G}) \rightarrow \Omega(S, G) \rightarrow 1 \) in which the first map and third maps are given by the identifications \( \hat{S} \rightarrow \hat{T} \) and \( \Omega(S, G) = \Omega(\hat{S}, \hat{G}) \rightarrow \Omega(\hat{T}, \hat{G}) \) induced by \( L_j \).

**Lemma 4.3.3.** In the exact sequence for \( W_F \)-cohomology

\[ 1 \rightarrow \hat{S}^1 \rightarrow N(\hat{T}, \hat{G})_{\varphi(W_F)} \rightarrow \Omega(S, G)(F) \rightarrow H^1(W_F, \hat{S}) \]
an element \( w \in \Omega(S, G)(F) \) is mapped to the parameter of the character \( w\theta/\theta \), provided the \( \chi \)-data is \( w \)-invariant.

Proof. The \( w \)-invariance of the \( \chi \)-data and [LS87, (2.6.2)] imply the existence of a lift \( n \in N(\hat{T}, \hat{G}) \) such that \( \text{Ad}(n) \circ L_j = L_j \circ w \). The connecting homomorphism sends \( w \) to the class of the 1-cocycle of \( W_F \) valued in \( \hat{T} \) for the action of \( W_F \) via \( \text{Ad}(\varphi(-)) \) given by

\[
x \mapsto \varphi(x)^{-1} n \varphi(x) n^{-1} = L_j(\varphi_S(x)^{-1} \cdot w \varphi_S(x)).
\]

Via the identification of \( \hat{T} \) with \( \hat{S} \) this 1-cocycle becomes \( x \mapsto \varphi_S(x)^{-1} \cdot w \varphi_S(x) \), which is the parameter for \( w\theta/\theta \).

Proof of Proposition 4.3.2. Let \( \hat{M} = \text{Cent}(\varphi(P_F), \hat{G}) \). According to [Kal19, Lemma 5.2.2] we have \( \text{Cent}(\varphi(W_F), \hat{G}) \subset N(\hat{T}, \hat{M}) \subset N(\hat{T}, \hat{G}) \). We thus consider the exact sequence

\[
1 \to \hat{T} \to N(\hat{T}, \hat{G}) \to \Omega(\hat{T}, \hat{G}) \to 1
\]

with action of \( W_F \) via \( \text{Ad}(\varphi(-)) \). Since \( N(\hat{T}, \hat{G})^{\varphi(W_F)} \) is contained in \( N(\hat{T}, \hat{M}) \), its image in \( \Omega(\hat{T}, \hat{G}) \) is contained in \( \Omega(\hat{T}, \hat{M}) \). Any \( w \in \Omega(\hat{S}, \hat{M})^+ \) preserves the \( \chi \)-data in the datum \((S, \hat{j}, \chi, \theta)\) so we may apply Lemma 4.3.3 and see that the image of \( N(\hat{T}, \hat{G})^{\varphi(W_F)} \) in \( \Omega(S, G)(F) \) is precisely \( \Omega(S, G^0)(F)_\theta \). Now apply Lemma 3.6.5.

The functoriality of the exact sequence follows by a straightforward unwinding of the definitions.

We set \( \hat{S} = S/Z, \hat{G} = G/Z \) and obtain the covers \( \hat{S} \to \hat{S} \) and \( \hat{G} \to \hat{G} \). Let \( [\hat{S}]^+ \) be the preimage of \( \hat{S}^\Gamma \) and \( S^+_\varphi \subset \hat{G} \) be the preimage of \( S^\varphi \). Both of these are functors in \( (S, \hat{j}, \chi, \theta) \).

Corollary 4.3.4. We have the functorial exact sequence

\[
1 \to \pi_0([\hat{S}]^+) \to \pi_0(S^+_\varphi) \to \Omega(S, G)(F)_\theta \to 1.
\]

(4.5)

It is tempting to expect that this extension, or at least the simpler extension (4.4), has the multiplicity 1 property in the sense of Definition A.7. While this does hold in many cases, it turns out that it doesn’t always hold, as we now discuss.

Lemma 4.3.5. Assume that \( \varphi \) is of depth zero. The extension (4.5) has multiplicity 1 in the following cases.

1. \( G \) is simply connected.

2. \( G \) is unramified.

Proof. If \( G \) is simply connected we can write it as a product of \( F \)-simple factors, and assume that \( G \) is \( F \)-simple. Then it is of the form \( \text{Res}_{E/F} H \) for an absolutely simple simply connected group \( H \) defined over a finite tamely ramified extension \( E/F \). We may thus assume that \( G \) is absolutely simple. The claim now follows from Lemma 3.1.4 and Lemma A.11 in all cases except when \( H \) is
split of type $D_{2n}$ and $\Omega(S,G)(F)_\eta \cong (\mathbb{Z}/2\mathbb{Z})^2$. In the latter case we let $s \in \hat{G}$ be the image of the topological generator of $I_P/P_F$ under $\varphi$, and $f \in \hat{G}$ the image of a Frobenius element. The extension (4.5) is then the push-out of the extension of Lemma 1.1 along the inclusion $\hat{S}_{sc} \to \hat{S}_{sc}'$, and the claim follows from that Lemma.

Assume now that $G$ is unramified. We shall repeatedly modify the extension (4.5) and use Corollary A.12, without explicitly referring to it. For example, we may replace the kernel $Z$ of $G \to \hat{G}$ by a larger one, because the extension for the smaller $Z$ is a push-out of the extension for the larger $Z$. We may thus assume $Z(G_{der}) \subset Z$, so that $\hat{G} = G_{ad} \times Z(\hat{G})$ and $\hat{G} = \hat{G}_{sc} \times Z(\hat{G})^\circ$. Since every irreducible representation of $\pi_0(S^+_{sc})$ transforms under $\pi_0(Z(\hat{G})^\circ)$ by a character, it is enough to fix a character $\zeta$ of $\pi_0(Z(\hat{G})^\circ)$ and consider only those irreducible representations of $\pi_0(S^+_{sc})$ that transform by that character. Applying the bijection [Kal18b, (6.7)] we may assume that this character is trivial on the kernel of the morphism $Z(\hat{G})^\circ \to Z(\hat{G}_{sc})$ induced by projecting onto the first factor of $\hat{G} = \hat{G}_{sc} \times Z(\hat{G})^\circ$: Indeed, we have the diagram

$$
\begin{array}{ccc}
Z(\hat{G}_{sc}) & \longrightarrow & Z(\hat{G}_{sc}) \\
\downarrow & & \downarrow \\
Z(\hat{G})^\circ & \longrightarrow & Z(\hat{G})^\circ
\end{array}
$$

We may thus restrict $\zeta$ to $Z(\hat{G}_{sc})$, extend to $Z(\hat{G}_{sc})$, and via the vertical arrow obtain another character $\zeta'$ of $Z(\hat{G})^\circ$. By construction this character is trivial on $Z(\hat{G})^\circ \Gamma$. Upon further enlarging $Z$ we may assume that $\zeta$ is trivial on $Z(\hat{G})^\circ \Gamma$ (see erratum to [Kal18b]). Thus the difference between $\zeta$ and $\zeta'$ is trivial on $Z(\hat{G})^\Gamma$ and thus factors through the differential $d : Z(\hat{G})^\circ \to Z^1(\Gamma, \hat{Z})$ and can be extended to a character $\eta$ of $Z^1(\Gamma, \hat{Z})$. This differential is the restriction of the differential $d : \pi_0(S^+_{sc}) \to Z^1(\Gamma, \hat{Z})$. We can pull back $\eta$ to a character of $\pi_0(S^+_{sc})$. Tensoring with this character gives a bijection between the irreducible representations transforming under $\pi_0(Z(\hat{G})^\circ)$ by $\zeta$ and those transforming by $\zeta'$, and this bijection preserves the property of having multiplicity 1 upon restriction to $\pi_0(\hat{S}^\circ)$.

We have thus arranged that $\zeta$ is trivial on the kernel of the morphism $Z(\hat{G})^\circ \to Z(\hat{G}_{sc})$. Let $\zeta_{sc}$ be an extension of this character to $Z(\hat{G}_{sc})$. Write $Irr(\pi_0(S^+_{sc}), \zeta)$ for the set of irreducible representations of $\pi_0(S^+_{sc})$ that transform under the central group $\pi_0(Z(\hat{G})^\circ)$ by $\zeta$. If $\rho \in Irr(\pi_0(S^+_{sc}), \zeta)$, then the representation $\rho \boxtimes \zeta_{sc}$ of $\pi_0(S^+_{sc}) \times Z(\hat{G}_{sc})$ descends to the push-out $S^+_{sc} \times_{Z(\hat{G})^\circ} Z(\hat{G}_{sc})$. The restriction of $\rho$ to $\pi_0(\hat{S}^\circ)$ has multiplicity 1 if and only if the restriction of $\rho \boxtimes \zeta_{sc}$ to $\pi_0((S^\circ)_{sc} \times_{Z(\hat{G})^\circ} Z(\hat{G}_{sc}))$ has multiplicity 1.

Let $S^\circ_{sc}$ be the preimage in $\hat{G}_{sc}$ of $S^\circ_{sc}/Z(\hat{G})^\Gamma \subset G_{ad}$. Recall the bijection $S^\circ_{sc} \to S^\circ_{sc} \times_{Z(\hat{G})^\circ} Z(\hat{G}_{sc})$ from [Kal18a, (4.6)]. It sends $s_{sc} \in S^\circ_{sc}$ to $(s_{sc}, y', (y')^{-1})$, where $y \in Z(\hat{G})$ is chosen so that $s_{der}y \in S^\circ_P$, where $s_{der} \in \hat{G}_{der}$ is the image of $s_{sc}$ under $\hat{G}_{sc} \to \hat{G}_{der}$, $y' \in Z(\hat{G}_{der})$ and $y'' \in Z(\hat{G})^\circ$ are chosen so that $y = y'y''$, and $y' \in Z(\hat{G}_{sc})$ and $y'' \in Z(\hat{G})^\circ$ are lifts of $y'$ and $y''$. Then $(s_{sc}, y', (y')^{-1}) \in$
\( \tilde{G}_{sc} \times Z(\tilde{G})^\circ = \tilde{G} \) belongs to \( S^+_1 \). Let \( \tilde{S}^{+}_{sc} \) denote the preimage of \( \tilde{S}^{+}_{ad} \). In the same way we obtain the isomorphism \( \tilde{S}^{+}_{sc} \rightarrow [\tilde{S}^{+}_{sc} \times Z(\tilde{G})^\circ, Z(\tilde{G}_{sc})] \). We have thus reduced the problem to showing that the extension
\[
1 \rightarrow \tilde{S}^{+}_{sc} \rightarrow S^{+}_{sc} \rightarrow \Omega(S,G)(F)_\theta \rightarrow 1
\]
has multiplicity 1. All groups in this extension are finite.

Write again \( s \in \tilde{G} \) and \( f \in \tilde{G} \times \text{Frob} \) for the images of the topological generator of \( I_F/P_F \) and a Frobenius element in \( W_F/P_F \) under \( \varphi \). Then \( S^{+}_{sc} = \{ \tilde{x} \in \tilde{G}_{sc} | \exists z \in Z(\tilde{G}) : \text{Ad}(s)(x) = xz, \text{Ad}(f)(xz) = xz \} \), where \( x \tilde{\in} \tilde{G} \) is the image of \( \tilde{x} \). The element \( s \) is regular semi-simple. Let \( \tilde{T} \) be its connected centralizer.

Then \( \tilde{x} \in N(\tilde{T}_{sc}, \tilde{G}_{sc}) =: \tilde{N}_{sc} \). Let \( \tilde{N}^{+}_{sc} = \{ \tilde{x} \in \tilde{N}_{sc} | \exists z \in Z(\tilde{G}) : \text{Ad}(s)(x) = xz \} \). We have the extension \( 1 \rightarrow \tilde{T}_{sc} \rightarrow \tilde{N}^{+}_{sc} \rightarrow \Omega_s \rightarrow 1 \), where \( \Omega_s \) is the stabilizer in \( \Omega(\tilde{T}, \tilde{G}) \) of \( s \). Since \( (1-f) : \tilde{T}_{sc} \rightarrow \tilde{T}_{sc} \) has finite kernel, it is surjective, and hence taking \( \text{Ad}(f) \)-fixed points gives an exact sequence \( 1 \rightarrow \tilde{T}_{sc} \rightarrow \tilde{N}^{+}_{sc} \rightarrow \Omega_s^f \rightarrow 1 \).

We claim that \( \Omega_s^f = \Omega(S,G)(F)_\theta \). Indeed, we know that \( \Omega(S,G)(F)_\theta \) is the image of \( N(\tilde{T}, \tilde{G})^{s,f} \) under \( N(\tilde{T}, \tilde{G}) \rightarrow \Omega(\tilde{T}, \tilde{G}) \). Clearly this image is contained in \( \Omega_s^f \). The converse inclusion follows by taking \( \text{Ad}(f) \)-fixed points in the exact sequence \( 1 \rightarrow \tilde{T}_{der} \rightarrow N(\tilde{T}_{der}, \tilde{G}_{der})^{s,f} \rightarrow \Omega_s \rightarrow 1 \) and using the surjectivity of \( 1-f : \tilde{T}_{der} \rightarrow \tilde{T}_{der} \).

Pushing out \( 1 \rightarrow \tilde{T}_{sc} \rightarrow \tilde{N}^{+}_{sc} \rightarrow \Omega_s \rightarrow 1 \) along the inclusion \( \tilde{T}_{sc} \rightarrow \tilde{T}_{sc} = \{ \tilde{x} \in \tilde{T}_{sc} | \exists z \in Z(\tilde{G}) : \text{Ad}(f)(xz) = xz \} \) we obtain the extension
\[
1 \rightarrow \tilde{T}_{sc} \rightarrow S^{+}_{sc} \rightarrow \Omega(S,G)(F)_\theta \rightarrow 1,
\]
of which we want to show that it satisfies multiplicity 1. It is thus enough to show this for the extension \( 1 \rightarrow \tilde{T}_{sc} \rightarrow \tilde{N}^{+}_{sc} \rightarrow \Omega_s^f \rightarrow 1 \).

Let \( \tilde{N}_{sc}^{+} \) be the preimage of the centralizer of \( s \) in \( \tilde{N}_{ad} \) and let \( \Omega_l \) be the image of \( \tilde{N}_{sc}^{+} \) in \( \Omega(T, \tilde{G}) \). Then \( \tilde{N}_{sc}^{+} \) is the preimage in \( \tilde{N}_{sc}^{+} \) of \( \Omega_s \subset \Omega_l \). Taking \( \text{Ad}(f) \)-fixed points we obtain the extension \( 1 \rightarrow \tilde{T}_{sc} \rightarrow \tilde{N}^{+}_{sc} \rightarrow \Omega_s^f \rightarrow 1 \). It is enough to show the multiplicity 1 property for this extension, because pulling back along the inclusion \( \Omega_l \rightarrow \Omega_s^f \) we obtain the extension \( 1 \rightarrow \tilde{T}_{sc} \rightarrow \tilde{N}^{+}_{sc} \rightarrow \Omega_s \rightarrow 1 \).

We now break \( \tilde{G}_{sc} \) into a product of simple factors. The extension \( 1 \rightarrow \tilde{T}_{sc} \rightarrow \tilde{N}_{sc} \rightarrow \Omega_l \rightarrow 1 \) breaks accordingly. The action of \( \text{Ad}(f) \) permutes the simple factors, and the extension \( 1 \rightarrow \tilde{T}_{sc} \rightarrow \tilde{N}_{sc} \rightarrow \Omega_s^f \rightarrow 1 \) breaks up according to orbits of simple factors under \( \text{Ad}(f) \). By Corollary A.12 we may assume that there is only one orbit. Shapiro’s lemma then reduces to the case where the orbit is a singleton. Lemma 3.1.4 completes the proof in all cases except when \( G \) is of split type \( D_{2n} \) and \( \Omega_s = \Omega_l = (\mathbb{Z}/2\mathbb{Z})^2 \), in which case we appeal to Lemma I.1.

In the following example we will show that the extension (4.4) can fail to have multiplicity 1, even for parameters of depth zero, when the group in question is ramified. As discussed in the proof of Proposition 4.1.7, there is an isomorphism \( S^+_1(G) = S^+_1(G^0) \), where \( G^0 \subset G \) is a tame twisted Levi, for which \( \varphi \) is essentially of depth zero. Even if \( G \) is taken to be unramified, or simply connected, \( G^0 \) will be neither of these. Thus, despite Lemma 4.3.5, one cannot
expect the multiplicity 1 property for the extension (4.4) for general parameters, even after placing restrictions on \( G \).

**Example 4.3.6.** Let \( \Gamma = (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \) be a quotient of \( W_F / P_F \), with \((1, 0)\) being an image of the topological generator of \( I_F / P_F \), and \((0, 1)\) being an image of a Frobenius element. Consider the following complex algebraic groups with \( \Gamma \)-action: An algebraic torus \( Z_1 = \mathbb{C}^\times \times \mathbb{C}^\times \) with \((1, 0)(z_1, z_2) = (z_1^{1}, z_2^{-1})\) and \((0, 1)(z_1, z_2) = (z_1 z_2^{-1}, z_2^{-1})\). An algebraic torus \( Z_2 = \mathbb{C}^\times \) with \((1, 0)z = z^{-1}\) and \((0, 1)z = z\). The group \( G_1 = \text{SL}_4(\mathbb{C}) \) with both \((1, 0)\) and \((0, 1)\) acting as \( \theta \), where \( \theta \) is the pinned non-trivial outer automorphism of \( \text{SL}_4(\mathbb{C}) \) relative to the standard pinning. The group \( G_2 = \text{SL}_4(\mathbb{C}) \) with \((1, 0)\) acting as \( \theta \) and \((0, 1)\) acting as the identity.

We embed \( \mu_4 \to Z_1 \) via \( z \mapsto (1, z) \). Let \( G_1 = (G'_1 \times Z_1) / \mu_4 \) and \( G_2 = (G'_2 \times Z_2) / \mu_4 \), where in both cases \( \mu_4 \) is embedded anti-diagonally. Thus \( G_2 \) is the dual group of a ramified unitary group splitting over a ramified quadratic extension, while \( G_1 \) is the dual group of a reductive group whose base-change to \( F^{ur} \) is the product of a ramified unitary group and \( \mathbb{C}_m \).

We embed \( \mu_4 \to \mu_4 \times \mu_4 \) by \( z \mapsto (z, z^2) \) and then further \( \mu_4 \times \mu_4 \to Z_1 \times Z_2 \) by \((z_1, z_2) \mapsto (1, z_1, z_2)\) and form \( G = (G_1 \times G_2) / \mu_4 \).

Consider the elements

\[
s = \left[ \begin{array}{cc} \zeta_4 & \zeta_6 \\ \zeta_6^{-1} & \zeta_4^{-1} \end{array} \right], \quad \left[ \begin{array}{cc} \zeta_4 & \zeta_6 \\ \zeta_6^{-1} & \zeta_4^{-1} \end{array} \right] \times (1, 0)
\]

and

\[
f = \left[ \begin{array}{cc} \zeta_4 & \zeta_6^{-1} \\ -\zeta_6^{1} & \zeta_4 \end{array} \right], \quad \left[ \begin{array}{cc} \zeta_4 & \zeta_6^{-1} \\ -\zeta_6^{1} & \zeta_4 \end{array} \right] \times (0, 1)
\]

of \((G'_1 \times G'_2) \rtimes \Gamma\), where \( \zeta_k = \exp(2\pi i / k) \). We have \( fsf^{-1} = s^{11} \) and we set \( p = q = 11 \). These two elements together give a depth zero supercuspidal parameter for the algebraic group over \( \mathbb{Q}_p \), with dual group \( G \).

We now compute the mutual centralizer of \( s \) and \( f \) in \( G \) and its image in \( G_{ad} \). Let \( T_{ad} \) denote the standard diagonal torus in \( \text{PGL}_4(\mathbb{C}) \times \text{PGL}_4 \). Then

\[
T_{ad} = \left\{ \begin{array}{cc} a & b \\ b^{-1} & a^{-1} \end{array} \right\} \times \left\{ \begin{array}{cc} c & d \\ d^{-1} & c^{-1} \end{array} \right\},
\]

where \((a, b)\) and \((c, d)\) run over \((\mathbb{C}^\times \times \mathbb{C}^\times) / \mu_2\), with \( \mu_2 \) embedded diagonally. The preimage in \( G_{sc} \) is

\[
= \left\{ \begin{array}{cc} x & a \\ b & b^{-1} \\ a^{-1} \end{array} \right\} \times \left\{ \begin{array}{cc} y & c \\ d & d^{-1} \\ c^{-1} \end{array} \right\}
\]

where now \((x, a, b)\) and \((y, c, d)\) run over \((\mu_4 \times \mathbb{C}^\times \times \mathbb{C}^\times) / \mu_2\), with \( \mu_2 \) embedded diagonally. Applying \((s - 1)\) to such an element gives \((x^{-2}, y^{-2}) \in \mu_4 \times \mu_4 = \)
The centralizer of $s$ in $G_{ad}$ has four cosets under the centralizer in $T_{ad}$ and they are represented by the matrices $(1, n), (n, 1)$, and $(n, n)$, where

$$n = \begin{bmatrix} -\zeta_8 & \zeta_8^{-1} \\ -\zeta_8^{-1} & -\zeta_8 \end{bmatrix} \in \text{SL}_4(\mathbb{C}),$$

where as before $\zeta_k = \exp(2\pi i/k)$. We compute $(s - 1)(n, 1) = (\zeta_4^{-1}, 1)$, and $(s - 1)(1, n) = (1, \zeta_4^{-1}), (f - 1)(n, 1) = (\zeta_4^{-1}, 1), (f - 1)(1, n) = (1, 1)$.

We can now compute the image in $G_{ad}$ of $G^{s,f}$. First consider an element of $T_{sc}$ with image in $T_{ad}$. It is given by $(x, a, b, (y, c, d))$. It will belong to the image of $T^{s,f}$ in $T_{ad}$ if and only if there exists $(z_1, z_2) \in Z_1 \times Z_2$ s.t. $(s - 1)(x, a, b, y, c, d) = (s - 1)(z_1, z_2, z_3)$ and $(f - 1)(x, a, b, y, c, d) = (f - 1)(z_1, z_2, z_3)$, where we have used $Z_1 \times Z_2 = C^\times \times C^\times \times C^\times$, and the equalities are to hold in $G$. By definition $(s - 1)(z_1, z_2, z_3) = (z_1^{-2}, z_2^{-2}, z_3^{-2})$ and $(f - 1)(z_1, z_2, z_3) = (z_2^4, z_2^{-2}, 1)$.

Looking at $(f - 1)$ we see that the term (4.6) must belong to the center of $G_{ad}$, which forces $a^2 = a^{-2} = b^2 = b^{-2}$, i.e. $a, b \in \mu_4$ and $a^2 = b^2$, and the same for the pair $(c, d)$. Looking at both $(s - 1)$ and $(f - 1)$ we see that we must find $(z_1, z_2, z_3) \in C^\times$ such that the equalities $(1, x^{-2}, y^{-2}) = (z_1^{-2}, z_2^{-2}, z_3^{-2})$ and $(1, x^{-2}a^{-2}, c^{-2}) = (z_2^2, z_2^{-2}, 1)$ hold in the quotient of $C^\times \times C^\times \times C^\times$ by the subgroup $\{(1, z, z^2) | z \in C^\times\}$. Using that subgroup and the fact that $a \in \mu_4$ we can rewrite the second equation as $(1, x^{-2}, c^{-2}) = (z_2^4, z_2^{-2}, 1)$. This forces $z_2 \in \mu_4$ and $x^{-2}c = \pm z_2^{-2}$, implying $c \in \mu_2$ and hence $d \in \mu_2$. Conversely, for any tuple $(x, a, b, y, c, d)$ satisfying $a, b \in \mu_4, c, d \in \mu_2, a^2 = b^2$, we can find $(z_1, z_2, z_3)$ that satisfy these equations. We conclude that the image of $T^{s,f}$ in $T_{ad}$ is given by

$$\begin{bmatrix} 1 & \epsilon & \eta \\ \eta & \epsilon \eta \\ \epsilon \eta & \epsilon \eta & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \delta \\ \delta \end{bmatrix},$$

for $\epsilon, \eta, \delta \in \mu_2$.

We now come to the non-trivial $T^{s,f}$-cosets in $G^{s,f}$ and their image in $G_{ad}$. Since there are precisely four cosets of $T_{ad}$ in $G_{ad}$, there can be at most four $T^{s,f}$-cosets in $G^{s,f}$. The element $(1, n)$ of $G_{ad}$ is fixed by both $s$ and $f$, and it is the image of the element $(1, \zeta_4^{-1}n) \in G^{s,f}$. Let $t$ be the diagonal matrix with entries $(\zeta_4, \zeta_4, -\zeta_4, -\zeta_4)$. Then $(s - 1)(n, t) = (1, \zeta_4^{-1}, 1)$ and $(f - 1)(n, t) = (1, \zeta_4^{-1}, -1)$. Modulo $(1, z, z^2)$ these elements become $(1, 1, 1)$ and $(1, 1, 1)$ respectively, and thus equal to $(z_1^{-2}, z_2^{-2}, z_3^{-2})$ and $(z_4^2, z_2^{-2}, 1)$ if we take $z_1 = z_2 = 1$ and $z_3 = \zeta_4$. We conclude that $(n, t)$ also belongs to the image of $G^{s,f}$ in $G_{ad}$. We conclude that there are exactly four cosets of $T^{s,f}$ in $G^{s,f}$. The image of $G^{s,f}$ in $G_{ad}$ is thus an extension

$$1 \to T^{s,f} / Z(G)^{s,f} \to G^{s,f} / (Z(G)^{s,f} \to (Z/2Z)^2 \to 0,$n

and the elements $(1, n)$ and $(n, t)$ map to a basis of $(Z/2Z)^2$. We compute the commutator and find that it is given by $\epsilon = \eta = 1$ and $\delta = -1$. Since the actions
of \((1, 0)\) and \((0, 1)\) send \((\epsilon, \eta, \delta)\) to \((\eta, \epsilon, \delta)\) and \((\epsilon, \eta, \delta)\) respectively, we see that \((1, 1, -1)\) does not vanish in the quotient of coinvariants in \(T^\times/(Z(G))^\times\) for the action of \((\mathbb{Z}/2\mathbb{Z})^2\). Lemma A.11 implies that the extension (4.7) does not have the multiplicity 1 property.

### 4.4 Internal structure I: Reduction to depth zero DL-packets

Having gained some understanding of the structure of \(\pi_0(S^+_{\hat{\mathbb{Z}}})\), we turn to establishing a bijection between \(\text{Irr}(\pi_0(S^+_{\hat{\mathbb{Z}}}))\) and the compound \(L\)-packet \(\Pi_{\varphi}\) constructed in §4.2. Such a bijection is expected to depend only on the choice of a Whittaker datum for the quasi-split group \(G\), and to satisfy stability and endoscopic character identities, as described in [Kal16a, Conjecture G]. Our construction in this paper will be less precise – we will make some auxiliary choices and show that they lead to a bijection \(\text{Irr}(\pi_0(S^+_{\hat{\mathbb{Z}}})) \to \Pi_{\varphi}\). Then we will sketch an argument showing that this bijection satisfies stability as well as endoscopic transfer for \(s \in [\hat{\mathbb{S}}]^+ \subset S^+_{\hat{\mathbb{Z}}}\). The discussion of how the auxiliary choices involved in the construction of the bijection relate to the choice of a Whittaker datum, the details of the argument of endoscopic transfer, and its extension to all \(s \in S^+_{\hat{\mathbb{Z}}}\), will be given in a forthcoming paper.

Before we begin, we summarize here the construction of the compound \(L\)-packet \(\Pi_{\varphi}\). The parameter \(\varphi\) corresponds to an isomorphism class of torally wild \(L\)-packet data \((S, j, \chi, \theta)\) by Proposition 4.1.7. The compound \(L\)-packet \(\Pi_{\varphi}\) is a disjoint union of non-singular Deligne-Lusztig packets. There is a surjective map from the set of those isomorphism classes of non-singular Deligne-Lusztig packet data that map to the isomorphism class of \((S, j, \chi, \theta)\) to the set of non-singular Deligne-Lusztig packets inside of \(\Pi_{\varphi}\). The former are of the form \((S, j, \chi, \theta, G', \xi, z, j)\) and are a torsor under \(H^1(u \to W, Z(G) \to S)\), see (4.3). The surjection does not depend on any choices, and maps two such data \((S, j, \chi, \theta, (G', \xi, z, j_i))\), \(i = 1, 2\), to the same non-singular Deligne-Lusztig packet if and only if there is \(w \in \Omega(S, G)(F)_0\) such that \((S, j, \chi, \theta, (G', \xi, z, j_2 \circ w))\) is isomorphic to \((S, j, \chi, \theta, (G', \xi, z, j_1))\).

The first auxiliary choice that we make is an admissible embedding \(j_0 : S \to G\) having the following property: The pair \((S, \theta)\) and the embedding \(j_0\) determine a tame twisted Levi \(G^0 \subset G\) as in Definition 3.6.1 and Remark 3.6.3. Recall that \(G^0\) contains \(j_0(S)\) and its root system is \(\{\alpha \in R(S, G)\mid \theta \circ N_{E/F} \circ \alpha^\vee(E_0^+) = 1\}\), where \(E/F\) is the splitting field of \(S\). The maximal torus \(j_0(S)\) of \(G^0\) is maximally unramified. It follows that the point in \(B(G^0, F)\) determined by it by Prasad’s theorem [Pra01] is a vertex, [Kal19, Lemma 3.4.3]. The property of \(j_0\) that we require is that \(G^0\) is quasi-split and this vertex is absolutely special. Such an admissible embedding exists by [Kal19, Lemma 3.4.12].

The choice of \(j_0\) establishes a bijection between \(H^1(u \to W, Z(G) \to S)\) and the set of isomorphism classes of non-singular Deligne-Lusztig packet data mapping to the isomorphism class of \((S, j, \chi, \theta)\), and thus a surjection from \(H^1(u \to W, Z(G) \to S)\) to the set of non-singular Deligne-Lusztig packets inside of \(\Pi_{\varphi}\). Lemma E.1 and [Kal19, Lemma 3.4.10] imply that \(\eta_1, \eta_2 \in H^1(u \to W, Z(G) \to S)\) map to the same non-singular Deligne-Lusztig packet if and only if there exists \(w \in \Omega(S, G)(F)_0\) s.t. \(\eta_2 = w\eta_1\). Thus we obtain a bijection between \(H^1(u \to W, Z(G) \to S)/\Omega(S, G)(F)_\theta\) and the set of non-singular Deligne-Lusztig packets inside of \(\Pi_{\varphi}\). Let us write \([\pi_\eta]\) for the Deligne-Lusztig
Corollary 4.3.4 gives us the extension (4.5)

\[ 1 \to \pi_0([\hat{S}]^+) \to \pi_0(S_\varphi^+) \to \Omega(S, G)(F)_{\theta} \to 1, \]

functorially assigned to a torally wild \( L \)-packet datum \((S, \hat{j}, \chi, \theta, (G', \xi, z), j)\). Restriction of representations gives a surjection \( \text{Irr}(\pi_0(S_\varphi^+)) \to \pi_0([\hat{S}]^+)/\Omega(S, G)(F)_{\theta} \to \pi_0([\hat{S}]^+)\) of \([\text{Kal}16b, \text{Corollary 5.4}]\) we see that the construction of the bijection \( \text{Irr}(\pi_0(S_\varphi^+)) \to \Pi_{\varphi} \) reduces to the construction of a bijection \( \text{Irr}(\pi_0(S_\varphi^+), \eta) \to [\pi_\eta] \), where \( \text{Irr}(\pi_0(S_\varphi^+), \eta) \) is the set of irreducible representations of \( \pi_0(S_\varphi^+) \) whose restriction to \( \pi_0([\hat{S}]^+) \) contains the character \( \eta \).

We shall now reduce the construction of the bijection \( \text{Irr}(\pi_0(S_\varphi^+), \eta) \to [\pi_\eta] \) to the case where \( \varphi \) is essentially of depth zero, by which we mean \( \varphi(P_F) \subset Z(\hat{G}) \). It would be convenient to fix a finite \( Z \subset Z(G) \) and form \( S_\varphi^+ \) with respect to that \( Z \), rather than the full \( Z(G) \).

We have fixed the embedding \( j_0 : S \to G^0 \to G \), with \( G^0 \subset G \) quasi-split. Let \((S, \hat{j}, \chi_0, \theta, (G', \xi, z), j)\) be the unique non-singular Deligne-Lusztig packet datum s.t. \( \text{inv}(j_0, j) = \eta \). Then \( \pi_{\eta} \) is the supercuspidal representation associated to the tame elliptic \( k_F \)-non-singular pair \((jS, \hat{j})\), where we recall that \( \theta \) is the character \( j_0 \cdot j \cdot \hat{\chi}_0 \cdot e_{f, \text{ram}} \cdot e_{\text{ram}} \), \( \hat{\chi}_0 \) is the character of \( S(F) \) associated to the \( \xi \)-data \( \zeta_0 = \chi_0 \cdot (\chi_\alpha')^{-1} \) and \( \chi_\alpha' \) is computed in terms of \( \theta \).

Let \( G^0 \subset G' \) be the twisted Levi subgroup with maximal torus \( jS \) and root system \( R_{0+} \). It is an inner form of \( G^0 \). In fact, the inner twist \( \xi : G \to G' \) restricts to an inner twist \( \xi : G^0 \to G'^0 \) and \( (G^0, \xi, z) \) becomes a rigid inner twist of \( G^0 \).

Recall the subgroup \( \hat{M} = \text{Cent}(\varphi(P_F), \hat{G}) \). In Lemma 4.1.8 we showed that it is a dual group to \( G^0 \). Let \( L_{j_{G^0}} : L G^0 \to L G \) be the \( L \)-embedding associated to \( \chi_0 \) and let \( \varphi_{G^0} : W_F \to L G^0 \) be s.t. \( \varphi = L_{j_{G^0}} \circ \varphi_{G^0} \). Then \( S_{\varphi} \subset G \) lies in \( \hat{M} \) and equals \( S_{\varphi_{G^0}} \subset \hat{M} \). In particular, \( \varphi_{G^0} \) is discrete and hence supercuspidal. Moreover \( \varphi_{G^0} \) is essentially of depth zero, by construction. The identification of \( S_{\varphi_{G^0}} \) with \( S_{\varphi} \) extends to an identification of \( S_{\varphi_{G^0}}^+ \) with \( S_\varphi^+ \), where both are taken with respect to the fixed finite \( Z \subset Z(G) \). For any \( \eta \in H^1(u \to W, Z \to S) \) we thus have an identification between \( \text{Irr}(\pi_0(S_{\varphi_{G^0}}^+, \eta)) \) and \( \text{Irr}(\pi_0(S_\varphi^+, \eta)) \).

Let \( L_{j_{S,G^0}} : L S \to L G^0 \) be the \( L \)-embedding associated to unramified \( \chi \)-data. Let \( \varphi_{S,G^0} : W_F \to L S \) be the parameter s.t. \( \varphi_{G^0} = L_{j_{S,G^0}} \circ \varphi_{S,G^0} \). Let \( \theta_{G^0} \) be the character of \( S(F) \) corresponding to \( \varphi_{S,G^0} \). Then \((S, \hat{j}, \emptyset, \theta_{G^0})\) is a torally wild \( L \)-packet datum associated to \( \varphi_{G^0} \) and \((S, \hat{j}, \emptyset, \theta, (G^0, \xi, z), j)\) is the unique non-singular Deligne-Lusztig packet datum with \( \text{inv}(j_0, j) = \eta \). The supercuspidal representation \( \pi_{G^0}^{\eta} \) of \( G^0(F) \) associated to this datum is \( \pi_{G^0}^{\eta} = \pi_{(j_{S,G}, j_0, j)}^{\eta} \), where \( \theta_{G^0} = j_0 \cdot \theta_{G^0} \).

Recall that \( \theta \) is the character associated to the parameter \( \varphi_S \) obtained by \( \varphi = L_{j_{S,G}} \circ \varphi_S \), where \( L_{j_{S,G}} \) is constructed by lifting to \( R(S, G) \times R(S, G^0) \) the \( \chi \) data \( \chi_0 \) and then combining it with unramified \( \chi \)-data for \( R(S, G^0) \). According to [Kala, §6.2] we have \( L_{j_{S,G}} = L_{j_{G^0}} \circ L_{j_{S,G^0}} \). Therefore \( \theta_{G^0} = \theta \).
Write $\delta = j_\ast \zeta \cdot e_{f, \text{ram}} \cdot e_{\text{ram}}$ so that $\theta_j = j_\ast \theta \cdot \delta$. We will now compare the Deligne-Lusztig packets $\pi_{(jS,j,\theta)}^P$ and $\pi_{(jS,j,\theta-\delta)}^P$. The first is parameterized by $\text{Irr}(N(jS,G^0)(F)_{j_\ast \theta,j_\ast \theta})$, while the second by $\text{Irr}(N(jS,G')(F)_{j_\ast \theta-\delta,j_\ast \theta \cdot \delta})$. We have $j_\ast \theta \cdot \delta |_{\theta^0} = j_\ast \theta |_{\theta^0}$, and $\delta$ is $N(S,G^0)(F)$-invariant. Then [Kal19, Lemma 3.6.5] implies that $N(jS,G')(F)_{j_\ast \theta-\delta} = N(jS,G^0)(F)_{j_\ast \theta}$. It is shown in [FKS] that there exists an extension $\hat{\delta}$ of $\delta$ to $G^0(F)_j$, which contains $N(jS,G^0)(F)_{j_\ast \theta}$. Multiplication by $\hat{\delta}$ gives a dimension-preserving isomorphism

$$\text{Irr}(N(jS,G^0)(F)_{j_\ast \theta,j_\ast \theta}) \to \text{Irr}(N(jS,G')(F)_{j_\ast \theta-\delta,j_\ast \theta \cdot \delta}).$$

4.5 Internal structure II: Depth zero DL-packets

We now assume that $\varphi$ is essentially of depth zero, i.e. $\varphi(P_F) \subset Z(\hat{G})$. This implies $\hat{M} = \hat{G}$ and dually $G^0 = G$. For $\eta \in H^1(u \to W, Z \to G)$ our goal is to construct a bijection $\text{Irr}(\pi_0(S'_\eta^+), \eta) \to [\pi_\eta]$. Let $\Omega(S,G)(F)_{\theta, \eta}$ be the mutual stabilizer in $\Omega(S,G)(F)$ of $\theta$ and $\eta$. We pull back the extension (4.5) along the inclusion $\Omega(S,G)(F)_{\theta, \eta} \to \Omega(S,G)(F)_\theta$ and obtain

$$1 \to \pi_0([\hat{S}]^+) \to \pi_0(S'_\eta^+) \to \Omega(S,G)(F)_{\theta, \eta} \to 1.$$  

We then push out along the character $\eta : \pi_0([\hat{S}]^+) \to \mathbb{C}^\times$ and obtain an extension

$$1 \to \mathbb{C}^\times \to \square_1 \to \Omega(S,G)(F)_{\theta, \eta} \to 1. \quad (4.8)$$

By Lemma C.5 the set $\text{Irr}(\pi_0(S'_\eta^+), \eta)$ is in canonical bijection with the set of id-isotypic irreducible representations of $\square_1$.

Write $(S, j_\ast, \emptyset, (G', \xi, z), j)$ for the datum, unique up to isomorphism, such that $\text{inv}(j_0, j) = \eta$. We have the extension

$$1 \to jS(F) \to N(jS,G')(F)_\theta \to N(jS,G')(F)_{j_\ast \theta}/jS(F) \to 1.$$  

By Lemma E.1 and [Kal19, Lemma 3.4.10] the embedding $j$ gives an isomorphism $\Omega(S,G)(F)_{\theta, \eta} \to N(jS,G')(F)_{j_\ast \theta}/jS(F)$. Thus, pushing out the above extension by $\theta \circ j^{-1}$ we obtain an extension

$$1 \to \mathbb{C}^\times \to \square_2 \to \Omega(S,G)(F)_{\theta, \eta} \to 1. \quad (4.9)$$

By Lemma C.5 and Corollary 3.6.7 the set of id-isotypic irreducible representations of $\square_2$ is in bijection with $[\pi_\eta]$. This bijection depends on the choice of normalization $\epsilon$ of the geometric intertwining operators. We are thus left with proving the following:

**Proposition 4.5.1.** The extensions (4.8) and (4.9) are isomorphic.

The rest of this section is devoted to the proof of Proposition 4.5.1. It will follow from an application of [Kalb, Proposition 6.2] once we have established the appropriate framework.

Fix a $\Gamma$-invariant pinning $(\hat{T}, \hat{B}, \{X_{\alpha}\})$ of $\hat{G}$ and modify the torally wild $L$-packet datum $(S, j_\ast, \emptyset, \theta)$ within its isomorphism class so that $j_\ast(\hat{S}) = \hat{T}$. Let $L_j : LS \to N(\hat{T}, \hat{G}) \rtimes W_F$ be the extension of $\hat{j}$ determined by unramified $\chi$.
We obtain two actions of $W_F$ on $N(\widehat{T}, \widehat{G})$, one by $\text{Ad}(\varphi(w))$, and another by $\text{Ad}(\hat{j}(1 \times w))$. The first action is the twist of the second action by $\hat{j} \circ \varphi_{S,0}$, where $\varphi_{S,0} \in Z^1(W_F, \widehat{S})$, is determined by $\varphi_S(w) = \varphi_{S,0}(w) \times w$. In particular, both actions induce the same action on $\widehat{T}$ and $\Omega(\widehat{T}, \widehat{G})$, and this is the action that makes $\hat{j} : \widehat{S} \to \widehat{T}$ equivariant. But on $N(\widehat{T}, \widehat{G})$ the two actions differ. The group of fixed points in $N(\widehat{T}, \widehat{G})$ for the first action is $S_{\varphi}$.

**Fact 4.5.2.** The second action extends to $\Gamma$.

**Proof.** It is enough to find a finite field extension $L/F$ s.t. $L_j(1 \times w) = 1$ for $w \in W_L$, but if $L$ contains the splitting field of $\mathcal{S}$ then the value $L_j(1 \times w)$ is given by the formula for $r_q(x)$ on [LS87, p.237], with $x \in L^\times$ being the image of $w$ under the local reciprocity map $W_F^ab \to L^\times$. Since all our $\chi$-data have finite order there exists $L$ for which this formula evaluates to 1 for all $x \in L^\times$. 

**Lemma 4.5.3.** Consider the exact sequence

$$1 \to \widehat{T}_{sc} \to N(\widehat{T}_{sc}, \widehat{G}_{sc}) \to \Omega(\widehat{T}, \widehat{G}) \to 1$$

with the action of $\Gamma$ given by $\text{Ad}(L_j(1 \times w))$, $w \in W_F$. Taking invariants we obtain the exact sequence

$$1 \to \widehat{S}_{sc}^\Gamma \to N(\widehat{T}_{sc}, \widehat{G}_{sc})^\Gamma \to \Omega(S, G)(F) \to 1.$$

**Proof.** That the first and third term have this shape follows from the fact that this action makes $\hat{j} : \widehat{S} \to \widehat{T}$ equivariant, as discussed above.

To show that an element $\mu \in \Omega(S, G)(F)$ lifts, define $\hat{j}' = \hat{j} \circ \mu^{-1}$. We extend $\hat{j}'$ to $L_j' : L_{S_{sc}} \to N(\widehat{T}_{sc}, \widehat{G}_{sc}) \times W_F$ using the $\chi$-data for $R(S, G)$ from the torally wild $L$-packet datum $(\mathcal{S}, \hat{j}, \chi, \theta)$. Note that the transport of this $\chi$-data to $\widehat{T}$ via $\hat{j}$ is the same as the transport via $\hat{j}'$, because this $\chi$-data is $\Omega(S, G)(F)$-invariant. In particular, the restrictions to $W_F$ of $L_j$ and $L_j'$ agree. On the other hand, [LS87, (2.6.2)] says $L_j' = \text{Ad}(n_{\mu}^{-1} \circ L_j)$, for a suitable $n_{\mu} \in N(\widehat{T}_{sc}, \widehat{G}_{sc})$ lifting $\mu$. Thus, $n_{\mu}$ commutes with the restriction of $L_j$ to $W_F$, i.e. $n_{\mu} \in N(\widehat{T}_{sc}, \widehat{G}_{sc})^\Gamma$. 

Define $T$ and $\bar{T}$ to be the complex algebraic groups $N(\widehat{T}, \widehat{G})$ and $N(\widehat{T}, \widehat{G})$ equipped with the $\Gamma$-action that extends the action $\text{Ad}(L_j(1 \times w))$ for $w \in W_F$.

Define $A$ to be the finite group $\Omega(S, G)$ with the natural $\Gamma$-action. Then $\bar{T}$ is an extension of $\widehat{S}$ by $A$ that remains exact after taking $\Gamma$-fixed points by Lemma 4.5.3. We have $S_{\varphi} = T^{\varphi \in (W_F)}$.

Let $x \in Z^1(u \to W, Z \to S)$ represent the class $\eta = \text{inv}(j_0, j)$. Then $A[x] = \Omega(S, G)(F)_{\eta} = N(jS, G')(F)/jS(F)$, $A[\varphi_{S,0}] = \Omega(S, G)(F)_\theta$, and $A[x][\varphi_{S,0}] = [N(jS, G')(F)/jS(F)]_\theta$.

We write $C_{[x]}^{\Gamma}$ for the extension obtained by taking the preimage $T^{\varphi} + \Gamma$ of $T^{\Gamma}$ in $\bar{T}$, pulling back along the inclusion $A[x] \to A^{\Gamma}$, and pushing out along $[x] : [\widehat{S}]^+ \to \mathbb{C}^\times$. We write $C_{[x]}^{\varphi_{S,0}}$ for the extension obtained by pulling back the extension $S_{\varphi}^+$ along the inclusion $A[x][\varphi_{S,0}] \to A[\varphi_{S,0}]$ and pushing out along $[x] : [\widehat{S}]^+ \to \mathbb{C}^\times$, i.e. the extension (4.8).
We now let $\tilde{T}$ be the algebraic group $N(j_{0}, S, G)$ with its natural $F$-structure. It is an extension of $S$ by $A$. It remains exact after taking $\Gamma$-fixed points by [Kalb, Lemma 3.4.10]. Write $E_{[x]}^{0}$ for the extension obtained by pulling back $\tilde{T}(F)$ along the inclusion $A[x_{*}] \to A$ and pushing out along $\theta$. The group $N(j S, G')$ is the inner twist $T_{x}$ of $\tilde{T}$ by the 1-cocycle $x$ in the sense of [Kalb]. Write $E_{[x]}^{0}$ for the extension obtained by pulling back $\tilde{T}_{x}(F)$ along the inclusion $A[x_{*}]_{[x]} \to A[x]$ and pushing out along $\theta$, i.e. the extension (4.9).

Finally, let $E_{[x]}^{0, [\phi S]}$ be obtained by pulling back $E_{[x]}^{0}$ along the inclusion $A[x_{*}]_{[\phi S]} \to A[x]$, and $E_{[x]}^{0, [\phi S]}$ be obtained by pulling back $E_{[x]}^{0}$ along the inclusion $A[x_{*}]_{[\phi S]} \to A[\phi S]$. Then [Kalb, Proposition 6.2] states that an isomorphism $E_{[x]}^{0, [\phi S]} \to E_{[x]}^{0, [\phi S]}$ determines an isomorphism $E_{[x]}^{0} \to E_{[x]}^{0}$.

It therefore remains to establish an isomorphism $E_{[x]}^{0, [\phi S]} \to E_{[x]}^{0, [\phi S]}$, which we shall do by showing that both extensions are split. The choice of an isomorphism is then given by choices of splittings.

Write $N(\tilde{T}_{sc}, \tilde{G}_{sc})_{\theta}^{\Gamma}$ for the preimage in $N(\tilde{T}_{sc}, \tilde{G}_{sc})_{\theta}^{\Gamma}$ of $\Omega(S, G)(F)_{\theta}$, and analogously $N(\tilde{T}_{sc}, \tilde{G}_{sc})_{\hat{\eta}}^{\Gamma}$.

** Lemma 4.5.4.** The extension

$$1 \to \tilde{S}_{sc}^{\Gamma} \to N(\tilde{T}_{sc}, \tilde{G}_{sc})_{\theta}^{\Gamma} \to \Omega(S, G)(F)_{\theta} \to 1$$

has multiplicity 1 in the sense of Definition A.7.

**Proof.** This is an extension of one finite abelian group by another, so applying Lemma A.11 it is enough to show that it has trivial commutator in the sense of Definition A.8. Let $\theta_{sc}$ be the restriction of $\theta$ to $S_{sc}(F)$. The extension we are considering is the restriction of the extension

$$1 \to \tilde{S}_{sc}^{\Gamma} \to N(\tilde{T}_{sc}, \tilde{G}_{sc})_{\theta_{sc}}^{\Gamma} \to \Omega(S, G)(F)_{\theta_{sc}} \to 1,$$  \hspace{1cm} (4.10)

so it is enough to show that this extension has trivial commutator. This extension breaks up according to the $F$-simple factors of $G$, and its commutator breaks up accordingly, so we may assume that $G$ is $F$-simple. Since $\theta_{sc}$ is still non-singular, $\Omega(S, G)(F)_{\theta_{sc}}$ is cyclic, and hence the commutator is trivial, except when $G$ is the restriction of scalars of a group of split type $D_{2n}$ by Lemma 3.1.4, in which case $\Omega(S, G)(F)_{\theta_{sc}}$ may be isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2}$.

In that latter case we argue as follows. The formation of $\tilde{T}$ respects restriction of scalars, so we may assume that $G$ is split of type $D_{2n}$. The isomorphism class of the extension does not change if we conjugate $\tilde{T}$ by an element of $\tilde{T}$, so we may apply Lemma 4.1.10 to conclude that $\tilde{T}_{j}$ restricts trivially to $I_{F}$. Let $f \in N(\tilde{T}_{ad}, \tilde{G}_{ad})$ be the image of the Frobenius element in $W_{F}/I_{F}$ under $\tilde{j}$|$_{W_{F}}$. Then $f$ is an elliptic element and the extension (4.10) is exactly the extension of Lemma I.1. \hfill $\square$

** Corollary 4.5.5.** The extension $E_{[x]}^{0}$ is split. Any extension of $\eta$ to $N(\tilde{T}_{sc}, \tilde{G}_{sc})_{\theta}^{\Gamma}$ is a splitting.

**Proof.** Pulling back the extension of Lemma 4.5.4 along the inclusion $A[x] \to A^{\Gamma}$ and then pushing out along $[x]: \tilde{S}_{sc}^{\Gamma} \to \mathbb{C}^{*}$ is another way to obtain $E_{[x]}^{0}$. \hfill $\square$
Lemma 4.5.6. The character $\theta$ extends to $N(j_0S,G)(F)$. 

Proof. The proof is the same as for Lemma 2.2.3, the only difference being that the equality $\Omega(S,G)(k) = N(S,G)(k)/S(k)$ used there and implied by Lang’s theorem is replaced here by the equality $\Omega(S,G)(F) = N(j_0S,G)(F)/S(F)$ due to [Kal19, Lemma 3.4.10], which uses the fact that the point of $j_0S$ is absolutely special. $\square$

Corollary 4.5.7. The extension $E_0[\varphi_S]$ is split. Any extension of $\theta$ to $N(j_0S,G)(F)$ is a splitting of $E_0[\varphi_S]$. 

Proof. Immediate. $\square$

The choices that we are left to account for in the depth-zero case are: The embedding $j_0$, the extension $\tilde{\theta}$ of $\theta$, the extension $\tilde{\eta}$ for each $\eta$, and the normalization $\epsilon$ for each $j$. The choices of $\tilde{\theta}$ and normalization of $\epsilon$ for $j_0$ are linked, while $\eta_0 = 1$ has the natural extension $\tilde{\eta}_0 = 1$. For $\eta \neq 1$, the choices of $\tilde{\eta}$ and normalization $\epsilon$ are linked.

Once existence and uniqueness of generic member is known, the Whittaker datum $w$ pins down $j_0$ and provides a second link for the normalization $\epsilon$ at the vertex of $j_0$ and $\tilde{\theta}$. So the only thing left is to find a second link between the normalization $\epsilon$ for each $j \neq j_0$ and the extension $\tilde{\eta}_j$ for $\eta_j$.

In the positive depth case, there is also the choice of extension $\hat{\delta}$ of $\delta$.

4.6 Remarks about stability and transfer

In the construction of the bijection $\text{Irr}(\pi_0(S^+_\varphi)) \rightarrow \Pi_\varphi$ in the previous subsection we made a number of auxiliary choices. In this subsection we will sketch an argument showing that the resulting parameterized $L$-packet satisfies stability, and more generally endoscopic transfer for elements $s \in S^+_\varphi$ that lie in the subgroup $[\hat{S}]^+$. The details of this argument, its extension to all $s \in S^+_\varphi$, and a canonical choice for the bijection $\text{Irr}(\pi_0(S^+_\varphi)) \rightarrow \Pi_\varphi$, will be the subject of a forthcoming paper.

Thus we fix a rigid inner twist $(G', \xi, z)$ and consider the part $\Pi_\varphi(G', \xi, z)$ of the compound $L$-packet $\Pi_\varphi$ corresponding to it. We form the $s$-stable character

$$\sum_{\rho \in \text{Irr}(\pi_0(S^+_\varphi))]^{\pi_\rho} \rho \rightarrow [z]} \text{tr}\rho(s)\Theta_{\pi_\rho}.$$ 

Since $\text{tr}\rho(s)$ depends only on $\rho|_{[\hat{S}]^+}$ we may break the sum as follows

$$\sum_{\eta \in \pi_0([\hat{S}]^+)\backslash \Omega(S,G)(F)\rho \in \text{Irr}(\pi_0(S^+_\varphi))]^{\pi_\eta} \rho \rightarrow [z]} \text{tr}\rho(s)\Theta_{\pi_\rho},$$

and then use the fact that $\rho|_{[\hat{S}]^+} = [\Omega(S,G)(F)\rho \cdot \eta]^{\oplus m(\eta, \rho)}$. According to Corollary C.4 and Proposition 4.5.1, $m(\eta, \rho)$ is equal to the dimension of the representation of $N(jS,G)(F)$ corresponding to $\rho$, which in turn is equal to the
multiplicity in \( \pi_j \) of the irreducible constituent corresponding to \( \rho \), i.e. \( \pi_\rho \). Altogether the above double sum becomes

\[
\sum_{\eta \in \pi_0(\hat{S}^+)^*} \eta(s) \Theta_{\pi_j},
\]

where \( j \) is determined by \( \text{inv}(j_0, j) = \eta \). In other words, we have

\[
\sum_{j: S \to G'} (\text{inv}(j_0, j), s) \Theta_{\pi_{(jS, \theta_j)}},
\]

where \( j \) runs over the set of \( G'(F) \)-conjugacy classes of admissible embeddings \( S \to G' \), and \( j_0 : S \to G \) is the admissible embedding giving the base point in \( \Pi_\rho \).

To compute the character of \( \pi_{(jS, \theta_j)} \) we apply recent [Spi17] and ongoing work of DeBacker and Spice. It provides an inductive formula for the character of an irreducible supercuspidal representation obtained from J.K.Yu’s construction in terms of the character of the corresponding representation of lower depth. Since the formula is additive, we may apply it to the reducible representation \( \pi_{(jS, \theta_j)} \) and obtain an expression for its character in terms of the character of the depth-zero representation \( \pi_{(G^{\text{p}, S, \sigma_{-1}})} \), which in turn reduces to the character formula for the reducible Deligne-Lusztig induction \( R_{\bar{\theta}} \) and its extension \( R_\theta \) from §2 and §3.5. Altogether the formula we obtain for \( \Theta_{\pi_{(jS, \theta_j)}} \) is virtually the same as that of [Kal19, §4.8] and the arguments of [Kal19, §6] apply.

The situation when \( s \in S_\rho^+ \) does not lie in \( \hat{S}^+ \) is rather different and much more subtle. For then \( \text{tr}_\rho(s) \) depends not just on the restriction of \( \rho \) to \( \hat{S}_\rho^+ \). This means that different members of a given non-singular Deligne-Lusztig packet will contribute to the \( s \)-stable character with different weights.

**Appendix**

**A Basic Clifford theory**

We recall here basic facts about Clifford theory and offer some mild generalizations needed in this paper.

Let \( G \) be a group (not necessarily finite) and \( \pi : G \to \text{GL}(V) \) a representation (not necessarily finite-dimensional). We say that \( V \) is irreducible if the only \( G \)-invariant subspaces are \( V \) and \( \{0\} \).

**Lemma A.1.** The following are equivalent.

1. \( V \) is the sum of its irreducible subrepresentations.
2. \( V \) is a direct sum of irreducible subrepresentations.
3. For every subrepresentation \( W_1 \subset V \) there is a subrepresentation \( W_2 \subset V \) s.t. \( V = W_1 \oplus W_2 \).

**Proof.** This is [BH06, Proposition 2.2], where the assumption is made that \( G \) is locally profinite and \( V \) is smooth, but this assumption is not used in the proof. \( \square \)
Definition A.2. A representation $V$ satisfying the assumptions of above Lemma is called semi-simple.

Fact A.3. Let $H \subset G$ a subgroup of finite index.

1. A representation of $G$ is semi-simple if and only if its restriction to $H$ is.
2. A representation of $H$ is semi-simple if and only if its induction to $G$ is.

Proof. This is [BH06, Lemma 2.7], where again the assumption that $G$ is locally profinite, $H$ is open, and $V$ is smooth, is not used. The only part stated here but not in loc. cit. is that if $\sigma$ is a representation of $H$ s.t. $\text{Ind}_H^G \sigma$ is semi-simple, then $\sigma$ is semi-simple. But since there is an $H$-equivariant embedding $\sigma \to \text{Ind}_H^G \sigma$ the statement is obvious. □

Lemma A.4. Let $B$ be a group, $A$ a normal subgroup of $B$, $\pi$ a finite-dimensional irreducible representation of $B$ whose restriction to $A$ is semi-simple. We make no finiteness assumptions on $A$, $B$, or $C = B/A$. Then

1. The set $S_{\pi}$ of irreducible constituents of $\pi|_A$ is a single $B/A$-orbit and each member of $S_{\pi}$ occurs with the same multiplicity $m_{\pi A}$ in $\pi|_A$.
2. If $B/A$ is abelian and $m_{\pi A} = 1$, then $\{\chi \in (B/A)^* | \chi \otimes \pi \cong \pi\}$ is the annihilator of the kernel of the action of $B/A$ on $S_{\pi}$.
3. If $B/A$ is abelian and $\pi'$ is another finite dimensional irreducible representations of $B$ with semi-simple restriction to $A$ s.t. $\text{Hom}_A(\pi, \pi') \neq 0$, then $\pi' = \chi \otimes \pi$ for a character $\chi \in (B/A)^*$.

Proof. By assumption $\text{Res}_A^B \pi = \bigoplus_{\pi \in S_{\pi}} \sigma^{m_{\pi A}}$ for a (necessarily finite) set $S_{\pi}$ of irreducible representations of $A$ and positive integers $m_{\pi A}$. If $S' \subset S_{\pi}$ is $B$-invariant subset, then $\bigoplus_{\pi \in S'} \sigma^{m_{\pi A}}$ is a $B$-subrepresentation, contradicting the irreducibility of $\pi$.

Since $\pi$ is finite-dimensional, $m_{\pi A} = \dim \text{Hom}_A(\sigma, \pi)$ by Schur’s lemma. Since $\pi|_A \cong \pi \text{Ad}(b)|_A$ for any $b \in B$ we see that $m_{\pi A} = m_{\pi A \circ \text{Ad}(b)}$ and the claim follows from the previous paragraph.

Assume now that $C = B/A$ is abelian and $m_{\pi A} = 1$. The kernel of the action of $C$ on $S_{\pi}$ is the stabilizer $C_{\pi}$ of one, hence any, $\sigma \in S_{\pi}$. It is a finite index subgroup of $C$. Let $B_\sigma$ be its preimage in $B$, a finite index subgroup of $B$. By Fact A.3 the representation $\pi|_{B_\sigma}$ is semi-simple, so we can write it as $\pi|_{B_\sigma} = \sigma_1 \oplus \cdots \oplus \sigma_k$ with $\sigma_i$ and irreducible representation of $B_\sigma$. Then $\sigma_i|_A$ is a subrepresentation of $\pi|_A$ and hence semi-simple, so we can write it as $\sigma_{i,1} \oplus \cdots \oplus \sigma_{i,k_i}$, where $\sigma_{i,j}$ are irreducible representations of $A$. The group $C_{\pi}$ acts transitively on the set $\{\sigma_{i,1}, \ldots, \sigma_{i,k_i}\}$ by the previous point. On the other hand, $\sigma_{i,1}$ is an irreducible constituent of $\pi|_A$ and hence fixed by $C_{\pi}$. It follows that $k_i = 1$, i.e. $\sigma_i|_A$ is irreducible. Therefore Schur’s lemma implies

$$\text{End}_A(\pi) = \text{End}_{B_\sigma}(\pi) = \text{Ind}_{B_\sigma}^B \mathbb{C} = \text{Ind}_{C/C_{\pi}}^B \mathbb{C} = \bigoplus_{\chi \in (C/C_{\pi})^*} \chi.$$  

Finally let $\pi'$ be a finite-dimensional irreducible representation of $B$ s.t. $\pi'|_A$ is semi-simple and $\text{Hom}_A(\pi, \pi') \neq \{0\}$. Now $\text{Hom}_C(\pi, \pi')$ is a finite-dimensional
representation of \( B \), isomorphic to \( \pi \otimes \pi' \). By a theorem of Chevalley [Ser94] this is a semi-simple representation of \( B \). Therefore, the subrepresentation \( \text{Hom}_A(\pi, \pi') \) is also semi-simple. But this is a representation of \( C \). Since \( C \) is abelian Schur’s lemma implies that every finite-dimensional irreducible representation is 1-dimensional. Let \( \chi \) be a character of \( C \) that occurs in \( \text{Hom}_A(\pi, \pi') \). Then \( \text{Hom}_A(\pi \otimes \chi, \pi') \) contains the trivial character of \( C \), i.e. \( \text{Hom}_B(\pi \otimes \chi, \chi') \neq 0 \).

**Remark A.5.** If we write \( \text{Irr}_{A \text{-ss}}(B) \) for the set of finite-dimensional irreducible representations of \( B \) whose restriction to \( A \) is semi-simple, the above lemma shows that restriction to \( A \) gives a well-defined map \( \text{Irr}_{A \text{-ss}}(B) \to \text{Irr}(A)/C \) and shows that \( C^* \) acts transitively on the fibers of this map when \( C \) is abelian. When \( m_s = 1 \), the kernel of the action of \( C^* \) on the fiber through \( \pi \) and the kernel of the action of \( C \) on \( S_s \) are mutually annihilators.

**Example A.6.** The simplest example that shows the necessity of the assumption \( m_s = 1 \) above is given by the quaternion group \( Q \), which is a non-abelian central extension

\[
0 \to \mathbb{Z}/2\mathbb{Z} \to Q \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to 0.
\]

It has five irreducible representations, four of which are the characters of \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \), pulled back to \( Q \), and one of which is 2-dimensional and on which \( \mathbb{Z}/2\mathbb{Z} \) acts by its non-trivial character. This 2-dimensional \( \pi \) is preserved under the product by all characters of \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \), while the unique irreducible constituent of its restriction to \( \mathbb{Z}/2\mathbb{Z} \) is preserved by all elements of \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \). Moreover, \( \text{End}_{\mathbb{Z}/2\mathbb{Z}}(\pi) \) is the regular representation of \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \).

By a finite-dimensional projective representation of a finite group \( C \) we shall understand a set-theoretic map \( \tau : C \to \text{GL}(V) \), where \( V \) is a finite-dimensional complex vector space, such that for \( c_1, c_2 \in C \) there exists \( z(c_1, c_2) \in \mathbb{C}^\times \) with \( \tau(c_1)\tau(c_2) = z(c_1, c_2)\tau(c_1c_2) \). It is immediate that then \( z \in Z^2(C, \mathbb{C}^\times) \).

Let \( 1 \to A \to B \to C \to 1 \) be an extension of finite groups.

**Definition A.7.** We shall say that the extension has the multiplicity 1 property if for every irreducible representation \( \pi \) of \( B \), every irreducible constituent of \( \pi|_A \) occurs with multiplicity 1.

Of particular interest for us will be the situation where both \( A \) and \( C \) are abelian. To this extension we can associate a commutator function as follows. Let \( s : C \to B \) be a set-theoretic section. Then \( s(c_1)s(c_2)s(c_1)^{-1}s(c_2)^{-1} \) is an element of \( A \), which we call \( f(c_1, c_2) \). If the action of \( C \) on \( A \) is trivial, then \( f(c_1, c_2) \) does not depend on the choice of \( s \). In general it does, but the image of \( f(c_1, c_2) \) in the group \( A_{c_1,c_2} \) of coinvaraints of \( A \) for the action of the subgroup of \( C \) generated by \( c_1 \) and \( c_2 \) does not. It thus makes sense to ask if the image of \( f(c_1, c_2) \) in \( A_{c_1,c_2} \) is trivial.

**Definition A.8.** We shall say that the extension \( B \) has trivial commutator if for all \( c_1, c_2 \in C \) the image of \( f(c_1, c_2) \) in \( A_{c_1,c_2} \) is trivial.

**Lemma A.9.** A central extension of a finite abelian group by \( \mathbb{C}^\times \) is split if and only if it is abelian.

**Proof.** Clearly a split central extension of \( \mathbb{C}^\times \) is abelian. Conversely let \( 1 \to \mathbb{C}^\times \to B \to C \to 1 \) be a central extension and assume that \( B \) is abelian. Given
Lemma A.11. Let $B$ be a finite group, $A \subset B$ a normal subgroup, $C = B/A$.

1. The map $	ext{Irr}(B) \to \text{Irr}(A)/C$ obtained by restriction is surjective.
2. We have $\dim(\pi) = |S_\pi| \cdot m_\pi \cdot \dim(\rho)$, for any $\rho \in S_\pi$.
3. Assume that $C$ is abelian. Let $\rho \in S_\pi$ and let $C_\rho \subset C$ be its stabilizer. If $H^2(C_\rho, \mathbb{C}^\times) = 0$ then $m_\pi = 1$.

Assume now that both $A$ and $C$ are abelian.

4. We have $m_\pi = 1$ if and only if $\rho(f(c_1, c_2)) = 1$ for all $c_1, c_2 \in C_\rho$.
5. The extension has the multiplicity one property if and only if its commutator is trivial.

Proof. For surjectivity, let $\rho \in \text{Irr}(A)$, acting on a complex vector space $W$, and let $B_\rho$ be the stabilizer of the isomorphism class of $\rho$ for the action of $B$ on $\text{Irr}(A)$. Choose a set of representatives $b_1, \ldots, b_k$ for $C_\rho := B_\rho/A$. For each $b_i$ choose a $T_i \in \text{Aut}_C(W)$ giving an isomorphism $\rho \circ \text{Ad}(b_i)^{-1} \to \rho$ of representations of $A$. Define a map $\hat{\rho} : B_\rho \to \text{Aut}_C(W)$ by $\hat{\rho}(b_i a) = T_i \circ \rho(a)$ for all $a \in A$. Then $\hat{\rho}$ is a projective representation whose associated 2-cocycle $z \in Z^2(B_\rho, \mathbb{C}^\times)$, defined by $z(b_1 b_2) = \hat{\rho}(b_1) \circ \hat{\rho}(b_2) \circ \hat{\rho}(b_1 b_2)^{-1}$, is immediately checked to be inflated from $C_\rho$. Let $\tau$ be an irreducible projective representation of $C_\rho$ with 2-cocycle $z^{-1}$; it exists, c.f. [Tap77, Theorem 1.3]. Then $\hat{\rho} \otimes \tau$ is a linear representation of $B_\rho$. The map $\text{End}_C(\tau) \to \text{End}_A(\hat{\rho} \otimes \tau)$ given by $f \mapsto \text{id} \otimes f$ is an isomorphism of $C_\rho$-representations, hence $\hat{\rho} \otimes \tau$ is irreducible. By Mackey’s test, the induction $\pi$ of $\hat{\rho} \otimes \tau$ to $B$ remains irreducible and is a preimage of $\rho$ under the map $\text{Irr}(B) \to \text{Irr}(A)/C$.

We have $\text{Res}_{B_\rho}^B \pi = \bigoplus_{b \in B_\rho \setminus B} (\hat{\rho} \otimes \tau)^b$, all summands being pairwise non-isomorphic. We see $|S_\pi| = |B : B_\rho|$, $m_\pi = \dim(\tau)$, and

$$\dim(\pi) = |B : B_\rho| \cdot \dim(\tau) \cdot \dim(\rho) = |S_\pi| \cdot m_\pi \cdot \dim(\rho).$$

Assume now that $C = B/A$ is abelian. If $H^2(C_\rho, \mathbb{C}^\times) = 0$, then $\tau$ is 1-dimensional by Fact A.10.
Lemma A.9 implies that this pushout is a split extension of $C$ only if $\rho$ is the composition of $f|_{C_p \times C_p}$ with $\rho$. Lemma A.9 implies that this pushout is a split extension of $C_p$ by $C^\times$ if and only if $\rho(f(c_1, c_2)) = 1$ for all $c_1, c_2 \in C_p$. The cocycle $\zeta$ represents the class of this extension. Fact A.10 implies that $n_\pi = \dim(\tau) = 1$ is equivalent to $\rho(f(c_1, c_2)) = 1$ for all $c_1, c_2 \in C_p$.

Given a subgroup $C' \subset C$ we can pull back the extension $1 \to A \to B \to C \to 1$ along the inclusion $C' \to C$ and then push it out along the projection $A \to A_{C'}$, where $A_{C'}$ is the group of $C'$-coinvariants of $A$. The result is a central extension $B'$ of $C'$ by $A_{C'}$. This extension is abelian, for every subgroup $C' \subset C$, if and only if the image of $f(c_1, c_2)$ in $A_{(c_1, c_2)}$ is trivial for all $c_1, c_2 \in C$. If these equivalent statements hold, then what we just proved implies $n_\pi = 1$ for all $\pi \in \Irr(B)$, because $\rho \in S_\pi$ factors through $A_{C'}$.

Assume now the converse – there exists $C' \subset C$ such that the extension $B'$ is not abelian. Hence there exist $c_1, c_2 \in C'$ s.t. $1 \neq f(c_1, c_2) \in A_{C'}$. Let $\rho : A_{C'} \to C^\times$ be a character s.t. $\rho(f(c_1, c_2)) \neq 1$. We inflate $\rho$ to a character of $A$ and let $C_p$ be its stabilizer in $C$. By construction $C' \subset C_p$. Now $\rho : A \to C^\times$ descends to a character of $A_{C'}$. Let $B_\rho$ be the central extension of $C_p$ by $A_{C'}$. Pushing it out by $\rho$ we obtain a central extension of $C_p$ by $C^\times$, non-abelian because its commutator at $c_1, c_2$ is $\rho(f(c_1, c_2)) \neq 1$. Applying Lemma A.9 we see that this extension is non-split. Thus the 2-cocycle $\zeta \in Z^2(C_p, C^\times)$ that corresponds to $\rho$ is not cohomologically trivial. By Fact A.10 the irreducible projective representations $\tau$ of $C_p$ with this cocycle have dimension greater than 1, implying $n_\pi > 1$ for any $\pi \in \Irr(B)$ with $\rho \in S_\pi$.

**Corollary A.12.** The multiplicity 1 property for extensions of finite abelian groups is stable under pull-backs, push-outs, and cartesian products.

**Proof.** Given an extension $1 \to A \to B \to C \to 1$ with commutator function $f$, the commutator function of the pull-back along a homomorphism $i : C' \to C$ is the restriction $f \circ (i \times i)$, that of the push-out along a $C$-equivariant homomorphism $\rho : A \to A'$ is the composition $\rho \circ f$, and the commutator function of the product of two extensions $B_1$ and $B_2$ is the product $(f_1, f_2)$. \hfill \Box

**B  A basis theorem**

We record here some remarks on an abstract form of the Harish-Chandra basis theorem.

Let $G$ and $N$ be locally profinite groups. Let $S \subset N$ be an open normal subgroup of finite index. Let $\Pi$ be a smooth semi-simple finite length representation of $G \times N$ on a complex vector space $V$ and assume that $S$ acts via a character $\theta$. Then for any $n \in N/S$ the subspace $C\Pi(1 \times n) \subset \End_G(\Pi|_G)$ is well-defined (of dimension at most 1). Furthermore, we can decompose $\Pi = \bigoplus (\pi \otimes \rho)^{m_{\pi, \rho}}$, where $\pi$ and $\rho$ run over the set of irreducible smooth representations of $G$ and $N$, respectively, and $m_{\pi, \rho}$ are natural numbers. Then the condition $m_{\pi, \rho} \neq 0$ defines a correspondence

$$[\Pi|_G] \overset{m}{\leftarrow} \Irr(N, \theta)$$
between the set $[\Pi|_G]$ of irreducible constituents of the $G$-representation $\Pi|_G$, and the set $\text{Irr}(N, \theta)$ of irreducible representations of $N$ whose restriction to $S$ is $\theta$-isotypic.

**Lemma B.1.** The following two statements are equivalent.

1. The subspaces $\mathbb{C} \Pi(1 \times n)$ of $\text{End}_G(\Pi|_G)$ indexed by $n \in N/S$ are both linearly independent and generating.

2. We have $m_{\pi, \rho} \in \{0, 1\}$ and the correspondence $m$ is a bijection.

**Proof.** Write $M_{\pi, \rho} = \text{Hom}_{G \times N}(\pi \otimes \rho, \Pi)$, so that

$$\Pi = \bigoplus_{\pi, \rho} \pi \otimes \rho \otimes M_{\pi, \rho}, \quad m_{\pi, \rho} = \dim_{\mathbb{C}}(M_{\pi, \rho}).$$

Then

$$\text{End}_G(\Pi) = \bigoplus_{\pi, \rho, \pi', \rho'} \text{Hom}_G(\pi \otimes \rho, \Pi) \otimes \text{Hom}_{\mathbb{C}}(\rho \otimes \rho', \Pi) = \bigoplus_{\pi, \rho, \rho'} \text{Hom}_{\mathbb{C}}(\rho \otimes \rho', \Pi) \otimes \text{Hom}_{\mathbb{C}}(M_{\pi, \rho}, M_{\pi, \rho'}).$$

and for $n \in N$ the image of $\Pi(1 \times n)$ under these isomorphisms is $\rho(n) \otimes \text{id}$ in the components indexed by $(\pi, \rho, \rho')$ with $m_{\pi, \rho} > 0$ and 0 in the components indexed by $(\pi, \rho, \rho')$ with $\rho \neq \rho'$ or in the components indexed by $(\pi, \rho, \rho)$ with $m_{\pi, \rho} = 0$.

$2 \Rightarrow 1$: Then we have $\text{End}_G(\Pi) = \bigoplus_{\rho} \text{End}_\mathbb{C}(\rho)$ and the claim follows from the orthogonality relations for projective representations of the finite group $N/S$.

$1 \Rightarrow 2$: We first use the fact that $\{\Pi(1 \times n)|n \in N\}$ is generating to bound $m_{\pi, \rho}$. Consider the subspace given by the condition $\rho' = \rho$, i.e.

$$\bigoplus_{\rho} \text{End}_\mathbb{C}(\rho) \otimes \left(\bigoplus_{\pi} \text{End}_\mathbb{C}(M_{\pi, \rho})\right).$$

The set $\{\Pi(1 \times n)|n \in N\}$ is contained in that subspace, so this must then be the whole space. Therefore for fixed $\pi$ we have $m_{\pi, \rho} > 0$ for at most one $\rho$. Fixing now $\rho$, the only elements obtained from $\Pi(1 \times n)$ are of the form $\rho(n) \otimes (\oplus \pi \text{id}_{M_{\pi, \rho}})$, so 1 implies again that for each $\rho$ there is at most one $\pi$ with $m_{\pi, \rho} > 0$ and moreover that then $m_{\pi, \rho} = 1$.

It is clear that for each $\pi$ there is $\rho$ with $m_{\pi, \rho} > 0$, by virtue of $\pi \subset \Pi|_G$. It remains to show that conversely for a given $\rho$ there does exist a $\pi$ with $m_{\pi, \rho} > 0$, and for this we use the linear independence of $\{\Pi(1 \times n)|n \in N/S\}$. We consider again the above displayed space. Each $\rho$-summand has dimension $\dim(\rho)^2$, so the entire space has dimension $\sum_{\rho} m_{\pi, \rho} \dim(\rho)^2 \leq |N/S|$ and linear independence implies that equality must hold, i.e. for every $\rho$ there does exist a $\pi$ with $m_{\pi, \rho} = 1$.

**Remark B.2.** The second statement in the Lemma above can be equivalently stated as follows: For each $\rho \in \text{Irr}(N, \theta)$ the $\rho$-isotypic component $\pi_\rho$ of $\Pi|_N$ is $G$-irreducible and the map $\rho \mapsto \pi_\rho$ is a bijection $\text{Irr}(N, \theta) \rightarrow [\Pi|_G]$. 

63
Let $G$ be a locally profinite group, $H \subset G$ an open (and hence closed) normal subgroup. Let $N \subset G$ be a closed subgroup, write $N_H = N \cap H$, and let $S \subset N_H$ be an abelian open normal subgroup of $N$ of finite index.

The group $N$ acts on $G$ by conjugation and we can form $G \rtimes N$. Note that $N \to G \rtimes N, n \mapsto (n^{-1}, n)$ is an injective group homomorphism that embeds $N$ as a normal subgroup of $G \rtimes N$ that commutes with $G$ and therefore provides an isomorphism $G \rtimes N \to G \times N$.

We have the subgroup $H \rtimes N$ of $G \rtimes N$, normal if $G/H$ is abelian. Let $\sigma$ be a smooth finite-length semi-simple representation of $H \rtimes N$ on a complex vector space $V$, such that the central subgroup $\{(s^{-1}, s) | s \in S\}$ acts by a smooth character $\theta$ of $S$.

**Proposition B.3.** Assume that

1. The set $\{\sigma(n^{-1} \rtimes n) | n \in N_H/S\}$ forms a basis of $\text{End}_H(\sigma)$;
2. For each $g \in G$ the representation $\sigma|_{^g\times 1}$ is isomorphic to $\sigma|_H$ if $g \in N$ and disjoint from $\sigma|_H$ otherwise.

Then the set $\{\text{Ind}_{H \rtimes N}^G(\sigma)(n^{-1} \rtimes n) | n \in N/S\}$ forms a basis of $\text{End}_G(\text{Ind}_H^G \sigma)$.

**Proof.** The proof is just a matter of unwinding the definitions. For $n \in N$ let $\beta(n) \in \text{End}_G(\text{Ind}_H^G \sigma)$ denote $\text{Ind}_{H \rtimes N}^G(\sigma)(n^{-1} \rtimes n)$.

Choose a set of representatives $\dot{g}$ for the coset space $G/H$ such that the cosets in $N/N_H$ are represented by elements $\dot{g}$ of $N$. By Frobenius reciprocity and the Mackey theorem we have the isomorphism

$$\text{End}_G(\text{Ind}_H^G \sigma) \to \bigoplus_{g \in G/H} \text{Hom}_H(\sigma^{\dot{g}}, \sigma) = \bigoplus_{n \in N/N_H} \text{End}_H(\sigma),$$

where the equality is due to our second assumption.

For a representative $\dot{n} \in N$ and $h \in N_H$ this isomorphism translates $\beta(\dot{n} h)$ to the tuple of homomorphisms that has all coordinates trivial except for the coordinate corresponding to $\dot{n} h \in G/H$, where it is

$$\sigma(1 \times \dot{n}) \sigma(h^{-1} \times h).$$

The first assumption now implies that as $\dot{n} h$ runs over $N/S$ these elements form a basis. 

**C Representations of extensions with abelian quotient**

Let $G$ be a locally profinite group and $N \subset G$ an open (and hence closed) normal subgroup such that $A = G/N$ is finite and abelian. We will collect some basic facts about the relationship between the finite-dimensional smooth irreducible representations of $G$ and those of $N$, writing $\text{Irr}(G)$ and $\text{Irr}(N)$ for the respective sets of isomorphism classes. For $\rho \in \text{Irr}(N)$ we write $G_{\rho}$ and
\[ A_\rho = G_\rho / N \] for the stabilizers in \( G \) and \( A \) of the isomorphism class \( \rho \). For \( \pi \in \text{Irr}(G) \) we write
\[ m(\sigma, \pi) = \dim \text{Hom}_G(\pi, \text{Ind}_N^G(\sigma)) = \dim \text{Hom}_N(\sigma, \text{Res}_N^G(\pi)), \]
noting that \( \text{Ind}_N^G \sigma \) and \( \text{Res}_N^G \pi \) are semi-simple representations [BH06, Lemma 2.7]. Here \( \text{Ind}_N^G \sigma \) is defined as in the case of finite groups, and coincides with both smooth induction and compact induction due to the finiteness of \( A \), implying that the functor \( \text{Ind}_N^G \) is both left and right adjoint to \( \text{Res}_N^G \), see [BH06, §2].

**Lemma C.1.** Given \( \sigma, \sigma' \in \text{Irr}(N) \) the representations \( \text{Ind}_N^G \sigma \) and \( \text{Ind}_N^G \sigma' \) are either equal or disjoint. They are equal if and only if there exists \( g \in G \) s.t. \( \sigma' = \sigma \circ \text{Ad}(g) \).

**Proof.** If \( \sigma' = \sigma \circ \text{Ad}(g) \) then clearly \( \text{Ind}_N^G \sigma \) and \( \text{Ind}_N^G \sigma' \) have a common irreducible constituent \( \pi \). Then both \( \sigma \) and \( \sigma' \) are irreducible constituents of \( \text{Res}_N^G \pi \), hence conjugate under \( G \) by Lemma A.4. \( \square \)

We now want to describe, for a given \( \sigma \), the function \( \text{Irr}(G) \to \mathbb{Z}, \pi \mapsto m(\sigma, \pi) \), i.e. the decomposition of \( \text{Ind}_N^G \sigma \).

**Lemma C.2.** Let \( \sigma \in \text{Irr}(N) \) and let \( N \subset H \subset G_\sigma \) be a subgroup to which \( \sigma \) extends. Then \( H \) is maximal with this property if and only if for one, hence any, extension \( \sigma_H \) of \( \sigma \) to \( H \) we have \( G_{\sigma_H} = H \).

**Proof.** We first claim that \( G_{\sigma_H} \) depends only on \( H \) and \( \sigma \), but not on \( \sigma_H \). For this, let \( \sigma_H \) and \( \sigma'_H \) be two extensions of \( \sigma \) to \( H \). By Lemma A.4 there exists a character \( \chi_H \in (H/N)^* \) s.t. \( \sigma'_H = \sigma_H \otimes \chi_H \). Since \( A \) acts trivially on \( H/N \) by conjugation we have \( \chi_H(g h g^{-1}) = \chi_H(h) \) for all \( g \in G \) and \( h \in H \). Thus for \( f \in \text{End}_C(V_\rho), g \in G \), and \( h \in H \) the conditions \( f \circ \sigma'_H(g h g^{-1}) = \sigma'_H(h) \circ f \) and \( f \circ \sigma_H(g h g^{-1}) = \sigma_H(h) \circ f \) are equivalent, and the claim follows.

Assume \( G_{\sigma_H} = H \). Let \( H < H' \subset G_\sigma \) be such that there is an extension \( \sigma_{H'} \) of \( \sigma \) to \( H' \). Then \( \sigma_{H'} \mid_H \) is an extension of \( \sigma \) to \( H \) that is clearly stabilized by \( H' \), hence \( H' \subset G_{\sigma_H} \), by which our assumption implies \( H' = H \).

Assume conversely that \( H \) is a maximal subgroup to which \( \sigma \) extends. Choose an extension \( \sigma_H \) of \( \sigma \) to \( H \). Let \( g \in G \) be such that \( \sigma_H \circ \text{Ad}(g) \cong \sigma_H \). Let \( H' \subset G_{\sigma_H} \) be the group generated by \( H \) and \( g \). Then \( \sigma_H \) extends to a projective representation of \( H' \) whose associated cohomology class lies in \( H^2(H'/H, \mathbb{C}^\times) \). Since \( H'/H \) is finite and cyclic we have \( H^2(H'/H, \mathbb{C}^\times) = 0 \) and this projective representation linearizes, providing an extension of \( \sigma_H \) to \( H' \). The maximality of \( H \) implies \( H' = H \), i.e. \( g \in H \), and we conclude \( G_{\sigma_H} = H \). \( \square \)

**Lemma C.3.** Let \( \sigma \in \text{Irr}(N) \) and \( \pi \in \text{Irr}(G) \). Let \( N \subset H \subset G \) be a maximal subgroup to which \( \sigma \) extends. Then \( m(\sigma, \pi) > 0 \) if and only if there exists an extension \( \sigma_H \) of \( \sigma \) to \( H \) such that \( \pi = \text{Ind}_H^G \sigma_H \).

**Proof.** If \( \sigma_H \) is such an extension and \( \pi = \text{Ind}_H^G \sigma_H \) then Frobenius reciprocity implies \( m(\pi, \sigma_H) > 0 \), hence \( m(\pi, \sigma) > 0 \). Conversely assume \( m(\pi, \sigma) > 0 \). Pick arbitrarily an extension \( \sigma'_H \) and let \( \pi' = \text{Ind}_H^G \sigma'_H \). Then \( \pi' \) is semi-simple and we can check its irreducibility by computing the dimension of its space of
self-intertwiners, which is 1 according to [Kut77] and the fact that \( H \subset G_\sigma \). Moreover \( m(\pi', \sigma) > 0 \) as just argued. According to Lemma A.4 there \( \chi \in A^* \) s.t. \( \pi = \pi' \otimes \chi = \text{Ind}_H^G(\sigma' \otimes \chi|_H) \). Then \( \sigma_H = \sigma'_H \otimes \chi|_H \) satisfies the requirement. \( \square \)

Let \( \sigma \in \text{Irr}(N) \), let \( N \subset H \subset G \) be a maximal subgroup to which \( \sigma \) extends and let \( A' = H/N \). We will construct an injective group homomorphism \( A_\sigma/A' \rightarrow (A')^* \).

Consider \( \text{Ind}_H^G \sigma \). The set of its irreducible constituents is precisely the set of irreducible representations of \( H \) whose restriction to \( N \) contains \( \sigma \). Thus \( A_\sigma \) acts on this set and this action factors through \( A_\sigma/A' \). Fix an extension \( \sigma_H \) of \( \sigma \) to \( H \). By Lemma C.3 the stabilizer of \( \sigma_H \) in \( A_\sigma \) equals \( A' \). At the same time, we can apply Lemma A.4 and to conclude that the set of irreducible representations of \( H \) whose restriction to \( N \) contains \( \sigma \) is precisely \( \{ \sigma_H \otimes \chi | \chi \in (A')^* \} \) and since \( m(\sigma, \sigma_H) = 1 \) the map \( \chi \rightarrow \sigma_H \otimes \chi \) is injective. Thus, for a given \( a \in A_\sigma/A' \) there is a unique \( \chi_a \in (A')^* \) with \( \sigma_H \circ \text{Ad}(a)^{-1} = \sigma_H \otimes \chi_a \). Note that \( \chi_a \) does not depend on the choice of extension \( \sigma_H \) of \( \sigma \). This implies that \( a \rightarrow \chi_a \) is a homomorphism. It is injective because the action of \( A_\sigma/A' \) on the set \( \{ \sigma_H \otimes \chi \} \) is simple.

**Corollary C.4.** Let \( \sigma \in \text{Irr}(N) \) and \( \pi \in \text{Irr}(G) \) be s.t. \( m(\sigma, \pi) > 0 \). Let \( N \subset H \subset G \) be a maximal subgroup to which \( \sigma \) extends and let \( A' = H/N \). Then

1. \( m(\sigma, \pi) = [A' : A_\sigma] \).
2. \( \dim(\pi) = \dim(\sigma)m(\sigma, \pi)[A_\sigma : A] = \dim(\sigma)[A' : A] \).
3. We have the cartesian square

\[
\begin{array}{ccc}
\text{Stab}(\pi, A) & \longrightarrow & A^* \\
\downarrow & & \downarrow \\
A_\sigma/A' & \longrightarrow & (A')^*
\end{array}
\]

**Proof.** Lemma C.3 immediately gives \( \dim(\pi) = \dim(\sigma)[A' : A] \), while \( \dim(\pi) = \dim(\sigma)m(\sigma, \pi)[A_\sigma : A] \) is immediate from Lemma A.4.

To compute \( \text{Stab}(\pi, A) \), let \( \chi \in A^* \). Write \( \pi = \text{Ind}_H^G \sigma_H \) as in Lemma C.3. Then \( (\text{Ind}_H^G \sigma_H) \otimes \chi = \text{Ind}_H^G(\sigma_H \otimes \chi|_A') \) and by [Kut77] this is isomorphic to \( \text{Ind}_H^G \sigma_H \) if and only if there exists \( a \in A \) s.t. \( \sigma_H \circ \text{Ad}(a)^{-1} = \sigma_H \otimes \chi_a \). Restricting this relation from \( H \) to \( N \) we see \( a \in A_\sigma \), and this relation becomes equivalent to \( \chi|_A' = \chi_a \). \( \square \)

**Lemma C.5.** Assume that \( N \) is abelian and \( \sigma \) is a character. Then the set of \( \pi \in \text{Irr}(G) \) with \( m(\sigma, \pi) > 0 \) is in canonical bijection with the set of id-isotypic representations of the pushout of \( G_\sigma \) by \( \sigma \).

**Proof.** Write \( \text{Irr}(G, \sigma) \) for the subset of \( \pi \in \text{Irr}(G) \) with \( m(\sigma, \pi) > 0 \). By Lemma A.4 the \( \pi_\sigma \in \text{Irr}(G_\sigma, \sigma) \) are precisely those whose restriction to \( N \) is \( \sigma \)-isotypic. This implies that if \( g \in G \) is s.t. \( \pi_\sigma \cong \pi'_\sigma \circ \text{Ad}(g) \) then \( g \in G_\sigma \) and hence \( \pi_\sigma \cong \pi'_\sigma \). In particular \( G_{\pi_\sigma} = G_\sigma \). Since \( \pi = \text{Ind}_G^G \pi_\sigma \) is semi-simple, this implies via [Kut77] that it is irreducible, and moreover that \( \pi \cong \pi' \) implies
D  DL-varieties and homomorphisms with abelian kernel and cokernel

We review here the material [DL76, §1.21-§1.27]. Let $\tilde{G} \rightarrow G$ be a homomorphism of connected reductive groups defined over a finite field $k$, with abelian kernel and cokernel. Let $S \subset G$ be a maximal torus, $\tilde{S} \subset \tilde{G}$ its inverse image, $\theta : S^F \rightarrow \tilde{Q}_l^\times$ a character, and $\tilde{\theta} : \tilde{S}^F \rightarrow \tilde{Q}_l^\times$ its pullback. Let $U \subset \tilde{G}$ be the unipotent radical of a Borel subgroup containing $\tilde{S}$. Then $U \subset G$ is also the unipotent radical of a Borel subgroup containing $S$.

The following results are proved in loc. cit. when $\tilde{G}$ is the simply connected cover of the derived subgroup of $G$. The proof given there works without change for the more general $G$ considered here:

1. The natural map $\text{cok}((\tilde{S}^F \rightarrow S^F) \rightarrow \text{cok}(\tilde{G}^F \rightarrow G^F)$ is bijective.
2. The $(G \times Z S)^F$-space $Y^G_U$ is the induction of the $(\tilde{G} \times \tilde{S})^F$-space $Y^G_{\tilde{U}}$.
3. We have an isomorphism $H^1_i(Y^G_{\tilde{U}}, \tilde{Q}_l) = \text{Ind}^{S^F}_{\tilde{S}^F}(H^1_i(Y^G_{\tilde{U}}, \tilde{Q}_l))$ as modules for the action of $S^F$ on the right.
4. The natural map $Y^G_{\tilde{U}} \rightarrow Y^G_U$ induces an isomorphism $H^1_i(Y^G_{\tilde{U}}, \tilde{Q}_l) \rightarrow H^1_i(Y^G_U, \tilde{Q}_l)$.
5. For $\chi : \text{cok}(\tilde{G}^F \rightarrow G^F) \rightarrow \tilde{Q}_l^\times$ we have $H^1_i(Y_U, \tilde{Q}_l)_{\chi} = \chi \otimes H^1_i(Y_U, \tilde{Q}_l)_{\theta}$.

E  Remarks about embeddings of tori

Consider two reductive groups $G_1$ and $G_2$ and a rigid inner twist $(\xi, z) : G_1 \rightarrow G_2$, maximal tori $S_1 \subset G_1$, and $g \in G(F)$ such that $\xi \circ \text{Ad}(g) : S_1 \rightarrow S_2$ is defined over $F$. Then $\xi \circ \text{Ad}(g)$ induces an isomorphism $\Omega(S_1, G_1) \rightarrow \Omega(S_2, G_2)$ defined over $F$. We have the action of $\Omega(S_1, G_1)(F)$ on $H^1(F, S_1)$ induced by the action on $S_1$. Write $\delta : \Omega(S_1, G_1)(F) \rightarrow H^1(F, S_1)$ for the connecting homomorphism. Write $\eta_g$ for the class in $H^1(w \rightarrow W, Z \rightarrow S_1)$ of the 1-cocycle $w \mapsto g^{-1}z(w)\sigma_w(g)$. Given $w \in \Omega(S_1, G_1)(F)$ the class $\eta_{gw}$ is independent of the choice of lift $w \in N(S_1, G_1)(F)$ of $w$ and will be denoted by $\eta_{gw}$. The proof of the following lemma is elementary and left to the reader.

Lemma E.1. Let $w \in \Omega(S_1, G_1)(F)$. Then

1. $\eta_{gw} = w^{-1}\eta_g \cdot \delta w$;
2. The image of $w$ in $\Omega(S_2, G_2)(F)$ belongs to the subgroup $\text{N}(S_2, G_2)(F)/S_2(F)$ if and only if $\eta_{gw} = \eta_g$;
3. Assume that $\text{N}(S_1, G_1)(F)/S_1(F) = \Omega(S_1, G_1)(F)$. Then $\xi \circ \text{Ad}(g)$ identifies the stabilizer of $\eta_g$ in $\Omega(S_1, G_1)(F)$ with the subgroup $\text{N}(S_2, G_2)(F)/S_2(F)$ of $\Omega(S_2, G_2)(F)$.
F Parahoric subgroups and restriction of scalars

Let \( E/F \) be a finite extension of non-archimedean local fields, not necessarily tamely ramified. Let \( H \) be a connected reductive group over \( E \) and \( G = \text{Res}_{E/F} H \). There is a natural identification \( B(G, F) = B(H, E) \). Let \( x \) be a point in this building, and \( \mathfrak{g}_x^o \) and \( \mathfrak{y}_x^o \) the corresponding (connected) parahoric group schemes defined over \( O_F \) and \( O_E \), respectively.

**Fact F.1.** The identity \( G = \text{Res}_{E/F} H \) extends to an isomorphism \( \mathfrak{g}_x^o = \text{Res}_{O_E/O_F} \mathfrak{y}_x^o \).

**Proof.** The \( O_F \)-group scheme \( \mathfrak{g}_x^o \) is affine according to [BT84, §4.6.2]. Since Weil restriction preserves affineness, so is \( \text{Res}_{O_E/O_F} \mathfrak{y}_x^o \). It will thus be enough to show that the \( F \)-algebra isomorphism between the coordinate rings of \( G \) and \( \text{Res}_{E/F} H \) maps the coordinate ring of \( \mathfrak{g}_x^o \) bijectively onto the coordinate ring of \( \text{Res}_{O_E/O_F} \mathfrak{y}_x^o \). Since \( \mathfrak{g}_x^o \) is smooth by loc. cit. and then so is \( \text{Res}_{O_E/O_F} \mathfrak{y}_x^o \) by [CGP15, Proposition A.5.2]. We may thus apply [BT84, Proposition 1.7.6] to compute the coordinate rings and see that it is enough to show that under the identification \( G(F^u) = \text{Res}_{E/F} H(F^u) \) the subgroups \( \mathfrak{g}_x^o(O_{F^u}) \) and \( \text{Res}_{O_E/O_F} \mathfrak{y}_x^o(O_{F^u}) \) become identified.

Let \( F' \subset E \) be the maximal unramified subextension of \( F \). We have the compatible isomorphisms \( O_E \otimes_{O_F} O_{F^u} \to O_{E^u}^{[F: E]} \) and \( E \otimes_{F^u} F' \to (E^u)^{[F': F]} \), giving rise to the compatible isomorphisms \( \text{Res}_{O_E/O_F} \mathfrak{y}_x^o(O_{F^u}) \to \mathfrak{y}_x^o(O_{E^u})^{[F: E]} \) and \( \text{Res}_{E/F} H(F^u) \to H(E^u)^{[F': F]} \), which show that \( \text{Res}_{O_E/O_F} \mathfrak{y}_x^o(O_{F^u}) \) is the parahoric subgroup of \( \text{Res}_{E/F} H(F^u) \) corresponding to the point \( x \). Under the equality \( G = \text{Res}_{E/F} H \) this group is identified with \( \mathfrak{g}_x^o(O_{F^u}) \).

Recall that we denote by \( G_x^o \) and \( H_x^o \) the reductive quotients of the special fibers of \( \mathfrak{g}_x^o \) and \( \mathfrak{y}_x^o \). The proof of the following Lemma was communicated to us by Brian Conrad.

**Lemma F.2.** The isomorphism \( \mathfrak{g}_x^o = \text{Res}_{O_E/O_F} \mathfrak{y}_x^o \) induces an isomorphism \( G_x^o = \text{Res}_{k_E/k_F} H_x^o \).

**Proof.** We apply base change to \( k_F \) and use that Weil restriction of scalars commutes with base change to reduce to showing that the reductive quotient of \( \text{Res}_{A/k} \mathfrak{y}_x^o = \text{Res}_{k_E/k_F} H_x^o \), where \( A = O_E \otimes_{O_F} k_F \) and we are now reusing the symbol \( \mathfrak{y}_x^o \) to denote the base-change of the original \( \mathfrak{y}_x^o \) to \( A \). Note that \( \mathfrak{y}_x^o \) is still smooth connected affine and the reductive quotient of its special fiber is still \( H_x^o \).

Let \( \bar{\mathfrak{y}}_x^o \) denote the special fiber of \( \mathfrak{y}_x^o \). Reduction modulo the maximal ideal of \( A \) gives a surjective morphism \( \text{Res}_{A/k} \bar{\mathfrak{y}}_x^o \to \text{Res}_{k_E/k_F} \bar{\mathfrak{y}}_x^o \) of \( k_F \)-groups with connected unipotent kernel (apply [CGP15, Proposition A.5.12] to successive powers of the maximal ideal of \( A \)).

The projection \( \bar{\mathfrak{y}}_x^o \to H_x^o \) is a smooth surjective morphism of \( k_E \)-groups with connected unipotent kernel \( U \). Applying \( \text{Res}_{k_E/k_F} \) to it gives a surjective morphism \( \text{Res}_{k_E/k_F} \bar{\mathfrak{y}}_x^o \to \text{Res}_{k_E/k_F} H_x^o \) of \( k_F \)-groups with kernel given by the smooth affine \( k_F \)-group \( \text{Res}_{k_E/k_F}(U) \), see [CGP15, Proposition A.5.2(4) and Proposition A.5.14(3)]. This group is moreover connected and unipotent, for it is enough to check this over \( k_E \), where it becomes \( U[k_E:k_F] \).
Let $G$ be a connected reductive group defined over $F$.

**Definition G.1.** A point $x \in B(G, F)$ is called absolutely special if it is special in $B(G, E)$ for every finite Galois extension $E/F$.

Assume for a moment that $G$ is quasi-split. We recall some material due to Bruhat-Tits. A $\Gamma$-invariant pinning of $G$ provides a point in $B(G, F)$ – the Chevalley valuation corresponding to the pinning [BT84, 4.2.3]. Fix such a point $o \in B(G, F)$. For each $a \in R(A_T, G)_{\text{res}}$ we have the sets $\Gamma_a$ and $\Gamma'_a$, defined in [BT84, 4.2.20]. In the special case at hand where the valuation is discrete with image $\mathbb{Z}$ and the residual characteristic is not 2, they are given as follows. Let $a \in R(A_T, G)_{\text{res}}$ be a non-divisible root and let $\alpha \in R(T, G)$ be a lift. If $a$ is non-multiplicable then $\Gamma_a = \Gamma'_a = e_{\alpha}^{-1}\mathbb{Z}$. If $a$ is multiplicable then $\Gamma_a = \frac{1}{2}e_{\alpha}^{-1}\mathbb{Z}$, $\Gamma'_a = e_{\alpha}^{-1}\mathbb{Z}$, $\Gamma_{2a} = \Gamma'_{2a} = 2e_{\alpha}^{-1}(\mathbb{Z} + \frac{1}{2})$. Here $e_{\alpha}$ is the ramification index of the extension $F_o/F$. Note that the second case occurs only if $\alpha$ belongs to a component of type $A_{2n}$ and a power of the action of tame inertia preserves this component and maps $\alpha$ to $\alpha'$ s.t. $\beta = \alpha + \alpha'$ is also a root, in which case $2a$ is the image of $\beta$ and $e_{\beta} = \frac{1}{2}e_{\alpha}$. A point $x \in A(T, F)$ is called special if $\langle \alpha, x - o \rangle \in \Gamma_a$ for all non-divisible $a \in R(A_T, G)_{\text{res}}$ [BT84, 4.6.15], [BT72, 6.2.13]. It suffices to check this condition for the simple roots $a$ [BT72, 6.2.14] corresponding to some choice of positive roots.

It is thus clear that the Chevalley valuation $o$ is special, and in fact absolutely special. The next lemma shows that the absolutely special points are precisely the Chevalley valuations.

**Lemma G.2.** The following are equivalent

1. The point $x \in A(T, F)$ is absolutely special.
2. $\langle \alpha, x - o \rangle \in \Gamma'_a$ for all simple $a \in R(A_T, G)$ relative to a Borel subgroup $T \subset B$ defined over $F$.
3. $x = t \cdot o$ for some $t \in T_{\text{ad}}(F)$.
4. $x$ is a Chevalley valuation.

**Proof.** 1 $\Rightarrow$ 2: The point $x$ remains special in $B(G, F_o)$, thus $\langle \alpha, x - o \rangle \in \Gamma_a^{F_o}$, where $\Gamma_a^{F_o}$ is the set $\Gamma_a$ introduced above, but relative to the base field $F_o$ and the valuation on $F_o$ that extends the valuation on $F$. For the valuation on $F_o$ with image $\mathbb{Z}$ we would have $\Gamma_a^{F_o} = \mathbb{Z}$, because $o$ is now non-multiplicable, but upon rescaling the valuation so that its image is $e_{\alpha}^{-1}\mathbb{Z}$ the set $\Gamma_a^{F_o}$ also rescales and becomes $e_{\alpha}^{-1}\mathbb{Z}$ and thus equal to $\Gamma'_a$.

2 $\Rightarrow$ 3: The set of absolute simple roots $\Delta \subset R(T, G)$ provides an isomorphism

$$T_{\text{ad}} \rightarrow \prod_{\alpha \in \Delta / \Gamma} \text{Res}_{F_o / F} G_m, \quad t \mapsto (\alpha(t))_o.$$ Choose $t_o \in F_o^x$ with $\text{val}(t_o) = \langle \alpha, x - o \rangle$ and $t_{\sigma(\alpha)} = \sigma(t_o)$ for all $\sigma \in \Gamma$. The collection $(t_o)$ is an $F$-point of the right-hand side and determines $t \in T_{\text{ad}}(F)$ with $\alpha(t) = \langle a, x - o \rangle$ for all $\alpha \in \Delta$ with image $a \in R(A_T, G)$. Then $x = t \cdot o$.

3 $\Rightarrow$ 4: Immediate.

4 $\Rightarrow$ 1: Immediate.

□
Lemma G.3. If $B(G, F)$ has an absolutely special point, then $G$ is quasi-split.

Proof. Replacing $G$ by its adjoint group effects neither the assumption nor the conclusion of the lemma, so we assume that $G$ is adjoint. Let $T \subset G$ be a maximally unramified torus defined over $F$ such that its split subtorus $A_T$ is a maximal split torus. Such $T$ exists due to [BT84, Corollary 5.1.12]. Let $T'/T$ be the maximal unramified subtorus and let $F'/F$ be the splitting extension of $T'$, a finite unramified extension. The apartment $\mathcal{A}(A_T, F) \subset B(G, F)$ is equal to the Frobenius-fixed points of the apartment $\mathcal{A}(T', F')$ of $B(G, F')$. All apartments of $B(G, F)$ are of this form. Therefore we may assume that $x \in \mathcal{A}(A_T, F)$.

Now $T$ is a minimal Levi subgroup of the quasi-split group $G \times F''$. Since $T = \text{Cent}(T', G)$ and $T' \times F'$ is split, we see that $T$ is a minimal Levi subgroup of $G \times F'$. Thus $G \times F'$ is quasi-split.

Since $x$ is an absolutely special point of $B(G, F')$, Lemma G.2 applied to $G \times F'$ shows that $x$ is a Chevalley valuation. Thus there exists an $F'$-pinning $(T, B, \{X_a\})$ of $G \times F'$ giving rise to $x$. Let $\sigma$ denote Frobenius. There exists a unique $g \in G(F')$ such that $\text{Ad}(g) \sigma(T, B, \{X_a\}) = (T, B, \{X_a\})$. Since $x$ is $\sigma$-fixed, both pinnings $\sigma(T, B, \{X_a\})$ and $(T, B, \{X_a\})$ induce the same Chevalley point $x$. This implies $g \in G(F')_{x,0}$. Since $x$ is special, the stabilizer $G(F')_{x,0}$ equals the parahoric $G(F')_{x,0}$: By [BT84, Proposition 4.6.28] we have $G(F')_{x,0} = N(T, G)(F')_x \cdot G(F')_{x,0}$; since $x$ is special this product equals $T(F')_{x,0} \cdot G(F')_{x,0}$; since $G$ is adjoint we have $T(F')_h = T(F')_0 \subset G(F')_{x,0}$. The triviality of $H^1((\sigma), G(F')_{x,0})$ implies the existence of $h \in G(F')_{x,0}$ such that $h^{-1} \sigma(h) = g$. Then $h(T, B, \{X_a\})$ is another $F'$-pinning of $G \times F'$ giving rise to $x$, which is now moreover fixed by $\sigma$, and hence is an $F'$-pinning of $G$. \hfill $\square$

Corollary G.4. The simple roots in $R(A_T, G)_{\text{res}}$ are contained (and thus give a set of simple roots) in $R(A_T^0, G^\circ)$ if and only if $x$ is absolutely special.

Proof. This follows immediately from the description [BT84, 4.6.12+4.6.23] of $R(A_T^0, G^\circ)$ as the subset of $R(A_T, G)_{\text{res}}$ consisting of those $a$ for which $(a, x - o) \in \Gamma'_o$. \hfill $\square$

Remark G.5. In [Kal19] we introduced the notion of a superspecial point. We recall that $x \in B(G, F)$ is called superspecial if it is special in $B(G, F')$ for all finite unramified extensions $F'/F$. When $G$ is unramified, then the notions of absolutely special, superspecial, and hyperspecial, all agree. When $G$ is ramified, hyperspecial vertices do not exist, and the notions of absolutely special and superspecial are a replacement. Clearly an absolutely special point is superspecial. The converse however is false, as the following example shows.

Example G.6. Consider a ramified unitary group in 3 variables. Let $T$ be a maximally split maximal torus and $A \subset T$ the maximal split subtorus. Then $X^*(A) = \mathbb{Z}$. We have $X^*(T) = \mathbb{Z}^3$ with inertia acting by $(a, b, c) \mapsto (-c, -b, -a)$. Then $X^*(A)$ is the torsion-free quotient of the coinvariants of this action, so we have the isomorphism $X^*(A) \rightarrow \mathbb{Z}$ given by $(a, b, c) \mapsto a - c$. Let $e \in X^*(A)$ be the preimage of $1 \in \mathbb{Z}$. The relative root system is of type $BC_1$ and is given by $\{e, 2e, -e_1, -2e_3\} \subset X^*(A)$. The root $e$ is the restriction of both $e_1 - e_2$ and $e_2 - e_3$ in $\mathbb{Z}^3 = X^*(T)$. The root $2e$ is the restriction of $e_1 - e_3$.

We have $\Gamma_e = \frac{1}{4} \mathbb{Z}$, $\Gamma'_e = \frac{1}{2} \mathbb{Z}$, and $\Gamma_{2e} = \frac{1}{2} + \mathbb{Z}$. Let $o \in \mathcal{A}(T, F)$ be the absolutely special point given by an $F$-pinning. Identify $\mathcal{A}(T, F)$ with $X_*(A) \otimes \mathbb{Q}$. Then $x \in \mathcal{A}(T, F)$ is an absolutely special point giving rise to $e$. \hfill $\square$
Consider a special vertex $x$. We may assume without loss of generality that $x$ is a vertex, because it is a wall of an affine root with gradient $2e$. It is also special. However, it is not absolutely special. Indeed, the splitting field $E/F$ of the unitary group is a ramified quadratic extension and for all absolute roots $\alpha$ we have $\Gamma_\alpha = \Gamma_0 = \frac{1}{2}\mathbb{Z}$ with respect to the normalized valuation on $F$, so $\langle e_1 - e_2, x \rangle = \frac{1}{2} \not\in \Gamma_{e_1 - e_2}$.

Slightly more generally one can consider a ramified unitary group in $2n + 1$ variables. There are two special vertices, with connected reductive quotients $\text{SO}(2n + 1)$ and $\text{Sp}(2n)$, respectively. The first is absolutely special, while the second is superspecial but not absolutely special. There are also other vertices, which are non-special, and have connected reductive quotient $\text{SO}(2a + 1) \times \text{Sp}(2b)$ with $a + b = n$.

It turns out that the odd ramified unitary groups provide the only examples of superspecial vertices that are not absolutely special, as the following argument due to Gopal Prasad shows.

**Proposition G.7** (Gopal Prasad). Let $G$ be an absolutely almost simple group defined over $F$ that does not split over $F^u$. If $G$ is not of type $A_{2n}$ every superspecial vertex is absolutely special.

**Proof.** Write $K = F^u$ and consider the base change of $G$ to $K$. It is a quasi-split group and we let $L/K$ be the splitting extension of $G$. Write $H$ for $G \times L$. Consider a special vertex $x \in B(G, K)$ and let $G^s_x$ and $H^s_x$ be the semi-simple quotients of the special fibers of the parahoric groups schemes of $G$ and $H$ at $x$, respectively. These are connected reductive groups defined over the algebraic closure of a finite field and we have a natural embedding $G^s_x \to H^s_x$. Assume that $x$ is not a special vertex in $B(G, L)$. This, it is either a non-special vertex or is contained in a facet of positive dimension. We will show that if $G$ is not of type $A_{2n}$ then dimension considerations rule out the existence of such an embedding $G^s_x \to H^s_x$.

There are four possible cases to consider:

1. $G$ is of type $E_6^{(2)}$. Then $G^s_x$ is of type $F_4$, while $H^s_x$ is of type $D_5$ or $A_5$.

2. $G$ is of type $A_{2n+1}^{(2)}$. Then $G^s_x$ is of type $C_{n+1}$ while $H^s_x$ is either of type $A_{2n}$ or a product of groups of type $A_r$ and $A_{2n+1-r}$.

3. $G$ is of type $D_n^{(2)}$. Then $G^s_x$ is of type $B_{n-1}$, while $H^s_x$ is either of type $A_{n-1}$, or of type $D_{n-1}$, or a product of two groups of type $D_r$ and $D_{n-r}$.

4. $G$ is of type $D_4^{(3)}$ or $D_4^{(6)}$. In both of these cases $G^s_x$ is of type $G_2$, while $H^s_x$ is either product of four copies of a group of type $A_1$ or a group of type $A_3$.

\[ \square \]

**Corollary G.8.** Let $G$ be a connected reductive group over $F$. If $G$ has a superspecial vertex, then it is quasi-split.

**Proof.** We may assume without loss of generality that $G$ is adjoint. Then it is a product of $F$-simple factors and we may consider an individual such factor.
It is of the form $\text{Res}_{E/F} H$ for an absolutely simple adjoining group $H$ defined over a finite extension $E/F$. We have $B(G, F) = B(H, E)$ and $G$ is quasi-split over $F$ if and only if $H$ is quasi-split over $E$. Thus we may assume that $G$ is an absolutely simple group.

If $G$ is of type $A_{2n}^{(2)}$ over $F^u$, then it is automatically quasi-split over $F$. If $G$ splits over an unramified extension, then the vertex is hyperspecial, so $G$ is quasi-split. If $G$ does not split over an unramified extension, then Proposition G.7 shows that the vertex is absolutely special and the result follows from Lemma G.3. □

H Generic depth-zero supercuspidal representations

In this section we generalize [DR09, §6.1] to the case of ramified groups.

Let $G$ be a connected reductive group defined over $F$. Let $B = TU \subset G$ be a Borel subgroup defined over $F$ and let $\psi : U(F) \to \mathbb{C}^\times$ be a generic character. The $G(F)$-conjugacy class of the pair $(B, \psi)$ is called a Whittaker datum, and we shall denote it by $w$.

Let $x \in A(G, F)$. The image of $B \cap G(F^u)_{x,0}$ in $G_2^F(\bar{k}_F)$ is a Borel subgroup defined over $k_F$, call it $B$. Its unipotent radical $U$ is the image of $U \cap G(F^u)_{x,0}$. We say that $\psi$ has depth zero at $x$ if it is trivial on $U \cap G(F)_{x,0+}$ and the character $\psi_x$ of $U(k_F)$ it induces is generic.

**Lemma H.1.** Let $x \in B(G, F)$ be a vertex.

1. If $\psi$ has depth zero at $x$, then $x$ is absolutely special.
2. If $\psi$ has depth zero at $x$ and $y$, then $x = y$.
3. If $x$ is absolutely special and $\psi_x$ is a generic character of $U(k_F)$, then it is the restriction of some generic $\psi : U(F) \to \mathbb{C}^\times$ that has depth zero at $x$.

**Proof.** If $x$ is not absolutely special, above Corollary implies that there exists a simple root of $R(A_T, G)_{\text{res}}$ that is not contained in $R(A_T^\circ, G_2^\circ)$. Since $x$ is a vertex, the root system $R(A_T^\circ, G_2^\circ)$ has the same rank as the root system $R(A_T, G)_{\text{res}}$, so there exists a non-simple root of $R(A_T, G)$ that is simple in $R(A_T^\circ, G_2^\circ)$. The character $\psi$ is trivial on the corresponding root subgroup, and thus its restriction to $U(k_F)$ is not generic.

If $\psi$ has depth zero at $x$ and $y$, then both $x$ and $y$ are absolutely special, so there exists $t \in T_{\text{ad}}(F)$ with $y = tx$. For $a \in \Delta(A_T, G)$ with lift $\alpha \in \Delta(T, G)$ the images of $U_{1, y}(O_F)$ and $U_{1, x}(O_F)$ in the 1-dimensional $F_a$-vector space $U_a(F)/[U_a, U_a](F)$ are two $O_{F_a}$-lattices, the second being obtained from the first by multiplication by $a(t) \in F^\times_a$. If $\psi$ has depth zero at $x$ and $y$, then these lattices must agree, i.e. $a(t) \in O_{F_a}^\times$. This holds for all $a \in \Delta(A_T, G)$, thus $t \in T_{\text{ad}}(O_F)$, and hence $y = x$.

Assume now that $x$ is absolutely special and $\psi_x$ is a generic character of $U(k_F)$. The inclusions $U_a \to U$ combine to an isomorphism

$$\prod_{a \in \Delta(A_T, G)} U_a/[U_a, U_a] \to U/[U, U]$$
of \( F \)-groups, and the same is true over \( k_F \). Note that \([U_a, U_a] = U_{2a}\). For each \( a \in \Delta(A_T, G)_{\text{res}} \), the reduction map \( U_{a,x}(O_F) \to U_a(k_F) \) is surjective and its kernel contains \( U_{2a,x}(O_F) = U_{a,x}(O_F) \cap U_{2a}(F) \). It follows that the character \( \psi_x \) induces a character of \( \prod_{a \in \Delta(A_T, G)} U_{a,x}(O_F)/(U_{a,x}(O_F) \cap [U_a, U_a](F)) \) that is non-trivial on each factor. Since \( U_a(F)/[U_a, U_a](F) \) is locally compact abelian, and this character has finite order, it can be extended to \( \prod_a U_a(F)/[U_a, U_a](F) \) by Pontryagin duality.

Let \( \pi \) be a depth zero supercuspidal representation of \( G(F) \). According to [MP96, Proposition 6.8] there exists a vertex \( z \in B(G, F) \), an irreducible representation \( \rho \) of \( G(F)_x \), and a cuspidal irreducible representation \( \sigma \) of \( G^*_2(k_F) \) s.t. \( \pi = c\text{-Ind}_{G(F)}^{H(\pi)} \rho \) and \( \sigma \subset \rho|_{G(F)_x,a} \).

**Proposition H.2.** The following are equivalent

1. \( \pi \) is \( w \)-generic.
2. \( x \) is absolutely special, there exists \( t \in T(F) \) s.t. \( \psi' = \psi \circ \text{Ad}(t) \) has depth zero at \( x \), and \( \sigma \) is \( \psi'_x \)-generic.

**Proof.** By [Kut77], \( \text{Hom}_{U(F)}(\pi, \psi) \) is the product of \( \text{Hom}_{G(F)_x \cap gUg^{-1}(F)}(\rho, \psi^g) \) as \( g \) runs over \( G(F)_x \setminus G(F)/U(F) \). According to the Iwasawa decomposition this double coset space is equal to \( N(F)_x \setminus N(F) \), where \( N = N(T, G) \). The natural map \( G_{sc} \to G \) restricts to an isomorphism between the preimage of \( U \) in \( G_{sc} \) and \( U \), and this implies \( G(F)_x \cap U(F) = G(F)_{x,0} \cap U(F) \), and the same for \( U \) replaced by \( nUgn^{-1}, n \in N(F) \). The irreducible representations in the restriction \( \rho|_{G(F)_{x,0}} \) are the \( G(F)_x \)-conjugates of \( \sigma \), so we are looking at the product of \( \text{Hom}_{G(F)_{x,0} \cap gUg^{-1}(F)}(\sigma, \psi^g) \) as \( g \) runs over \( G(F)_{x,0} \setminus G(F)/U(F) = N(F)_{x,0} \setminus N(F) \).

If \( \pi \) is \( w \)-generic, then there exists \( g \in N(F) \) for which the corresponding factor is non-trivial. In particular, the restriction of \( \psi^g \) to \( G(F)_{x,0} \cap gUg^{-1}(F) \) is trivial, and thus \( \psi^g \) induces a character of \( gUg^{-1}(k_F) \). If \( x \) is not absolutely special, this character is not generic, and this contradicts the cuspidality of \( \sigma \).

Conversely, if \( x \) is absolutely special and \( \sigma \) is \( \psi'_x \) is generic, then the factor corresponding to \( t \in T(F) \) is non-zero. \( \square \)

### I A study of \( D_{2n} \)

We will consider two parallel situations involving the split Spin group in \( 4n \) variables. The first situation is the following:

Let \( \tilde{G}_{ad} \) be a complex semi-simple group of adjoint type \( D_{2n} \) and let \( \hat{G}_{sc} \) be its simply connected cover. Let \( s \in \hat{G}_{ad} \) be a regular semi-simple element whose centralizer \( \hat{G}_s^{ad} \) has component group \((\mathbb{Z}/2\mathbb{Z})^2\). Write \( \tilde{T}_{ad} \subset \hat{G}_{ad} \) for the connected centralizer of \( s \), a maximal torus. The centralizer \( \hat{G}_s^{ad} \) is then equal to \( \hat{N}_s^{ad} \) – the group of \( \text{Ad}(s) \)-fixed points in the normalizer of \( \tilde{T}_{ad} \). We have the exact sequence

\[
1 \to \tilde{T}_{ad} \to \hat{N}_s^{ad} \to (\mathbb{Z}/2\mathbb{Z})^2 \to 0.
\]
Let $\hat{T}_{sc}$ be the preimage of $\hat{T}_{ad}$ in $\hat{G}_{sc}$, a maximal torus, and let $\hat{N}_{sc}^+$ be the preimage of $\hat{N}_{ad}^+$. We thus have the extension

$$1 \to \hat{T}_{sc} \to \hat{N}_{sc}^+ \to (\mathbb{Z}/2\mathbb{Z})^2 \to 0.$$  

Let $f \in \hat{N}_{ad}$ be s.t. its image in $\hat{N}_{ad}/\hat{T}_{ad}$ commutes with every element of $\hat{N}_{ad}^+/\hat{T}_{ad} = (\mathbb{Z}/2\mathbb{Z})^2$ and has no fixed points in $X_*(\hat{T}_{ad})$. Then $\hat{T}_{ad}$ is finite. We write $(-)^f$ again for the groups of fixed points of $\text{Ad}(f)$ in $\hat{G}_{sc}$ as well as $\hat{G}_{ad}$.

**Lemma I.1.** The natural map $\hat{N}_{sc}^+/f \to (\mathbb{Z}/2\mathbb{Z})^2$ is surjective. The extension

$$1 \to \hat{T}_{sc}^f \to \hat{N}_{sc}^+/f \to (\mathbb{Z}/2\mathbb{Z})^2 \to 0$$

has trivial commutator.

The second situation is the following. Let $k$ be a finite field of characteristic different from 2 and let $G_{sc}$ be a simply connected group of type $D_{2n}$, defined over $k$. Let $S_{sc} \subset G_{sc}$ be an anisotropic torus and $\theta : S_{sc}(k) \to \mathbb{C}^\times$ a non-singular character whose stabilizer in $\Omega(S_{sc}, G_{sc})(k)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$.

**Lemma I.2.** The extension $1 \to S_{sc}(k) \to N(S_{sc}, G_{sc})(k) \to \Omega(S_{sc}, G_{sc})(k) \to 1$ has trivial commutator.

**Proof of Lemmas I.1 and I.2.** Let $\hat{x} \in \hat{N}_{sc}^+$ be a lift of some $x \in (\mathbb{Z}/2\mathbb{Z})^2$. Then $\hat{x}^{-1} \cdot \text{Ad}(f)(\hat{x})$ lies in $\hat{T}_{sc}$. The map $y \mapsto y^{-1} \cdot \text{Ad}(f)(y)$ is an endomorphism of $\hat{T}_{sc}$ with finite kernel, hence surjective. This allows us to modify the lift $\hat{x}$ by some $y \in \hat{T}_{sc}$ to achieve $\hat{x} \in \hat{N}_{sc}^+/f$. This proves the surjectivity claim of Lemma I.1. The corresponding implicit surjectivity claim in Lemma I.2 is immediate from Lang’s theorem.

Consider $R^+ = \{e_i - e_j | 1 \leq i < j \leq 2n\} \cup \{e_i + e_j | 1 \leq i < j \leq 2n\} \subset \mathbb{Z}^{2n}$, this is the standard presentation of the system of positive roots for type $D_{2n}$. Let $Q \subset \mathbb{Z}^{2n} \subset P$ be the root and weight lattices, respectively. Thus $Q$ is the span of $R = R^+ \cup -R^+$, or equivalently the sublattice of $\mathbb{Z}^{2n}$ consisting of vectors whose sum of coordinates is divisible by 2, while $P = \mathbb{Z}^{2n} + \frac{1}{2} \sum_{i=1}^{2n} e_i$.

The standard inner product on $\mathbb{R}^{2n}$ identifies each root $\alpha \in R$ with its coroot $\alpha^\vee \in R^\vee$, and in particular the root lattice $Q$ with the coroot lattice $Q^\vee$ and the weight lattice $P$ with the coweight lattice $P^\vee$.

Let $\hat{G} = \text{SO}_{2n}(\mathbb{C})$, so that we have the isogenies $\hat{G}_{sc} \to \hat{G} \to \hat{G}_{ad}$. We obtain $\hat{T}_{sc} = Q \otimes \mathbb{C}^\times$, $\hat{T} = \mathbb{Z}^{2n} \otimes \mathbb{C}^\times$, and $\hat{T}_{ad} = P \otimes \mathbb{C}^\times$. We use the exponential sequence $0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^\times \to 1$ and the isomorphisms $Q \otimes \mathbb{C} \to \mathbb{Z}^{2n} \otimes \mathbb{C} \to P \otimes \mathbb{C}$ to identify $\hat{T}_{sc} = C^{2n} / Q$ and $\hat{T}_{ad} = C^{2n} / P$. Of course, the isomorphism $X_*(\hat{T}) \cong \mathbb{Z}^{2n}$ used here involves a choice that in particular implies a choice of a positive Weyl chamber. We shall specify this choice further below.

We can do the same over the finite field $k$. For this, we fix arbitrarily an isomorphism of groups $k^\times \to (\mathbb{Q}/\mathbb{Z})_p'$ that will serve as a replacement of the exponential map. The action of Frobenius on $k^\times$ is translated by $q$ on $(\mathbb{Q}/\mathbb{Z})_p'$. Define again $\hat{G} = \text{SO}_{2n} / k$ as above, so that $\hat{G}_{sc} \to \hat{G} \to \hat{G}_{ad}$ are isogenies. Then we obtain $S_{sc}(\hat{k}) \cong \mathbb{Z}^{2n} \otimes k^\times \cong (\mathbb{Q}^{2n} / \mathbb{Q})_p'$.

Consider the group of signed permutations $\{\pm 1\}^{2n} \rtimes S_{2n}$ acting on $\mathbb{Z}^{2n}$. It preserves the root system $R$ and is the full group of automorphisms of $R$ unless...
There is a choice of isomorphism $X_*(\hat{T}) \cong \mathbb{Z}^{2n}$ so that the quotient $\hat{N}_{ad}/\hat{T}_{ad} \cong (\mathbb{Z}/\mathbb{Z})^2$ is the subgroup of $\Omega = N/T$ generated by the elements $w_1 := e_1e_{2n}$ and $w_2 := (-1) \cdot m$, where $e_i$ sends $e_i$ to $-e_i$ and fixes $e_j$ for $j \neq i$, $(-1)$ is multiplication by $-1$ on $\mathbb{Z}^{2n}$, and $m(e_i) = e_{2n+1-i}$. The image of $f$ in $\Omega$ is an elliptic element $w_0 \in \Omega$ that commutes with both $w_1$ and $w_2$. Such an element can be brought, by conjugation by elements commuting with $w_1$ and $w_2$, into the following form: $w_0 = w_0' \cdot mw_0'm^{-1}$, where $w_0'$ is a signed permutation of $\{1, \ldots, n\}$, acting on $\mathbb{Z}^{2n} = \mathbb{Z}^n \oplus \mathbb{Z}^n$ by the natural action on the first factor and the identity on the second factor, and given by the product of consecutive increasing negative cycles, the first of them having length 1. More precisely, there is a sequence of integers $1 = i_1 < 2 = i_2 < i_3 < \cdots < i_k < i_{k+1} = n + 1$ such that $w_0'(e_{i_{a+1}-1}) = -e_{i_a}$ and $w_0'(e_j) = e_{j+1}$ for $j + 1 \notin \{i_2, \ldots, i_{k+1}\}$. Thus we may adjust our choice of isomorphism $X_*(\hat{T}) \cong \mathbb{Z}^{2n}$ to ensure that $w_0$ is of this form.

The element $f$ is a lift of $w_0$ to $\hat{N}_{ad}$. We may further lift to $\hat{N}_{sc}$. Since $\hat{T}_{sc}^{w_0}$ is finite, all lifts of $w_0$ are conjugate under $\hat{T}_{sc}$. Conjugating $f$ by $\hat{T}_{sc}$ replaces the extension we are considering by an isomorphic extension. We may therefore arrange for $f$ to be any lift of $w_0$ we like. We choose a pinning of $\hat{G}_{sc}$, involving the maximal torus $\hat{T}_{sc}$, and we let $f = \hat{w}_0$ be the Tits lift of $w_0$ relative to that pinning [LS87, §2.1].

We have thus introduced coordinates into the situation of Lemma I.1 that will be helpful for our computations. We shall now do the same with the situation of Lemma I.2. For this, we fix a split maximal torus $T \subset G$ and choose $g \in G$ s.t. $gTg^{-1} = S$. Then $w_0 = g^{-1}\sigma(g) \in \Omega = N/T$ is an elliptic element. Let $w_1, w_2 \in \Omega$ generate the preimage under $Ad(g)$ of the stabilizer of $\theta$. Then $w_0, w_1, w_2$ all commute. As argued above, there is a choice of isomorphism $X_*(T) \cong \mathbb{Z}^{2n}$ so that $w_0, w_1, w_2$ have the coordinate form given above. Fix a pinning of $G$ involving the torus $T$ and let $\hat{w}_0$ be the Tits lift of $w_0$ relative to that pinning. Then $\hat{w}_0 \in N(T, G)(k)$ is of finite order, and hence $\sigma \mapsto \hat{w}_0$ determines a 1-cocycle of $Gal(\bar{k}/k)$ in $N(T, G)(\bar{k})$. Both $\sigma \mapsto \hat{w}_0$ and $\sigma \mapsto g^{-1}\sigma(g)$ map to the same element of $Z^1(Gal(\bar{k}/k), \Omega)$, so their difference is an element of $Z^1(Gal(\bar{k}/k), T_{w_0})$, where $T_{w_0}$ denotes the torus $T$ with Frobenius action twisted by $w_0$. By Lang’s theorem this latter element is of the form $t^{-1} \cdot w_0\sigma(t)w_0^{-1}$. Thus, after replacing $g$ by $gt^{-1}$ we obtain $g^{-1}\sigma(g) = \hat{w}_0$.

To prove Lemma I.1 we will find lifts of $w_1$ and $w_2$ in $\hat{N}_{sc}^{+}$ such that their commutator, which automatically lies in $\hat{T}_{sc}^{f}$, vanishes in the group of $(w_1, w_2)$-coinvariants. To prove Lemma I.2 we will find lifts in $N(S_{sc}, G_{sc})(k)$ of two generators of $\Omega(S_{sc}, G_{sc})(k)$ so that their commutator, again automatically belonging to $S_{sc}(k)$, vanishes in the group of coinvariants for the action of $\Omega(S_{sc}, G_{sc})(k)$. The latter is equivalent to finding lifts in $N(T_{sc}, G_{sc})(k)$ of $w_1$ and $w_2$ that are fixed by $Ad(\hat{w}_0) \circ \sigma$ and whose commutator, which lies in the $Ad(\hat{w}_0) \circ \sigma$-fixed points of $T_{sc}(k)$, vanishes in the group of $(w_1, w_2)$-coinvariants.

Let $\hat{w}_1, \hat{w}_2$ be the Tits lifts of $w_1$ and $w_2$ with respect to the chosen pinning. They automatically lie in $\hat{N}_{sc}^{+}$ (respectively $N(T_{sc}, G_{sc})(k)$), but may not commute with $\hat{w}_0$.

Using [LS87, Lemma 2.1.A] we see that for any two commuting $u, v \in \Omega$ the
commutator $[\dot{u}, \dot{v}] := \dot{u}\dot{v} - \dot{v}\dot{u}$ is given by $\lambda_{u,v}(-1)$, where $\lambda_{u,v}$ is the sum of the coroots for the set of roots $\Lambda_{u,v} := \{\alpha > 0, (uv)^{-1}\alpha > 0\} \cap \{(u^{-1}\alpha < 0, v^{-1}\alpha > 0) \cup (u^{-1}\alpha > 0, v^{-1}\alpha < 0)\}$.

The actions of $w_1^{-1}$ and $w_2^{-1}$ on $R^+$ are given by the following tables:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$w_1^{-1} \alpha$</th>
<th>$w_2^{-1} \alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$e_1 \pm e_{2n}$</td>
<td>$e_1 - e_j$, $j &lt; 2n$</td>
</tr>
<tr>
<td>$e_{2n}$</td>
<td>$-(e_1 \pm e_{2n})$</td>
<td>$-(e_{2n+1-j} + e_{2n+1-i})$</td>
</tr>
<tr>
<td>$e_i$</td>
<td>$e_i + e_j$</td>
<td>$e_i - e_j$</td>
</tr>
</tbody>
</table>

For $w_0^{-1}$ it is enough to record which positive roots are sent to negative. For this, let $t_a' = 2n + 1 - t_a$ and

$$B = \{i_a | a = 1, \ldots, k\} \cup \{i_a' | a = 1, \ldots, k\}.$$  

Then we have the following table:

<table>
<thead>
<tr>
<th>$w_0^{-1}(e_i - e_j)$</th>
<th>$w_0^{-1}(e_i + e_j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i, j \notin B$</td>
<td>$+$</td>
</tr>
<tr>
<td>$t_a &lt; i &lt; i_{a+1}$</td>
<td>$+$</td>
</tr>
<tr>
<td>$t_a' &lt; i &lt; i_{a+1}$</td>
<td>$+$</td>
</tr>
<tr>
<td>$i_{a+1} &lt; i &lt; i_{a+1}'$</td>
<td>$+$</td>
</tr>
<tr>
<td>$i_{a+1} &lt; i &lt; i_{a+1}'$</td>
<td>$+$</td>
</tr>
<tr>
<td>$i_{a+1} &lt; i &lt; i_{a+1}'$</td>
<td>$+$</td>
</tr>
<tr>
<td>$i_{a+1} &lt; i &lt; i_{a+1}'$</td>
<td>$+$</td>
</tr>
</tbody>
</table>

The element $\lambda_{u,v}(-1)$ is a torsion element of $\hat{T}_{sc}$ or $T_{sc}(\hat{k})$ respectively. In order to unify the treatment we define $q = p = 1$ in the situation of Lemma I.1 and interpret $(Q^{2n}/Q)^p$ to mean $(Q^{2n}/Q)$. Then $(Q^{2n}/Q)^p$ is the subgroup of torsion elements of $\hat{T}_{sc}$ in the case of Lemma I.1 and the full $T_{sc}(\hat{k})$ in the case of Lemma I.2. Moreover, $qw_0$ is the action of $f$ in the former case and the action of $\text{Ad}(\hat{w}_0) \circ \sigma$ in the latter case. We set $f = \text{Ad}(\hat{w}_0) \circ \sigma$ in the latter case.

In both cases the element $\lambda_{u,v}(-1)$ is represented by $\frac{1}{d} \lambda_{u,v}$. For $u = w_1$ and $v = w_0$ we see that $\Lambda_{w_1,w_0} = \emptyset$, and thus $w_0$ is fixed by $f$. For $u = w_2$ and $v = w_0$ we have

$$\Lambda_{u,v} = \{e_i \pm e_j | i = a, j < a_{i+1}\} \cup \{e_i - e_j | i = a, j \geq a_{i+1}, j \notin B\} \cup \{e_i - e_j | i = a', j \notin B\} \cup \{e_i + e_j | i = a, i \notin B\} \cup \{e_i + e_j | j = a', i \leq a_{i+1}, i \notin B\}.$$  

Thus $f w_2 f^{-1} = \lambda_{w_2,w_0}(-1)\hat{w}_2$ and we need to multiply $w_2$ by an element $t \in \hat{T}_{sc}$ (or $t \in T_{sc}$ respectively) such that $ftf^{-1} = \lambda_{w_2,w_0}(-1)t$. Since $qw_0 - 1$ is invertible on $Q^{2n}$ we can form $\mu = \frac{1}{d}(qw_0 - 1)^{-1}\lambda_{w_2,w_0} \in Q^{2n}$. All denominators of this vector are powers of the form $q^i + 1$, where $l_i$ are the lengths of the cycles in $w_0$. Since $Z^{2n}/Q \cong Z/2Z$ and $p \neq 2$ we see that the image $t \in (Q^{2n}/Q)$ of $r$ has order prime to $p$. Then $tw_2$ is fixed by $f$.

Now we have the lifts $\hat{w}_1$, $tw_2$ of $w_1$ and $w_2$ in $\hat{N}_{sc}^+$ (respectively $N(T_{sc}, G_{sc})(\hat{k})$) fixed by $f$. We now compute their commutator $[\hat{w}_1, tw_2] = (t^{-1}w_1 t) \cdot [\hat{w}_1, w_2]$. 

76
First consider $t^{-1} \cdot w_1 t$. It is the image in $(\mathbb{Q}^{2n}/Q)_{\rho'}$ of $(w_1 - 1)\mu = \frac{1}{2}(w_1 - 1)(qw_0 - 1)^{-1}\lambda_{w_2, w_0} \in \mathbb{Q}^{2n}$. To compute it, we decompose $\mathbb{Q}^{2n} = \mathbb{Q} \oplus \mathbb{Q}^{2n-2} \oplus \mathbb{Q}$. Both $w_1$ and $w_0$ respect this decomposition. We have $w_1 = (-1, \text{id}, -1)$ and $w_0 = (-1, *, -1)$, hence $(w_1 - 1)(qw_0 - 1)^{-1} = \left(\frac{2}{q+1}, 0, \frac{2}{q+1}\right)$.

To evaluate $\frac{1}{2}(w_1 - 1)(qw_0 - 1)^{-1}\lambda_{w_2, w_0}$ we thus need to only compute the contributions of $e_1$ and $e_{2n}$ to $\lambda_{w_2, w_0}$. Using the tables above we see that it is $b(e_1 + e_{2n})$, where $b = 2n - |B|$, so $\frac{1}{2}(w_1 - 1)(w_0 - 1)^{-1}\lambda_{w_2, w_0} = \frac{b}{q+1}(e_1 + e_{2n})$.

We note that $2|b$. Then the computation $(qw_0 - 1)^{-1}\frac{b}{q+1}e_1 = -be_1 \in Q$ shows that the image of $\frac{b}{q+1}e_1$ in $(\mathbb{Q}^{2n}/Q)_{\rho'}$ is fixed by $f$. Moreover, $(1 - w_2)^{\frac{b}{q+1}}e_1 = \frac{b}{q+1}(e_1 + e_{2n})$. We see that $(t^{-1} \cdot w_1 t)$ belongs in the $w_2$-coinvariants of $\mathcal{T}^f_{sc}$ (or $T_{sc}(k)^f$ respectively).

Next consider $[\tilde{w}_1, \tilde{w}_2]$. We have $A_{w_1, w_2} = \{e_1 - e_j | 1 < j < 2n\} \cup \{e_i + e_{2n} | 1 < i < 2n\}$ and hence $\frac{1}{2}\lambda_{w_1, w_2} = \frac{1}{2}(2n - 2)(e_1 + e_{2n}) \in Q$, thus $[\tilde{w}_1, \tilde{w}_2] = 1$. \hfill \qed

References


