ON CERTAIN SIGN CHARACTERS OF TORI AND THEIR EXTENSIONS TO BRUHAT–TITS GROUPS

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Abstract

We consider two sign characters defined on a tamely ramified maximal torus T of a twisted Levi subgroup M of a reductive p-adic group G. We show that their product extends to the stabilizer $M(F)_x$ of any point x in the Bruhat–Tits building of T, and give a formula for this extension. This result is used in the passage between zero and positive depth in [Kalb], as well as in forthcoming work on the Harish-Chandra character formula for supercuspidal representations.

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Let F be a non-archimedean local field with ring of integers O_F, maximal ideal $\mathfrak{p}_F$, residue field k_F with characteristic p and size q_F. We assume $p \neq 2$. Fix a separable closure $\bar{F}$ and let $\Gamma = \Gamma_F$ be the absolute Galois group, $W = W_F$ the Weil group, $I = I_F$ the inertia group, and $P = P_F$ the wild inertia group. Let $\Sigma = \Gamma \times \{\pm 1\}$. Let $F^u \subset F^{\text{tr}} \subset \bar{F}$ be the maximal unramified and tamely ramified extensions. Finite extensions of F will be assumed contained in $\bar{F}$.

Let G be a connected semi-simple adjoint group defined over F and split over F^{tr}, and let M be a twisted Levi subgroup split over F^{tr}. Note that the center Z_M of M is connected. We will write $\mathcal{B}$ and $\mathcal{B}^{\text{enl}}$ for the reduced and enlarged Bruhat–Tits buildings. Assume given a point $x \in \mathcal{B}^{\text{enl}}(M, F) \subset \mathcal{B}(G, F)$ and an element $X \in \text{Lie}(Z_M)^*(F)$ generic of depth $r \in \mathbb{R}$. Let $s = r/2$.

We denote by $M(F)_x$ the stabilizer of x and by M_x the (usually disconnected) k_F-group scheme with reductive connected component for which $\mathsf{M}_x(k_E) = M(E)_x/M(E)_{x,0+}$ for any finite unramified extension E/F. The connected component will be denoted by M_x° and satisfies $\mathsf{M}_x^\circ(k_E) = M(E)_{x,0}/M(E)_{x,0+}$. Here $M(E)_{x,0}$ and $M(E)_{x,0+}$ are the (connected) parahoric subgroup of $M(E)_x$ and its pro-unipotent radical.

Given an algebraic torus T defined over F write T for the special fiber of its ft-Neron model. Thus for any finite unramified extension E/F we have $\mathsf{T}(k_E) = T(E)_b/T(E)_{0+}$ and $\mathsf{T}^\circ(k_E) = T(E)_0/T(E)_{0+}$, where $T(E)_b$ is the maximal bounded subgroup of $T(E)$, $T(E)_0$ is the Iwahori subgroup, and $T(E)_{0+}$ the pro-unipotent radical. For $x \in \mathcal{B}^{\text{enl}}(T, F)$ we have $\mathsf{T} = \mathsf{T}_x$.

If $T \subset M$ is a maximal torus (always assumed defined over F) we write $R(T, G)$ and $R(T, M)$ for the absolute root systems of T in G and M, respectively, and $R(T, G/M) = R(T, G) \smallsetminus R(G, M)$. For $\alpha \in X^*(T)$ we write Γ_α and $\Gamma_{\pm\alpha}$ for the stabilizers in Γ of α and $\{\pm\alpha\}$, F_α and $F_{\pm\alpha}$ for the respective fixed fields, O_α and $O_{\pm\alpha}$ for their rings of integers, $\mathfrak{p}_\alpha$ and $\mathfrak{p}_{\pm\alpha}$ for their maximal ideals, and k_α and $k_{\pm\alpha}$ for the respective residue fields. The root α is called symmetric if $[F_\alpha : F_{\pm\alpha}] = 2$, and in addition unramified if $[k_\alpha : k_{\pm\alpha}] = 2$. We then denote by $F_\alpha^1 \subset F_\alpha^\times$ and $k_\alpha^1 \subset k_\alpha^\times$ the kernels of the norm maps for these quadratic extensions. Since $p \neq 2$, the groups $k_\alpha^\times$ and k_α^1 are cyclic of even order and we write sgn_{k_α} and $\text{sgn}_{k_\alpha^1}$ for the corresponding unique $\{\pm 1\}$-valued characters.

We will use fraktur letters to denote Lie algebras. Thus $\mathfrak{g}$, $\mathfrak{m}$, and $\mathfrak{t}$, are the Lie algebras of G, M, and T, respectively. The point x specifies an O_E-lattice $\mathfrak{g}(E)_{x,0}$ in the E-vector space $\mathfrak{g}(E)$, for every finite extension E/F. When E/F is tame and Galois we have $\mathfrak{g}(F)_{x,0} = \mathfrak{g}(E)_{x,0}^\Gamma$. We have $\mathfrak{g}(F^{\text{tr}})_{x,0} = \bigcup_E \mathfrak{g}(E)_{x,0}$, the union being over all finite tame extensions E/F.

Assume that T is tame, i.e. split over F^{tr}. If $\alpha \in R(T, G)$ and $x \in \mathcal{B}^{\text{enl}}(T, F) \subset \mathcal{B}(G, F)$ we write

$$\text{ord}_x(\alpha) = \{r \in \mathbb{R} | \mathfrak{g}_\alpha(F_\alpha)_{x,r+} \subsetneq \mathfrak{g}_\alpha(F_\alpha)_{x,r}\}.$$

For $\alpha \in R(T, G/M)$ we write α_0 for the restriction of α to Z_M. Then the set of non-zero weights $R(Z_M, G)$ for the action of Z_M on $\text{Lie}(G)$ is precisely the set $\{\alpha_0 | \alpha \in R(T, G/M)\}$.

Given $\alpha \in R(T, G/M)$ we denote by $e_\alpha = e(\alpha)$ and $e(\alpha_0)$ the ramification degrees of the extensions F_α/F and F_{α_0}/F, and by $e(\alpha/\alpha_0)$ their quotient, which is the ramification degree of F_α/F_{α_0}. Note that the set $\mathrm{ord}_x(\alpha)$ is a $e_\alpha^{-1}\mathbb{Z}$-torsor in $\mathbb{R}$.

Definition 1.0.1. For $T \subset M$ a tame maximal torus with $x \in \mathcal{B}^{\mathrm{enl}}(T, F)$ define the characters of $\mathsf{T}(k_F)$

$$\epsilon_\sharp^{G/M}(\gamma) = \prod_{\substack{\alpha \in R(T,G/M)_{\mathrm{sym,unram}}/\Gamma \\ s \in \mathrm{ord}_x(\alpha)}} \mathrm{sgn}_{k_\alpha^1}(\alpha(\gamma)) \cdot \prod_{\substack{\alpha \in R(T,G/M)_{\mathrm{asym}}/\Sigma \\ s \in \mathrm{ord}_x(\alpha)}} \mathrm{sgn}_{k_\alpha}(\alpha(\gamma)),$$

$$\epsilon_\flat^{G/M}(\gamma) = \prod_{\substack{\alpha \in R(T,G/M)_{\mathrm{sym,unram}}/\Gamma \\ \alpha_0 \in R(Z_M, G/M)_{\mathrm{sym,ram}} \\ 2 \nmid e(\alpha/\alpha_0)}} \mathrm{sgn}_{k_\alpha^1}(\alpha(\gamma)) \cdot \prod_{\substack{\alpha \in R(T,G/M)_{\mathrm{asym}}/\Sigma \\ \alpha_0 \in R(Z_M, G/M)_{\mathrm{sym,ram}} \\ 2 \nmid e(\alpha/\alpha_0)}} \mathrm{sgn}_{k_\alpha}(\alpha(\gamma)),$$

$$\epsilon_{f,r}^{G/M}(t) = \prod_{\substack{\alpha \in R(T,G/M)_{\mathrm{sym.ram}}/\Gamma \\ \alpha(t) \in -1+\mathfrak{p}_\alpha}} f_{(T,G)}(\alpha),$$

with $f_{(T,G)}$ being the toral invariant [Kal15, §4.1].

Remark: The character $\epsilon_\sharp$ was denoted by $\epsilon_{x,r/2}^{\mathrm{ram}}$ in [DS18, §4.3] and by ϵ^{ram} in [Kal19, (4.3.3)], while the character $\epsilon_\flat$ is that of [Kala, Proposition 5.21] and was denoted by δ in [Kalb].

Consider given a non-degenerate G-invariant quadratic form φ on $\mathfrak{g}$ whose restriction to the lattice $\mathfrak{g}(F^{\mathrm{tr}})_{x,0}$ in $\mathfrak{g}(F^{\mathrm{tr}})$ is integrally non-degenerate in the following sense: the values of φ on $\mathfrak{g}(F^{\mathrm{tr}})_{x,0}$ lie in $O_{F^{\mathrm{tr}}}$ and the restriction of φ to $\mathfrak{g}(F^{\mathrm{tr}})_{x,0}$ induces a non-degenerate form on $\mathfrak{g}(F^{\mathrm{tr}})_{x,0}/\mathfrak{g}(F^{\mathrm{tr}})_{x,0+}$.

Theorem 1.0.2. *Given such φ, there is a canonical sign character $\epsilon_\varphi^{G/M} : \mathsf{M}_x(k_F) \to \{\pm 1\}$ with the following property: For any tame maximal torus $T \subset M$ with $x \in \mathcal{B}^{enl}(T, F)$ the restriction of $\epsilon_\varphi^{G/M}$ to $\mathsf{T}(k_F)$ equals $\epsilon_\sharp^{G/M} \cdot \epsilon_\flat^{G/M} \cdot \epsilon_{f,r}^{G/M} \cdot \epsilon_{\varphi,r}^{G/M}$, where*

$$\epsilon_{\varphi,r}^{G/M}(t) = \prod_{\substack{\alpha \in R(T,G/M)_{sym.ram}/\Gamma \\ \alpha(t) \in -1+\mathfrak{p}_\alpha}} sgn_{k_\alpha^\times}(\varphi(H_\alpha)) \cdot (-1)^{[k_\alpha:k]+1} \cdot sgn_{k_\alpha^\times}(-1)^{e_\alpha s - \frac{1}{2}}.$$

Remark 1.0.3. It follows from [AR00, Proposition 4.1] that such a form φ exists when p does not divide the order of $\pi_1(G)$ and the order of the bonds of the Dynkin diagram of G.

2 Applications to the general case

In this section only we consider a connected reductive group G split over F^{tr} and a Yu tower $G^0 \subset \cdots \subset G^d = G$ that is part of a Yu datum, and a tame maximal torus $S \subset G^0$ with $x \in \mathcal{B}^{\mathrm{enl}}(S/Z_G, F)$; in applications S will often be assumed elliptic and maximally unramified in G^0. Let $S(F)_c$ denote the preimage of $[S/Z_G](F)_b$. Let S denote the k_F-group scheme with reductive connected component s.t. $\mathsf{S}(k_E) = S(E)_c/S(E)_{0+}$ for every finite unramified extension E/F.

For each i we can consider the adjoint group $G^{i+1}/Z_{G^{i+1}}$ and its twisted Levi subgroup $G^i/Z_{G^{i+1}}$. Definition 1.0.1 gives us characters $\epsilon_*^{(G^{i+1}/Z_{G^{i+1}})/(G^i/Z_{G^{i+1}})}$

on $[S/Z_{G^{i+1}}](F)_b$, which we may inflate to $S(F)_c$. We will denote the product over $i = 0, \ldots, d-1$ of $\epsilon_*^{(G^{i+1}/Z_{G^{i+1}})/(G^i/Z_{G^{i+1}})}$ without superscript. Thus we have the characters $\epsilon_\sharp$, $\epsilon_\flat$, and $\epsilon_{f,r}$, of $S(F)_c$.

Let $\mathfrak{z}$ denote the center of $\mathfrak{g}$, and $\mathfrak{z}^i$ the center of $\mathfrak{g}^i$.

Corollary 2.0.1. *Suppose there exists a non-degenerate G-invariant quadratic form φ on $\mathfrak{g}/\mathfrak{z}$ whose restriction to the lattice $(\mathfrak{g}/\mathfrak{z})(F^{tr})_{x,0}$ is integrally non-degenerate. Then the product $\epsilon_\sharp \cdot \epsilon_\flat$ extends from $S(F)_c$ to $G^0(F)_x$.*

Proof. The form φ induces a form on $\mathfrak{g}^i/\mathfrak{z}^i$ with the same property. From Theorem 1.0.2 we obtain the character $\epsilon_\varphi^{(G^{i+1}/Z_{G^{i+1}})/(G^i/Z_{G^{i+1}})}$ of $G^i(F)_x/Z_{G^{i+1}}(F)$ and the character $\epsilon_{\varphi,r}^{(G^{i+1}/Z_{G^{i+1}})/(G^i/Z_{G^{i+1}})}$ of $[S/Z_{G^{i+1}}](F)_b$. We inflate the character $\epsilon_\varphi^{(G^{i+1}/Z_{G^{i+1}})/(G^i/Z_{G^{i+1}})}$ to $G^i(F)_x$ and then restrict it to $G^0(F)_x$. We inflate the character $\epsilon_{\varphi,r}^{(G^{i+1}/Z_{G^{i+1}})/(G^i/Z_{G^{i+1}})}$ to $S(F)_c$. Taking the product over $i = 0, \ldots, d-1$ we obtain a character ϵ_φ of $G^0(F)_x$ and a character $\epsilon_{\varphi,r}$ of $S(F)_c$. Then Theorem 1.0.2 implies that the restriction of ϵ_φ to $S(F)_c$ equals $\epsilon_\sharp \cdot \epsilon_\flat \cdot \epsilon_{f,r} \cdot \epsilon_{\varphi,r}$.

It is very easy to see that the two characters $\epsilon_{\varphi,r}, \epsilon_{f,r} : \mathsf{S}(k) \to \{\pm 1\}$ extend to $\mathsf{G}_x^0(k)$. The argument for both is the same, so consider $\epsilon_{\varphi,r}$. It restricts trivially to $\mathsf{S}^\circ(k)$, because all symmetric ramified roots restrict trivially to S°. Therefore the exterior tensor product of $\epsilon_{\varphi,r}$ and the trivial character of $(\mathsf{G}_x^0)^\circ(k)$ descends to a character of $(\mathsf{G}_x^0)^\circ(k) \cdot \mathsf{S}(k)$. That this character extends to $\mathsf{G}^0(k)$ follows from Lemma 2.0.2 and the fact that the commutators of elements of G^0 lie in $(\mathsf{G}^0)^\circ$.

□

Lemma 2.0.2. *Let G be a locally profinite group, $H \subset G$ a closed normal subgroup of finite index, G/H abelian. Let $\chi : H \to \mathbb{C}^\times$ be a continuous character. The following are equivalent.*

1. *χ extends to G.*
2. *For any $g_1, g_2 \in G$ we have $\chi(g_1 g_2 g_1^{-1} g_2^{-1}) = 1$.*

Proof. First, taking $g_2 = h \in H$ we see that $\chi(g_1 h g_1^{-1}) = \chi(h)$, i.e. χ is G-invariant. Therefore we may form the push-out diagram

$$\begin{tikzcd} 1 \arrow[r] & H \arrow[r, "i"] \arrow[d, "\chi"] & G \arrow[r, "p"] \arrow[d, "d"] & G/H \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathbb{C}^\times \arrow[r, "a"] & \square \arrow[r, "b"] & G/H \arrow[r] & 1. \end{tikzcd}$$

Then the extensions $\tilde{\chi} : G \to \mathbb{C}^\times$ of χ are in bijection with the characters $s : \square \to \mathbb{C}^\times$ that satisfy $s \circ a = \mathrm{id}$, via the relation $\tilde{\chi} = s \circ d$. On the other hand, the bottom extension splits if and only if $\square$ is abelian, i.e. if and only if its commutator $G/H \times G/H \to \mathbb{C}^\times$ is trivial, as discussed in [Kalb, Appendix A]. But this commutator is precisely $(g_1, g_2) \mapsto \chi(g_1 g_2 g_1^{-1} g_2^{-1})$. □

3 A REPACKAGING OF $\epsilon_\sharp \cdot \epsilon_\flat$

From now on and for the rest of the paper we return to the notation of the first section. Thus we have a connected semi-simple adjoint group G over the p-adic field F and a tame twisted Levi subgroup $M \subset G$.

Fix a tame maximal torus $T \subset M$. For a finite Σ-invariant subset $0 \notin \Phi \subset X^*(T)$ such that each $\alpha \in \Phi$ is bounded on $T(F)_c$ (for example $\Phi \subset R(T,G)$) we can define a character $\epsilon_\Phi : \mathsf{T}(k_F) \to \{\pm 1\}$ by

$$\epsilon_\Phi(\gamma) = \prod_{\alpha \in \Phi_{\text{sym,unram}}/\Gamma} \operatorname{sgn}_{k_\alpha^1}(\alpha(\gamma)) \cdot \prod_{\Phi_{\text{asym}}/\Sigma} \operatorname{sgn}_{k_\alpha}(\alpha(\gamma)). \tag{3.1}$$

Example 3.0.1. The characters $\epsilon_\sharp$ and $\epsilon_\flat$ come via this construction from

$$\Phi_\sharp = \{\alpha \in R(T,G/M) | s \in \operatorname{ord}_x(\alpha)\}$$

and

$$\Phi_\flat = \{\alpha \in R(T,G/M) | \alpha_0 \in R(Z_M, G/M)_{\text{sym,ram}}, 2 \nmid e(\alpha/\alpha_0)\}.$$

Fact 3.0.2. *If $\Phi_1, \Phi_2 \subset X^*(T)$ are such subsets, then so is their symmetric difference*

$$\Phi_1 \Delta \Phi_2 = (\Phi_1 \cup \Phi_2) \smallsetminus (\Phi_1 \cap \Phi_2)$$

and moreover

$$\epsilon_{\Phi_1} \cdot \epsilon_{\Phi_2} = \epsilon_{\Phi_1 \Delta \Phi_2}.$$

Proof. Left to the reader. ☐

We now define three additional characters of $\mathsf{T}(k_F)$, called ϵ_0, $\epsilon_t^{\text{sym,ram}}$, and $\epsilon_{\text{sym,ram}}$, the second of which depends on a parameter $t \in \mathbb{R}$, via the following subsets of $R(T,G/M)$:

$$\Phi_0 = \{0 \in \operatorname{ord}_x(\alpha)\}. \tag{3.2}$$

$$\Phi_t^{\text{sym,ram}} = \{\alpha_0 \notin R(Z_M, G)_{\text{sym,ram}}, t \in \operatorname{ord}_x(\alpha)\}. \tag{3.3}$$

$$\Phi_{\text{sym,ram}} = \{\alpha_0 \in R(Z_M, G)_{\text{sym,ram}}, 2 \nmid e(\alpha/\alpha_0), 0 \notin \operatorname{ord}_x(\alpha) \not\ni s\} \tag{3.4}$$

Fact 3.0.3. *Let $\alpha \in R(T,G/M)$ be s.t. α_0 is symmetric ramified. The rational number $e(\alpha_0)r$ is an odd integer. If α itself is symmetric ramified, then the rational number $e(\alpha)r$ is an odd integer, and in particular the natural number $e(\alpha/\alpha_0)$ is odd.*

Proof. Consider $\langle H_\alpha, X\rangle \in F_\alpha^\times$. Since $-r = \operatorname{ord}(\langle H_\alpha, X\rangle) \in \operatorname{ord}(F_\alpha^\times) = e_\alpha^{-1}\mathbb{Z}$, we see that $re_\alpha \in \mathbb{Z}$.

In fact we claim that $\langle H_\alpha, X\rangle \in F_{\alpha_0}^\times$, and therefore $re(\alpha_0) \in \mathbb{Z}$. Indeed, taking a non-degenerate invariant symmetric bilinear form β on $\mathfrak{g}$ the isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ it provides restricts to an isomorphism $\mathfrak{z} \to \mathfrak{z}^*$, where $\mathfrak{z}$ is the Lie algebra of the center of M. Let $X_\beta \in \mathfrak{z}(F)$ correspond to $X \in \mathfrak{z}^*(F)$. Then

$\langle H_\alpha, X\rangle = \beta(H_\alpha, H_\alpha)\alpha(X_\beta)$. Since β is invariant the value $\beta(H_\alpha, H_\alpha) \in \bar{F}^\times$ depends only on the length of α, therefore for any $\sigma \in \Gamma$ we have $\sigma(\beta(H_\alpha, H_\alpha)) = \beta(H_{\sigma\alpha}, H_{\sigma\alpha}) = \beta(H_\alpha, H_\alpha)$. At the same time $\alpha(X_\beta) = \alpha_0(X_\beta) \in F_{\alpha_0}^\times$ and the claim is proved.

If α is symmetric then the $F_\alpha/F_{\pm\alpha}$-trace of $\langle H_\alpha, X\rangle \in F_\alpha^\times$ is zero. If the extension $F_\alpha/F_{\pm\alpha}$ is ramified, this implies that $-r = \langle H_\alpha, X\rangle \in \mathrm{ord}(F_\alpha^\times) \smallsetminus \mathrm{ord}(F_{\pm\alpha}^\times) = e_\alpha^{-1}\mathbb{Z} \smallsetminus 2e_\alpha^{-1}\mathbb{Z}$, and therefore re_α is odd. Since $e(\alpha/\alpha_0)$ is a divisor of re_α, it is also odd.

If only α_0 is symmetric ramified, we can apply the same reasoning to $\langle H_\alpha, X\rangle \in F_{\alpha_0}^\times$ and conclude that $re(\alpha_0)$ is odd. $\square$

Lemma 3.0.4. *We have the equality*

$$\epsilon_\sharp \cdot \epsilon_\flat = \epsilon_0 \cdot \epsilon_0^{\textit{sym,ram}} \cdot \epsilon_s^{\textit{sym,ram}} \cdot \epsilon_{\textit{sym,ram}}.$$

Proof. By Fact 3.0.2 it is enough to show

$$\Phi_\sharp \Delta \Phi_\flat = \Phi_0 \Delta \Phi_0^{\text{sym,ram}} \Delta \Phi_s^{\text{sym,ram}} \Delta \Phi_{\text{sym,ram}}.$$

The set $\Phi_\flat$ can be expressed as the disjoint union of the following subsets of $R(T, G/M)$, where "s.r." stands for "symmetric ramified", and "n.s.r" for "not symmetric ramified":

$$\begin{array}{ccc} \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \in \mathrm{ord}_x(\alpha) \ni s\} & \cup & \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \notin \mathrm{ord}_x(\alpha) \ni s\} \\ \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \in \mathrm{ord}_x(\alpha) \not\ni s\} & \cup & \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \notin \mathrm{ord}_x(\alpha) \not\ni s\} \end{array}$$

while $\Phi_\sharp$ has the analogous subdivision

$$\begin{array}{ccc} \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \in \mathrm{ord}_x(\alpha) \ni s\} & \cup & \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \notin \mathrm{ord}_x(\alpha) \ni s\} \\ \{\alpha_0 \text{ s.r.}, 2 \mid e(\alpha/\alpha_0), 0 \in \mathrm{ord}_x(\alpha) \ni s\} & \cup & \{\alpha_0 \text{ s.r.}, 2 \mid e(\alpha/\alpha_0), 0 \notin \mathrm{ord}_x(\alpha) \ni s\} \\ \{\alpha_0 \text{ n.s.r.}, 0 \in \mathrm{ord}_x(\alpha) \ni s\} & \cup & \{\alpha_0 \text{ n.s.r.}, 0 \notin \mathrm{ord}_x(\alpha) \ni s\}. \end{array}$$

The symmetric difference $\Phi_\sharp \Delta \Phi_\flat$ is therefore the disjoint union

$$\begin{array}{ccc} \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \in \mathrm{ord}_x(\alpha) \not\ni s\} & \cup & \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \notin \mathrm{ord}_x(\alpha) \not\ni s\} \\ \{\alpha_0 \text{ s.r.}, 2 \mid e(\alpha/\alpha_0), 0 \in \mathrm{ord}_x(\alpha) \ni s\} & \cup & \{\alpha_0 \text{ s.r.}, 2 \mid e(\alpha/\alpha_0), 0 \notin \mathrm{ord}_x(\alpha) \ni s\} \\ \{\alpha_0 \text{ n.s.r.}, 0 \in \mathrm{ord}_x(\alpha) \ni s\} & \cup & \{\alpha_0 \text{ n.s.r.}, 0 \notin \mathrm{ord}_x(\alpha) \ni s\}. \end{array}$$

By Fact 3.0.3, $e(\alpha_0)r$ is odd when α_0 is symmetric ramified, so the parities of $e(\alpha)r$ and $e(\alpha/\alpha_0)$ are the same. If $e(\alpha/\alpha_0)$ is even, then $e(\alpha)s$ is an integer, so the conditions $s \in \mathrm{ord}_x(\alpha)$ and $0 \in \mathrm{ord}_x(\alpha)$ are equivalent. If $e(\alpha/\alpha_0)$ is odd, then $e(\alpha)s$ is a half-integer, but not an integer, and so the conditions $0 \in \mathrm{ord}_x(\alpha)$ and $s \in \mathrm{ord}_x(\alpha)$ are mutually exclusive. Therefore, the above set of roots becomes the disjoint union

$$\begin{array}{ccc} \{\alpha_0 \text{ s.r.}, 0 \in \mathrm{ord}_x(\alpha)\} & \cup & \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \notin \mathrm{ord}_x(\alpha) \not\ni s\} \\ \{\alpha_0 \text{ n.s.r.}, s \in \mathrm{ord}_x(\alpha) \ni 0\} & \cup & \{\alpha_0 \text{ n.s.r.}, s \in \mathrm{ord}_x(\alpha) \not\ni 0\}. \end{array}$$

This set is the symmetric difference of

$$\{0 \in \mathrm{ord}_x(\alpha)\} \cup \{\alpha_0 \text{ s.r.}, 2 \nmid e(\alpha/\alpha_0), 0 \notin \mathrm{ord}_x(\alpha) \not\ni s\}$$

and

$$\{\alpha_0 \text{ n.s.r.}, s \notin \mathrm{ord}_x(\alpha) \ni 0\} \cup \{\alpha_0 \text{ n.s.r.}, s \in \mathrm{ord}_x(\alpha) \not\ni 0\},$$

which in turn equals the symmetric difference $\Phi_0 \Delta \Phi_0^{\text{sym,ram}} \Delta \Phi_s^{\text{sym,ram}} \Delta \Phi_{\text{sym,ram}}$. $\square$

To emphasize the torus T we may write ${}_T\epsilon_0$, ${}_T\epsilon_t^{\text{sym,ram}}$, and ${}_T\epsilon_{\text{sym,ram}}$. Our goal will be to define characters ${}_M\epsilon_0$, ${}_M\epsilon_t^{\text{sym,ram}}$, and ${}_M\epsilon_{\text{sym,ram}}$ of $\mathsf{M}_x(k_F)$ whose restriction to $\mathsf{T}(k_F)$ equals ${}_T\epsilon_0$, ${}_T\epsilon_t^{\text{sym,ram}}$, and ${}_T\epsilon_{\text{sym,ram}}$, respectively, whenever $T \subset M$ is a tame maximal torus with $x \in \mathcal{B}^{\text{enl}}(T, F)$. We will achieve this, up to the combined discrepancy of $\epsilon_{\varphi,r}$.

4 The extension of $\epsilon_t^{\text{SYM,RAM}}$

We begin our discussion with the character $\epsilon_t^{\text{sym,ram}}$. In this case we will construct ${}_M\epsilon_t^{\text{sym,ram}}$ and show that its restriction to $\mathsf{T}(k_F)$ equals ${}_T\epsilon_t^{\text{sym,ram}}$ for any tame maximal torus $T \subset M$ with $x \in \mathcal{B}^{\text{enl}}(T, F)$.

4.1 Sign characters from hypercohomology

Let G be an algebraic group defined over a field k of characteristic not 2 with absolute Galois group Γ. We do not assume that G is connected or reductive. Consider the abelian group of algebraic characters $X^*(\mathsf{G})$, written additively, and the multiplication by 2 map $X^*(\mathsf{G}) \to X^*(\mathsf{G})$. This forms a complex of Γ-modules of length 2, which we place in degrees 0 and 1. We recall the explicit description of the first Galois-hypercohomology group of this complex.

Definition 4.1.1.

1. The abelian group $Z^1(\Gamma, X^*(\mathsf{G}) \to X^*(\mathsf{G}))$ of *degree* 1 *hypercocycles* consists of pairs (ρ_σ, δ), where $\rho_\sigma \in Z^1(\Gamma, X^*(\mathsf{G}))$ and $\delta \in X^*(\mathsf{G})$ satisfy $(1-\sigma)\delta = 2\rho_\sigma$ for all $\sigma \in \Gamma$. Addition is inherited from $X^*(\mathsf{G})$.
2. The subgroup $B^1(\Gamma, X^*(\mathsf{G}) \to X^*(\mathsf{G}))$ of *degree* 1 *hypercoboundaries* consists of the pairs $((1-\sigma)\chi, 2\chi)$ for $\chi \in X^*(\mathsf{G})$.
3. The *first hypercohomology* group $H^1(\Gamma, X^*(\mathsf{G}) \to X^*(\mathsf{G}))$ is the quotient Z^1/B^1.

Definition 4.1.2. Given $(\rho_\sigma, \delta) \in Z^1(\Gamma, X^*(\mathsf{G}) \to X^*(\mathsf{G}))$ we define for each $g \in \mathsf{G}(k)$ and $\sigma \in \Gamma$ an element $\epsilon(g, \sigma) \in \bar{k}^\times$ by choosing arbitrarily a square root in $\bar{k}^\times$ of $\delta(g) \in \bar{k}^\times$ and setting

$$\epsilon(g, \sigma) = \rho_\sigma(g) \cdot \frac{\sigma\sqrt{\delta(g)}}{\sqrt{\delta(g)}}.$$

Fact 4.1.3.

1. *The element $\epsilon(g, \sigma) \in \bar{k}^\times$ is independent of the choice of square root.*
2. *For a fixed g the function $\sigma \mapsto \epsilon(g, \sigma)$ lies in $Z^1(\Gamma, \mu_2)$.*
3. *The function $g \mapsto Z^1(\Gamma, \mu_2)$ thus defined is a group homomorphism.*

Proof. Both $\rho_\sigma(g)$ and $(\delta/2)(g, \sigma) := (\sqrt{\delta(g)})^{-1}\sigma(\sqrt{\delta(g)})$ are, for a fixed $g \in \mathsf{G}(k)$, elements of $Z^1(\Gamma, \bar{k}^\times)$. The first one, $\rho_\sigma(g)$, is by definition multiplicative in g. The second, $(\delta/2)(g, \sigma)$, is independent of the choice of square root, for a different choice of square root will result in both numerator and denominator being multiplied by $-1 \in k$, so the result is unchanged. This independence then shows that $(\delta/2)(g \cdot g', \sigma) = (\delta/2)(g, \sigma) \cdot (\delta/2)(g', \sigma)$. It follows that $\epsilon(g, \sigma)$ lies for a fixed g in $Z^1(\Gamma, \bar{k}^\times)$ and is multiplicative in g. Now $\epsilon(m, \sigma)^2 = 1$ by the relation $(1-\sigma)\delta = 2\rho_\sigma$. $\square$

Let $\epsilon(g) \in k^\times/k^{\times,2}$ be the image of $\epsilon(g,-)$ under the isomorphism $H^1(\Gamma,\mu_2) \cong k^\times/k^{\times,2}$. Thus

$$\epsilon : \mathsf{G}(k) \to k^\times/k^{\times,2}$$

is a character. We can call it $\epsilon_{\rho,\delta}$ to emphasize its dependence on δ and ρ.

Fact 4.1.4. *The assignment $(\rho,\delta) \mapsto \epsilon_{\rho,\delta}$ is functorial. More precisely, if $f : \mathsf{H} \to \mathsf{G}$ is a morphism of algebraic groups over k, then*

$$\epsilon_{\rho\circ f,\delta\circ f} = \epsilon_{\rho,\delta} \circ f.$$

Proof. Immediate. □

Fact 4.1.5. *The assignment $(\rho,\delta) \mapsto \epsilon_{\rho,\delta}$ is a group homomorphism*

$$Z^1(\Gamma, X^*(\mathsf{G}) \to X^*(\mathsf{G})) \to Hom(\mathsf{G}(k), k^\times/k^{\times,2})$$

whose kernel contains $B^1(\Gamma, X^*(\mathsf{G}))$ and hence descends to a group homomorphism

$$H^1(\Gamma, X^*(\mathsf{G}) \to X^*(\mathsf{G})) \to Hom(\mathsf{G}(k), k^\times/k^{\times,2})$$

Proof. The homomorphism statement reduces to showing that for fixed g and σ the element $\epsilon_{\rho,\delta}(g,\sigma) \in \mu_2$ is multiplicative in the pair (ρ,δ), which follows from the independence of choice of square root. That this homomorphism kills coboundaries is immediate from the formula for $\epsilon(g,\sigma)$. □

Remark 4.1.6. The construction of the character $\epsilon_{\rho,\delta}$ can be rephrased more abstractly as follows. Consider G as a complex of length 1 placed in degree 0. Let $\mathbb{G}_m \to \mathbb{G}_m$ be the squaring map, considered as a complex placed in degrees 0 and 1. Let $\mathrm{Hom}(\mathsf{G}, \mathbb{G}_m \to \mathbb{G}_m)$ be the Hom-complex. It equals $X^*(\mathsf{G}) \to X^*(\mathsf{G})$. Now we have the obvious map

$$H^1(\Gamma, \mathrm{Hom}(\mathsf{G}, \mathbb{G}_m \to \mathbb{G}_m)) \to \mathrm{Hom}_{\mathrm{grp}}(\mathsf{G}(k), H^1(\Gamma, \mathbb{G}_m \to \mathbb{G}_m)).$$

But the squaring map on $\mathbb{G}_m$ being surjective, the complex $\mathbb{G}_m \to \mathbb{G}_m$ is quasi-isomorphic to its kernel μ_2.

4.2 A general construction of a sign character on $M(F)_x$

We now return to the p-adic ground field F. Assume given a Σ-invariant subset $\Psi \subset R(Z_M, G)$, a point $x \in \mathcal{B}^{\mathrm{enl}}(M,F)$, and $t \in \mathbb{R}$. We will associate a sign character $M(F)_x \to \{\pm 1\}$ and compute its restriction to any tame maximal torus $T \subset M$ with $x \in \mathcal{B}^{\mathrm{enl}}(T,F)$. Since we work with enlarged buildings, the groups $M(F)_x$ and $T(F)_x$ are compact.

For every I-orbit $O \subset \Psi$ we consider

$$V_O = \bigoplus_{\alpha_0 \in O} \mathfrak{g}_{\alpha_0}.$$

This is a vector subspace of $\mathfrak{g}$ defined over F^u and stable under the action of $M(F^u)$. The $\bar{k}_F$-vector space

$$V_O(F^u)_{x,t:t+}$$

is stable under the action of $M(F^u)_x$. Since $M(F^u)_{x,0+}$ acts trivially, this action descends to an algebraic action of M_x. Consider the character $\chi_O \in X^*(\mathsf{M}_x)$ defined by

$$\chi_O : M(F^u)_x/M(F^u)_{x,0+} \to \bar{k}_F^\times, \qquad \chi_O(m) = \det(m|V_O(F^u)_{x,t:t+}).$$

Lemma 4.2.1. $\chi_{-O} = \chi_O^{-1}$.

Proof. Since $M(F^u)_{x,0+}$ is open in $M(F^u)$, every element of $M(F^u)_x/M(F^u)_{x,0+}$ can be represented by a semi-simple, even regular, element of $M(F^u)_x$. Let $m \in M(F^u)_x$ be such. There exists a finite extension $F^u/F'/F$ such that $m \in M(F')_x$. The group $M(F')_x$ being compact, this element has a topological Jordan decomposition $m = m_s \cdot m_u$ by [Spi08, Proposition 1.8]. The order of m_u is pro-p, therefore $\chi_O(m_u) \in \bar{k}^\times$ equals 1. We may thus replace m by m_s, a topologically semi-simple element belonging to $M(F')_x$. According to [Spi08, Proposition 2.33, Corollary 2.37] $x \in \mathcal{B}(\mathrm{Cent}(m_s, M)^\circ, F)$. Choose a special maximal torus $T \subset \mathrm{Cent}(m_s, M)^\circ$ whose apartment contains x. Then $m_s \in T(F)$ and $x \in \mathcal{B}(T, F)$. Then

$$\chi_O(m) = \prod_{\substack{\alpha \in R(T,G)/I \\ \alpha|_{Z_M} \in O}} \alpha(m).$$

Multiplication by -1 is a bijection from the index set of the above product to the corresponding index set where O has been replaced by $-O$. ☐

Fact 4.2.2. *Let $\phi \in \Gamma/I$ be Frobenius. Then $\phi\chi_O = \chi_{\phi O}$.*

Proof. We have $\phi(V_O(F^u)) = V_{\phi O}(F^u)$ and hence $(\phi\chi_O)(m) = \phi\chi_O(\phi^{-1}m)$ equals

$$\phi(\det(\phi^{-1}m|V_O(F^u)_{x,t:t+})) = \mathrm{det}(m|\phi V_O(F^u)_{x,t:t+}) = \mathrm{det}(m|V_{\phi O}(F^u)_{x,t:t+}).$$

☐

Assumption 4.2.3. The elements of Ψ are not ramified symmetric.

Choose an I-invariant disjoint union decomposition $\Psi = \Psi^+ \cup \Psi^-$ and define

$$\delta = \prod_{O \in \Psi^+/I} \chi_O$$

and for $\sigma \in \Gamma/I$

$$\rho_\sigma = \prod_{\substack{O \in \Psi^+/I \\ \sigma^{-1}O \in \Psi^-/I}} \chi_O.$$

Fact 4.2.4. *$\sigma \mapsto \rho_\sigma$ lies in $Z^1(\Gamma/I, X^*(\mathsf{M}_x))$ and we have the relation $(1-\sigma)\delta = 2\rho_\sigma$ in $X^*(\mathsf{M}_x)$. Therefore $(\rho, \delta) \in Z^1(\Gamma/I, X^*(\mathsf{M}_x) \to X^*(\mathsf{M}_x))$. The class of this hypercocycle does no depend on the choice of decomposition $\Psi = \Psi^+ \cup \Psi^-$.*

Proof. The character ρ_σ is the product of χ_O for O running over the set $(\Psi^+/I) \cap \sigma(\Psi^-/I)$. We will write ρ_σ symbolically as $X^+ \cap \sigma X^-$. Then we compute

$$\begin{aligned}
\rho_\sigma + \sigma\rho_\tau &= [X^+ \cap \sigma X^-] + [\sigma X^+ \cap \sigma\tau X^-] \\
&= [\cancel{(X^+ \cap \sigma X^- \cap \sigma\tau X^+)} + (X^+ \cap \sigma X^- \cap \sigma\tau X^-)] \\
&+ [(X^+ \cap \sigma X^+ \cap \sigma\tau X^-) + \cancel{(X^- \cap \sigma X^+ \cap \sigma\tau X^-)}] \\
&= X^+ \cap \sigma\tau X^- \\
&= \rho_{\sigma\tau}.
\end{aligned}$$

For the second claim, we compute

$$\begin{aligned}
& (1-\sigma)\delta \\
= \quad & X^+ - \sigma X^+ \\
= \quad & [(X^+ \cap \sigma X^-) + (X^+ \cap \sigma X^+)] - \sigma[(X^+ \cap \sigma^{-1} X^+) + (X^+ \cap \sigma^{-1} X^-)] \\
= \quad & (X^+ \cap \sigma X^-) - (X^- \cap \sigma X^+) \\
= \quad & 2(X^+ \cap \sigma X^-)
\end{aligned}$$

For the independence statement it is enough to compare a given decomposition $\Psi = \Psi^+ \cup \Psi^-$ with the one obtained by replacing one $O \in \Psi^+/I$ with its negative. Then it follows from Lemma 4.2.1 that δ is replaced by $\delta \cdot \chi_O^{-2}$, while ρ_σ is replaced by $\rho_\sigma \cdot (\sigma - 1)\chi_O$. □

From the construction of §4.1 we obtain the sign character

$$\epsilon_M : \mathsf{M}_x(k_F) \to k_F^\times / k_F^{\times,2} \xrightarrow{\text{sgn}} \{\pm 1\},$$

which we may also view as a character of $M(F)_x$ via the surjection $M(F)_x \to \mathsf{M}_x(k_F)$.

4.3 Compatibility with tori and formula in terms of roots

Let $T \subset M$ be a tame maximal torus with $x \in \mathcal{B}(T, F)$. Let $\Phi \subset R(T, G/M)$ be the preimage of the given $\Psi \subset R(Z_M, G)$. The map $\Phi \to \Psi$ is surjective and Γ-equivariant. Since T is a tame twisted Levi subgroup of G in its own right, we can apply the same procedure to T and Φ in place of M and Ψ and in this way obtain a sign character $\epsilon_T : \mathsf{T}_x(k_F) \to \{\pm 1\}$.

Lemma 4.3.1. *The restriction of ϵ_M under the map $\mathsf{T}_x \to \mathsf{M}_x$ is equal to ϵ_T.*

Proof. Write δ_M and ρ_M in place of δ and ρ. We let $\Phi^\pm$ be the preimage of $\Psi^\pm$. Then we have the I-invariant disjoint union $\Phi = \Phi^+ \cup \Phi^-$. Let δ_T and ρ_T be defined in terms of T and Φ^+. By Fact 4.1.4 the restriction of ϵ_M to $\mathsf{T}_x(k_F)$ is given by the image of (ρ_M, δ_M) in $X^*(\mathsf{T})$ and our job is to show that that image is (ρ_T, δ_T).

Given $O_M \in \Psi^+/I$ let $\dot{O}_M \in \Phi^+/I$ be its preimage – an I-invariant subset that could consist of multiple I-orbits. By assumption neither of these I-orbits is symmetric. Let $\dot{O}_M/I$ be the set of I-orbits in $\dot{O}_M$. As a T-module

$$V_{O_M} = \bigoplus_{\alpha \in \dot{O}_M} \mathfrak{g}_\alpha = \bigoplus_{O_T \in \dot{O}_M/I} V_{O_T},$$

where V_{O_T} is defined in the same way as V_{O_M}. Therefore the restriction of χ_{O_M} to $T(F)_x$ equals the product

$$\prod_{O_T \in \dot{O}_M/I} \chi_{O_T}.$$

We see $\delta_T = \delta_M|_T$ and $\rho_{T,\sigma} = \rho_{M,\sigma}|_T$, as claimed. □

Proposition 4.3.2. *Let $\gamma \in T(F)_x$. Then*

$$\epsilon_T(\gamma) = \prod_{\substack{\alpha \in \Phi_{asym}/\Sigma \\ t \in ord_x(\alpha)}} sgn_{k_\alpha^\times}(\alpha(\gamma)) \cdot \prod_{\substack{\alpha \in \Phi_{sym,unram}/\Gamma \\ t \in ord_x(\alpha)}} sgn_{k_\alpha^1}(\alpha(\gamma)).$$

Proof. We choose an I-invariant decomposition $\Phi = \Phi^+ \cup \Phi^-$ in such a way that for an asymmetric element $\alpha \in \Phi^+$ we have $\Gamma \cdot \alpha \subset \Phi^+$. We have by construction for $\sigma \in \Gamma/I$

$$\epsilon(\gamma,\sigma) = \prod_{\substack{O \in \Phi^+/I \\ \sigma^{-1}O \in \Phi^-/I}} \chi_O(\gamma) \cdot \frac{\sigma\sqrt{\prod\limits_{O\in\Phi^+/I} \chi_O(\gamma)}}{\sqrt{\prod\limits_{O\in\Phi^+/I} \chi_O(\gamma)}} \in \mu_2(k_F).$$

We will find $\dot\gamma \in \bar k_F^\times$ s.t. $\dot\gamma^{-1}\sigma(\dot\gamma) = \epsilon(\gamma,\sigma)$. Then we have $\epsilon(\gamma) = \mathrm{sgn}_{k^\times}(\dot\gamma^2)$ and we will show that this equals the right hand side of the proposition.

Let us first compute $\chi_O(\gamma)$. Choose $\alpha \in O$. Then we have the isomorphism

$$\mathrm{Ind}_{I_\alpha}^I \mathfrak{g}_\alpha \to V_O$$

and hence the $T(F^u)$-equivariant isomorphism

$$\mathfrak{g}_\alpha(F^u_\alpha) \to V_O(F^u), \qquad X \mapsto (\sigma(X))_{\sigma \in I/I_\alpha}.$$

This isomorphism restricts to an isomorphism

$$\mathfrak{g}_\alpha(F^u_\alpha)_{x,t:t+} \to V_O(F^u)_{x,t:t+}.$$

The left-hand side is zero if $t \notin \mathrm{ord}_x(\alpha)$, and a one-dimensional $\bar k_F$-vector space if $t \in \mathrm{ord}_x(\alpha)$. On the latter, $\gamma \in T(F)_x$ acts by multiplication by the image in $\bar k_F$ of $\alpha(\gamma) \in O_{F^u_\alpha}$. Therefore

$$\chi_O(\gamma) = \begin{cases} \alpha(\gamma) \in \bar k_F^\times, & t \in \mathrm{ord}_x(\alpha) \\ 1, & \text{else.} \end{cases}$$

For any $O \in \Phi^+/I$ let k_O be the extension of k_F fixed by the stabilizer of O in Γ/I. We can call k_O the field of definition of O. Choose $\alpha \in O$ and let $\gamma_O = \alpha(\gamma) \in \bar k_O^\times$. Then γ_O does not depend on the choice of α. Note that $k_O = k_\alpha$, the residue field of the field F_α of definition of α.

We make now the following choices. For any $O \in \Phi^+/I$ we choose $\dot\gamma_O = \sqrt{\gamma_O} \in \bar k_F^\times$. In addition, for $O \in \Phi^+/I$ that is ϕ-symmetric we have the degree 2 subfield $k_{\pm O}$ that is fixed by the stabilizer of $\{\pm O\}$ in Γ/I. Let k_O^1 be the elements of k_O whose norm to $k_{\pm O}$ equals 1. Then $\gamma_O \in k_O^1$ and we choose $\delta_O \in k_O^\times$ s.t. $\delta_O/\sigma_{\pm O}\delta_O = \gamma_O$, where $\sigma_{\pm O}$ is the generator of the Galois group of the quadratic extension $k_O/k_{\pm O}$. We make the latter choice so that for any $\sigma \in \Gamma/I$ we have $\delta_{\sigma O} = \sigma(\delta_O)$. In particular, $\delta_{-O} = \delta_{\sigma_{\pm O}O} = \sigma_{\pm O}\delta_O$, so that $\delta_O/\delta_{-O} = \gamma_O$.

Let

$$\dot\gamma = \prod_{O \in (\Phi^+/I)_{\mathrm{sym}}} \delta_O^{-1} \cdot \prod_{O \in \Phi^+/I} \dot\gamma_O.$$

We claim that $\dot\gamma^{-1}\sigma(\dot\gamma) = \epsilon(\gamma,\sigma)$. It is clear that the second product in the definition of $\dot\gamma$ contributes to $\dot\gamma^{-1}\sigma(\dot\gamma)$ precisely the second product in the formula for $\epsilon(\gamma,\sigma)$ above. So we compute the contribution of the first product:

$$\begin{aligned}
&= \left(\prod_{O\in(\Phi^+/I)_{\mathrm{sym}}} \delta_O\right)\cdot\sigma\left(\prod_{O\in(\Phi^+/I)_{\mathrm{sym}}} \delta_O^{-1}\right)\\
&= \left(\prod_{O\in(\Phi^+/I)_{\mathrm{sym}}} \delta_O\right)\cdot\left(\prod_{\sigma^{-1}O\in(\Phi^+/I)_{\mathrm{sym}}} \delta_O^{-1}\right)\\
&= \left(\prod_{\substack{O\in(\Phi^+/I)_{\mathrm{sym}}\\ \sigma^{-1}O\in(\Phi^-/I)_{\mathrm{sym}}}} \delta_O\right)\cdot\left(\prod_{\substack{O\in(\Phi^-/I)_{\mathrm{sym}}\\ \sigma^{-1}O\in(\Phi^+/I)_{\mathrm{sym}}}} \delta_O^{-1}\right)\\
&= \prod_{\substack{O\in(\Phi^+/I)_{\mathrm{sym}}\\ \sigma^{-1}O\in(\Phi^-/I)_{\mathrm{sym}}}} (\delta_O/\delta_{-O})\\
&= \prod_{\substack{O\in(\Phi^+/I)_{\mathrm{sym}}\\ \sigma^{-1}O\in(\Phi^-/I)_{\mathrm{sym}}}} \chi_O(\gamma).
\end{aligned}$$

This is the first product in the formula for $\epsilon(\gamma,\sigma)$; even though in that formula there was no restriction to symmetric elements in Φ^+/I, the non-symmetric elements do not contribute to the product by our choice of Φ^+.

Finally we compute $\epsilon(\gamma)=\mathrm{sgn}_{k^\times}(\dot\gamma^2)$. We use the relations $\dot\gamma_O^2=\gamma_O$, $\sigma\gamma_O=\gamma_{\sigma O}$ and for symmetric O also $\gamma_O=\delta_O/\delta_{-O}$ and $\sigma\delta_O=\delta_{\sigma O}$.

$$\begin{aligned}
&\mathrm{sgn}_{k^\times}\left(\prod_{O\in(\Phi^+/I)_{\mathrm{sym}}} \delta_O^{-2}\cdot\prod_{O\in\Phi^+/I}\dot\gamma_O^2\right)\\
=\;&\mathrm{sgn}_{k^\times}\left(\prod_{O\in(\Phi^+/I)_{\mathrm{sym}}} (\delta_O^{-2}\cdot\gamma_O)\cdot\prod_{O\in(\Phi^+/I)_{\mathrm{asym}}}\gamma_O\right)\\
=\;&\mathrm{sgn}_{k^\times}\left(\prod_{O\in(\Phi^+/I)_{\mathrm{sym}}} (\delta_O\delta_{-O})^{-1}\right)\cdot\mathrm{sgn}_{k^\times}\left(\prod_{O\in(\Phi^+/I)_{\mathrm{asym}}}\gamma_O\right)\\
=\;&\mathrm{sgn}_{k^\times}\left(\prod_{O\in(\Phi/I)_{\mathrm{sym}}} \delta_O^{-1}\right)\cdot\mathrm{sgn}_{k^\times}\left(\prod_{O\in(\Phi^+/I)_{\mathrm{asym}}}\gamma_O\right)
\end{aligned}$$

We break the second product into Γ/I-orbits in $(\Phi^+/I)_{\mathrm{asym}}$. Each such orbit contributes

$$\mathrm{sgn}_{k^\times}\left(\prod_{\sigma\in(\Gamma/I)/(\Gamma/I)_O}\sigma\gamma_O\right)=\mathrm{sgn}_{k^\times}(N_{k_O^\times/k^\times}(\gamma_O))=\mathrm{sgn}_{k_O^\times}(\gamma_O).$$

We break the first product into Γ/I-orbits in $(\Phi/I)_{\mathrm{sym}}$. In the same fashion we obtain from each such orbit

$$\mathrm{sgn}_{k^\times}\left(\prod_{\sigma\in(\Gamma/I)/(\Gamma/I)_O}\sigma\delta_O^{-1}\right)=\mathrm{sgn}_{k^\times}(N_{k_O^\times/k^\times}(\delta_O^{-1}))=\mathrm{sgn}_{k_O^\times}(\delta_O^{-1})=\mathrm{sgn}_{k_O^1}(\gamma_O^{-1}).$$

Finally, since Γ/I orbits in $(\Phi/I)_{\text{sym}}$ are the same as Γ-orbits in Φ_{sym}, Γ/I-orbits in $(\Phi/I)_{\text{asym}}$ are the same as Σ-orbits in Φ_{asym}, and there are no ramified symmetric elements in Φ, the proof is complete. $\square$

4.4 The construction of $\epsilon_t^{\text{sym,ram}}$

Define ${}_M\epsilon_t^{\text{sym,ram}} : \mathsf{M}_x(k_F) \to \{\pm 1\}$ to be the character constructed in §4.2 from the subset $R(Z_M, G)^{\text{sym,ram}} \subset R(Z_M, G)$. According to §4.3 its restriction to $\mathsf{T}(k_F)$ equals ${}_T\epsilon_t^{\text{sym,ram}}$ for any tame maximal torus $T \subset M$ with $x \in \mathcal{B}(T, F)$.

5 THE EXTENSION OF ϵ_0

5.1 The spinor norm of the adjoint representation

In what follows k is an arbitrary field of characteristic not 2. For a quadratic space (V, φ) over k we have the spinor norm $O(V, \varphi) \to k^\times/k^{\times,2}$. It is a homomorphism determined by the following property: For an anisotropic vector $v \in V$ the reflection $\tau_v \in O(V, \varphi)$ is defined by $\tau_v(w) = w - 2\frac{\beta(v,w)}{\beta(v,v)}v$ and its spinor norm is $\varphi(v)$. Here $\beta(v, w) = \varphi(v + w) - \varphi(v) - \varphi(w)$.

Remark 5.1.1. The spinor norm also has a cohomological interpretation. If we consider $O(V, \varphi)$ as an algebraic group defined over k, it has a central extension $\text{Pin}(V, \varphi)$ with kernel μ_2. The connecting homomorphism $O(V, \varphi)(k) \to H^1(\Gamma, \mu_2) = k^\times/k^{\times,2}$ is the spinor norm.

It is possible that two distinct quadratic forms on V that give the same orthogonal group $O(V, \varphi)$ give different spinor norms. For example, if we replace φ by aq for $a \in k^\times$, the spinor norm of each reflection will be multiplied by $a \in k^\times/k^{\times,2}$. However, the restriction of the spinor norm to $SO(V, \varphi)$ depends only on $SO(V, \varphi)$ and not on φ, because $\text{Spin}(V, \varphi)$ is determined by $SO(V, \varphi)$ as its simply connected cover. For example, elements of $SO(V, \varphi)$ are products of an even number of reflections, so replacing φ by aq multiplies the spinor norm of such element by a power of a^2, hence leaves it unchanged.

Let G be a possibly disconnected reductive group defined over k. We have the adjoint representation of G on the Lie algebra of the adjoint group of G°. Let φ be a non-degenerate symmetric bilinear form invariant under G. Call the Lie algebra equipped with this form L. Then we obtain a homomorphism $\mathsf{G} \to O(L)$. Composing this homomorphism with the spinor norm $O(L)(k) \to k^\times/k^{\times,2}$ we obtain a homomorphism

$$\mathsf{G}(k) \to k^\times/k^{\times,2}. \tag{5.1}$$

When k is a finite field of odd characteristic we can further compose with the sign character $k^\times/k^{\times,2} \to \{\pm 1\}$.

We shall be particularly interested in the following situation: We have a connected reductive group H over a finite field k of size q with a semi-simple automorphism θ of finite order prime to q which satisfies $\phi\theta\phi^{-1} = \theta^q$, where ϕ is Frobenius, and $\mathsf{G} = \mathsf{H}^\theta$. We choose a non-degenerate H-invariant symmetric bilinear form on $\mathfrak{h}$ and restrict it to $\mathfrak{g}$, and consider the corresponding character of $\mathsf{G}(k)$. We want to compute the restriction of that character to $\mathsf{S}^\theta(k)$ for any

θ-invariant maximal torus $\mathsf{S} \subset \mathsf{H}$ defined over k. Note that $\mathsf{S}^{\theta,\circ}$ need not be a maximal torus of G°; in some cases S^θ will be a finite group. More precisely, $\mathsf{S}^{\theta,\circ}$ is a maximal torus of G° if and only if it is contained in a θ-invariant Borel subgroup $\mathsf{B}_{\bar{k}} \subset \mathsf{H}_{\bar{k}}$, and every maximal torus of G° is obtained this way. This is a result of Steinberg [Ste68]; see also the summary [KS99, Theorem 1.1.A].

Lemma 5.1.2. *The restriction to $\mathfrak{g}$ of a non-degenerate H-invariant symmetric bilinear form on $\mathfrak{h}$ is a non-degenerate G-invariant symmetric bilinear form on $\mathfrak{g}$.*

Proof. Invariance is obvious. For non-degeneracy it is enough to check that the restriction to a maximal toral subalgebra of $\mathfrak{g}$ is non-degenerate. We can extend scalars to $\bar{k}$. The maximal tori of G° are of the form $\mathsf{S}^{\theta,\circ}$ for θ-invariant Borel pairs (S, B) of H. The inclusion $\mathfrak{s}^\theta \to \mathfrak{s}$ has the section $\mathrm{ord}(\theta)^{-1} \sum_i \theta^i$, where we are using that $\mathrm{ord}(\theta)$ is non-zero in k. Let β denote the bilinear form on $\mathfrak{h}$. Its restriction to $\mathfrak{s}$ is non-degenerate, so for any $X \in \mathfrak{s}^\theta$ there exists $Y \in \mathfrak{s}$ such that $\beta(X,Y) \neq 0$. The invariance of β implies $\beta(X,Y) = \beta(\theta(X), \theta(Y)) = \beta(X, \theta(Y))$ from which we infer $\beta(X,Y) = \beta(X, \bar{Y})$, where $\bar{Y} \in \mathfrak{s}^\theta$ is the image of Y under the above section. $\square$

To get started with the computation, we return to the setting of a general field k of characteristic not 2 and collect some general observations. Our convention will be that to a quadratic form φ we associate the symmetric bilinear form $\beta(x,y) = \varphi(x+y) - \varphi(x) - \varphi(y)$ and to a symmetric bilinear form β we associate the quadratic form $\varphi(x) = \beta(x,x)/2$.

Fact 5.1.3. *Let H be the hyperbolic plane, i.e. $H = k \oplus k$ equipped with the quadratic form $\varphi(x,y) = xy$. Then $SO(\varphi)(k) = k^\times$ and the spinor norm of $t \in k^\times$ is equal to the class of t in $k^\times/k^{\times,2}$.*

Proof. A vector $(x,y) \in H$ is anisotropic if and only if $x \neq 0$ and $y \neq 0$. The corresponding reflection is given by

$$\tau_{(x,y)}(z,w) = (z,w) - \frac{xw + zy}{xy}(x,y)$$

and in terms of the basis $((1,0),(0,1))$ has matrix

$$\begin{bmatrix} 0 & -x/y \\ -y/x & 0 \end{bmatrix}$$

and its spinor norm is $\varphi(x,y) = xy$. For $x = y = 1$ we see that

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

has spinor norm 1, and hence

$$\begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -t \\ -t^{-1} & 0 \end{bmatrix}$$

has spinor norm t. $\square$

Fact 5.1.4. *Let l/k be a quadratic extension and let σ be its non-trivial Galois automorphism. We equip the k-vector space l with the quadratic form $\varphi(e) = e\sigma(e)$. Then $SO(\varphi)(k) = l^1$ and for $e \in l^\times$ the spinor norm of $e/\sigma(e)$ is the class of $e\sigma(e)$ in $l^\times/l^{\times,2}$.*

Proof. The reflection τ_e has the formula

$$\tau_e(x) = x - \frac{e\sigma(x) + x\sigma(e)}{e\sigma(e)}e = -\frac{e}{\sigma(e)}\sigma(x)$$

and therefore multiplication of $e/\sigma(e)$ equals $\tau_e \circ \tau_1$ and its spinor norm is $\varphi(e) = e\sigma(e)$. $\square$

Fact 5.1.5. *Let $V = V_1 \oplus V_2$ be an orthogonal decomposition of a quadratic space. The restriction of the spinor norm of $O(V)(k)$ to $O(V_1)(k) \times O(V_2)(k)$ is the product of the spinor norms of $O(V_i)(k)$.*

Proof. For $v \in V_1$ the image of the reflection $\tau_v \in O(V_1)(k)$ in $O(V)(k)$ is equal to the reflection $\tau_{(v,0)}$. But the norms of v and $(v, 0)$ are equal. $\square$

Lemma 5.1.6. *Let l/k be a separable field extension, (V_l, q_l) a non-degenerate quadratic space over l. Let V_k be the k-vector space obtained from V_l by restricting the scalar multiplication to k and let $q_k = tr_{l/k} \circ q_l$. Then $O(V_l, q_l)(l) \subset O(V_k, q_k)(k)$ and $SO(V_l, q_l)(l) \subset SO(V_k, q_k)(k)$. We can therefore speak of the k-spinor norm and the l-spinor norm of an element of $O(V_l, q_l)(l)$.*

1. *The k-spinor norm of any element of $SO(V_l, q_l)(l)$ is equal to $N_{l/k}$ applied to its l-spinor norm.*
2. *If there exists $v \in V_l$ with $q_l(v) = 1$, then the k-spinor norm of any reflection in $O(V_l, q_l)(l)$ is equal to the product of the discriminant of l/k with $N_{l/k}$ applied to its l-spinor norm.*

Proof. Note first that q_k is non-degenerate: Fix a basis $e_1, \ldots, e_n$ of l/k and let $e_1^*, \ldots, e_n^*$ be the dual basis for the trace form. Let $v_1, \ldots, v_m$ be a basis of V_l and let $v_1^*, \ldots, v_m^*$ be the dual basis for β_l. Then $e_i v_j$ and $e_i^* v_j^*$ are bases of V_k and $\beta_k(e_i v_j, e_k^* v_l^*) = \mathrm{tr}_{l/k} e_i e_k \beta_l(v_j, v_l) = \mathrm{tr}(e_i e_k)\delta_{jl} = \delta_{ik}\delta_{jl}$, so they are dual to each other.

The map $SO(V_l, q_l)(l) \to SO(V_k, q_k)(k)$ comes from a map of algebraic groups $\mathrm{Res}_{l/k} SO(V_l, q_l) \to SO(V_k, q_k)$ that, for any k-algebra R, is defined by the identification $\mathrm{Res}_{l/k}(\mathrm{GL}(V_l))(R) = \mathrm{Aut}_R(V_l \otimes_k R) = \mathrm{Aut}_R(V_k \otimes_k R) = \mathrm{GL}(V_k)(R)$. This map fits into the commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Res}_{l/k}\mu_2 \arrow[r] \arrow[d] & \mathrm{Res}_{l/k}\mathrm{Spin}(V_l, q_l) \arrow[r] \arrow[d] & \mathrm{Res}_{l/k}\mathrm{SO}(V_l, q_l) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_2 \arrow[r] & \mathrm{Spin}(V_k, q_k) \arrow[r] & \mathrm{SO}(V_k, q_k) \arrow[r] & 1
\end{tikzcd}$$

The middle vertical map exists because its source is simply connected and is unique because its source is connected. Looking at maximal tori we see that the left vertical map is the norm map, and hence the first part of the lemma is proved.

For the second part, it is enough to compute the k-spinor norm of one reflection in $O(V_l, q_l)(l)$. We take the reflection along $v \in V_l$ with $q_l(v) = 1$. Its image in $O(V_k, q_k)(k)$ is not a reflection – it acts by -1 on the l-line $\langle v \rangle$, which is now an n-dimensional k-subspace of V_k. Let $e_1, \ldots, e_n$ be a basis of l/k orthogonal for the trace form. Then $e_1 v, \ldots, e_n v$ is a basis of the k-subspace $\langle v \rangle$ and is orthogonal for the symmetric bilinear form β_k associated to q_k. We have used here that $q_l(v) \in k$. Since $q_l(v) = 1$, the k-spinor norm of that reflection is the product $\prod_{i=1}^n \mathrm{tr}_{l/k}(e_i^2)$, which is the discriminant of l/k. $\square$

Corollary 5.1.7. *Let l/k be a finite extension. Consider the quadratic form $\varphi(x,y) = tr_{l/k}(\gamma xy)$ on the k-vector space $l \oplus l$ for some $\gamma \in l^\times$. Then the spinor norm of $e \in l^\times \subset SO(\varphi)$ is equal to $N_{l/k}(e)$.*

Proof. Apply Lemma 5.1.6, Fact 5.1.3, and Remark 5.1.1. □

Fact 5.1.8. *Let $l/l_\pm/k$ be a tower of finite extensions with $[l : l_\pm] = 2$ and let σ be the non-trivial Galois automorphism of $l/l_\pm$. Consider the quadratic form $\varphi(x) = tr_{l_\pm/k}(\gamma x\sigma(x))$ on the k-vector space l for some $\gamma \in l_\pm^\times$. Then the spinor norm of $e/\sigma(e) \in l^1 \subset SO(\varphi)$ is equal to $N_{l/k}(e)$.*

Proof. Apply Lemma 5.1.6, Fact 5.1.4, and Remark 5.1.1. □

Fact 5.1.9. *Assume that k is a finite field of odd characteristic. Let l/k be a finite extension. Consider on the k-vector space l the quadratic form $\varphi(x) = tr_{l/k}(\gamma x^2)$, for some $\gamma \in l^\times$. The spinor norm of $-1 \in l^\times$ is equal to $N_{l/k}(\gamma) \cdot \eta^{[l:k]+1}$, where $\eta \in k^\times/k^{\times,2}$ is the non-trivial element.*

Proof. Every quadratic form over the finite field k represents $1 \in k$, see [JMV90, Lemma 1]. Therefore we can apply Lemma 5.1.6 to $V_l = l$ and $q_l(x) = \gamma x^2$ and see that the spinor norm of -1 is equal to $\mathrm{disc}(l/k) \cdot N_{l/k}(\gamma)$. From [JMV90, Theorem 1] we see that l/k has an orthonormal basis for the trace form, and hence trivial discriminant, if and only if $[l : k]$ is odd. □

After these general observations we return to the connected reductive group H over the finite field k and its semi-simple automorphism θ. Let $\mathsf{S} \subset \mathsf{H}$ be a θ-invariant maximal torus. Recall that we have fixed, for now arbitrarily, a non-degenerate H-invariant symmetric bilinear form β on $\mathfrak{h}$ and have the corresponding quadratic form φ. We will see that the particular choice of β will matter only for the contribution of the ramified symmetric roots.

Let $\Theta = \langle\theta\rangle$ and let Γ be the Galois group of k. Then $\Theta \rtimes \Gamma$ acts on $R(\mathsf{S},\mathsf{H})$. For $\alpha \in R(\mathsf{S},\mathsf{H})$ we have the exact sequence

$$1 \to \Theta_\alpha \to (\Theta \rtimes \Gamma)_\alpha \to \Gamma_A \to 1,$$

where $A = \Theta \cdot \alpha \subset R(\mathsf{S},\mathsf{H})$. Every element of A induces the same function on S^θ and therefore for $t \in \mathsf{S}^\theta(k)$ we have $\alpha(t) \in k_A^\times$, where k_A is the fixed field of Γ_A. The group Θ_α acts on the 1-dimensional $\bar{k}$-vector space $\mathfrak{h}_\alpha(\bar{k})$ by $\bar{k}$-linear automorphisms. We call α *relevant* if this action is trivial. Then the action of $(\Theta \rtimes \Gamma)_\alpha$ on $\mathfrak{h}_\alpha(\bar{k})$ descends to an action of Γ_A and this action endows the 1-dimensional $\bar{k}$-vector space $\mathfrak{h}_\alpha(\bar{k})$ with a descent datum to the field k_A. We write $\mathfrak{h}_\alpha(k_A)$ for the corresponding 1-dimensional k_A-vector space.

If $A \neq -A$, but $\Gamma \cdot A = -\Gamma \cdot A$, then $N_{k_A/k_{\pm A}}(\alpha(t)) = 1$, where $k_{\pm A}$ is the fixed field of $\Gamma_{\pm A}$, a subfield of k_A of degree 2. We will call such α *unramified symmetric*. We can choose $\delta_\alpha \in k_A^\times$ with $\delta_\alpha/\sigma_{\pm A}(\delta_\alpha) = \alpha(t)$, where $\sigma_{\pm A}$ is the non-trivial automorphism of $k_A/k_{\pm A}$. The class of $N_{k_A/k}(\delta_\alpha)$ in $k^\times/k^{\times,2}$ does not depend on the choice of δ_α.

If $A = -A$, then $\alpha(t) \in \{\pm 1\}$. We will call such α *ramified symmetric*. In that case choose $\tau \in \Theta_{\pm\alpha} \smallsetminus \Theta_\alpha$. For any non-zero $X \in \mathfrak{h}_\alpha(k_A)$ we have the element $\tau(X) \in \mathfrak{h}_{-\alpha}(k_A)$ and hence the element $\gamma_1 \in k_A^\times$ defined by $[X, \tau(X)] = \gamma_1 H_\alpha$.

The square class of γ_1 is independent of the choice of X. Let $\gamma_2 = \varphi(H_\alpha)$, where φ is the chosen quadratic form on $\mathfrak{h}$. Then $\gamma_2 \in k_A^\times$ and hence $\gamma_\alpha = \gamma_1\gamma_2 \in k_A^\times$.

If the $(\Theta \rtimes \Gamma)$-orbit of α is not equal to its negative we will call α *asymmetric*.

Proposition 5.1.10. *Let φ be a non-degenerate H-invariant quadratic form on $\mathfrak{h}$. The spinor norm of the action of $t \in \mathsf{S}^\theta(k)$ on $\mathfrak{h}^\theta(k)$ is equal to*

$$\prod_{\alpha \in R(\mathsf{S},\mathsf{H})_{asym}/\Theta\rtimes\Sigma} N_{k_A/k}(\alpha(t)) \cdot \prod_{\alpha \in R(\mathsf{S},\mathsf{H})_{sym.unram}/\Theta\rtimes\Gamma} N_{k_A/k}(\delta_\alpha) \cdot \prod_{\substack{\alpha \in R(\mathsf{S},\mathsf{H})_{sym.ram}/\Theta\rtimes\Gamma \\ \alpha(t)=-1}} (N_{k_A/k}(\gamma_\alpha) \cdot \eta^{[k_A:k]+1}),$$

where the products are taken only over relevant roots and $\eta \in k^\times \smallsetminus k^{\times,2}$.

Proof. We have

$$\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{n}, \qquad \mathfrak{n} = \bigoplus_{B \in R(\mathsf{S},\mathsf{H})/\Theta\rtimes\Gamma} \mathfrak{h}_B, \qquad \mathfrak{h}_B = \bigoplus_{\alpha \in B} \mathfrak{h}_\alpha$$

and therefore

$$\mathfrak{h}^\theta(k) = \mathfrak{s}^\theta(k) \oplus \mathfrak{n}^\theta(k), \qquad \mathfrak{n}^\theta(k) = \bigoplus_B \mathfrak{h}_B^\theta(k).$$

If we choose $\alpha \in B$ then as a $\Theta \rtimes \Gamma$-module we have

$$\mathfrak{h}_B(\bar{k}) = \mathrm{Ind}_{(\Theta\rtimes\Gamma)_\alpha}^{\Theta\rtimes\Gamma} \mathfrak{h}_\alpha(\bar{k}).$$

We have $\mathfrak{h}_B(k) \neq \{0\}$ if and only if α is relevant. Assuming that α is relevant, we have $\mathfrak{h}_B^\theta(k) = \mathfrak{h}_\alpha(k_A)$ and therefore

$$\mathfrak{n}^\theta(k) = \bigoplus_{\alpha \in R(\mathsf{S},\mathsf{H})/\Theta\rtimes\Gamma} \mathfrak{h}_\alpha^\theta(k_A).$$

The direct sum $\mathfrak{s}^\theta(k) \oplus \mathfrak{n}^\theta(k)$ is orthogonal with respect to the invariant symmetric bilinear form β. The action of $\mathsf{S}^\theta(k)$ also respects it, and is trivial on $\mathfrak{t}^\theta(k)$. We can apply Fact 5.1.5 and conclude that the spinor norm we are computing is for the action of $\mathsf{S}^\theta(k)$ on $\mathfrak{n}^\theta(k)$. The quadratic space $\mathfrak{n}^\theta(k)$ is a sum over $R(\mathsf{S},\mathsf{H})/\Theta \rtimes \Gamma$, but the summands are not necessarily orthogonal to each other. In order to achieve this we need to combine each asymmetric $\Theta \rtimes \Gamma$-orbit with its negative. Then the sum becomes orthogonal. Applying Fact 5.1.5 again we see that we are computing the product of the spinor norms for the action of $\mathsf{S}^\theta(k)$ on each individual summand of that sum.

Before we discuss the individual summands we note that $H_\alpha \in \mathfrak{s}(k_A)$ and therefore for $X \in \mathfrak{h}_\alpha(k_A)$ the unique $Y \in \mathfrak{h}_{-\alpha}(\bar{k})$ with $[X,Y] = H_\alpha$ lies in $\mathfrak{h}_{-\alpha}(k_A)$. In the following discussion, α will be tacitly assumed relevant.

Consider first $\alpha \in R(\mathsf{S},\mathsf{H})$ whose $\Theta \rtimes \Gamma$-orbit B is asymmetric. The contribution of the pair $\{B, -B\}$ to the quadratic space $\mathfrak{n}^\theta(k)$ equals $\mathfrak{h}_B^\theta(k) \oplus \mathfrak{h}_{-B}^\theta(k) = \mathfrak{h}_\alpha(k_A) \oplus \mathfrak{h}_{-\alpha}(k_A)$. Choose non-zero (X,Y) in that space with $[X,Y] = H_\alpha$. Then $\gamma := \varphi(H_\alpha) = \beta(H_\alpha, H_\alpha)/2 = \beta(X,Y) \in k_A$ and the basis (X,Y) gives an isomorphism between the quadratic space $(\mathfrak{h}_\alpha(k_A) \oplus \mathfrak{h}_{-\alpha}(k_A), \varphi)$ and $k_A \oplus k_A$ with the quadratic form $(x,y) \mapsto \mathrm{tr}_{k_A/k}(\gamma xy)$. The action of $t \in \mathsf{S}^\theta(k)$ on that space is via $(\alpha(t), \alpha(t)^{-1})$. Applying Corollary 5.1.7 we conclude that the contribution of this factor is $N_{k_A/k}(\alpha(t))$.

Next consider $\alpha \in R(\mathsf{S},\mathsf{H})$ whose Θ-orbit A is asymmetric, but whose $\Theta \rtimes \Gamma$-orbit B is symmetric. The contribution of B to the quadratic space $\mathfrak{n}^\theta(k)$ is $\mathfrak{h}_B^\theta(k) = \mathfrak{h}_\alpha(k_A)$. Choose a non-zero $X \in \mathfrak{h}_\alpha(k_A)$. Let $\dot\sigma_\pm \in (\Theta \rtimes \Gamma)_{\pm\alpha}$ be a lift of $\sigma_\pm \in \Gamma_{\pm A}$. Then the identifications $\mathfrak{h}_\alpha(k_A) = \mathfrak{h}_B^\theta(k) = \mathfrak{h}_{-\alpha}(k_A)$ send X to the element $Y \in \mathfrak{h}_{-\alpha}(k_A)$ given by $Y = \dot\sigma_\pm(X)$. Let $\gamma_1 \in k_A^\times$ be defined by $[X,Y] = \gamma_1 H_\alpha$. Since $\dot\sigma_\pm$ switches X and Y and negates H_α we see that it fixes γ_1, hence $\gamma_1 \in k_{\pm A}^\times$. We have $\beta(X,Y) = \gamma_1\gamma_2$, where $\gamma_2 = \varphi(H_\alpha)$. Note that $\gamma_2 = \varphi(H_\alpha) = \beta(H_\alpha,H_\alpha)/2$ is fixed by $\sigma_\pm$. Therefore $\gamma = \gamma_1\gamma_2 \in k_{\pm A}^\times$. The basis X gives an isomorphism from the quadratic space $(\mathfrak{h}_B^\theta(k),\varphi)$ to the space k_A with the quadratic form $x \mapsto \mathrm{tr}_{k_{\pm A}/k}(\gamma x \sigma_\pm(x))$. The action of $t \in \mathsf{S}^\theta(k)$ on this space is by multiplication by $\alpha(t) \in k_A^1$. According to Fact 5.1.8 we see that the contribution of this factor is $N_{k_A/k}(\delta_A)$, for $\delta_A \in k_A^\times$ with $\delta_A/\sigma_{\pm A}(\delta_A) = \alpha(t)$.

Finally consider $\alpha \in R(\mathsf{S},\mathsf{H})$ whose Θ-orbit A is symmetric. The contribution of its $\Theta \rtimes \Gamma$-orbit B to the quadratic space $\mathfrak{n}^\theta(k)$ is $\mathfrak{h}_B^\theta(k) = \mathfrak{h}_\alpha(k_A)$. Choose a non-zero $X \in \mathfrak{h}_\alpha(k_A)$ and $\tau \in \Theta_{\pm\alpha} \smallsetminus \Theta_\alpha$. Then the identifications $\mathfrak{h}_\alpha(k_A) = \mathfrak{h}_B^\theta(k) = \mathfrak{h}_{-\alpha}(k_A)$ send X to $Y = \tau(X) \in \mathfrak{h}_{-\alpha}(k_A)$. Let $\gamma_1 \in k_A^\times$ be defined by $[X,Y] = \gamma_1 H_\alpha$. Then we have $\beta(X,Y) = \gamma_1\gamma_2$, where $\gamma_2 = \varphi(H_\alpha)$. Set $\gamma = \gamma_1\gamma_2$. The basis X gives an isomorphism from the quadratic space $(\mathfrak{h}_B^\theta(k),\varphi)$ to the space k_A with the quadratic form $x \mapsto \mathrm{tr}_{k_{\pm A}/k}(\gamma x^2)$. The action of $t \in \mathsf{S}^\theta(k)$ on this space by multiplication by $\alpha(t) \in \{\pm 1\}$. If $\alpha(t) = +1$ then the spinor norm is 1. If $\alpha(t) = -1$ then by Fact 5.1.9 the spinor norm is equal to $N_{l/k}(\gamma)\eta^{[k_A:k]+1}$. $\square$

5.2 Inflating the spinor character to the p-adic field

Let now φ be a G-invariant non-degenerate quadratic form on $\mathfrak{g}$ whose restriction to $\mathfrak{g}(F^{\mathrm{tr}})_{x,0}$ is integrally non-degenerate. We consider the character (5.1) for the group G_x and compose it with $\mathrm{sgn}_{k^\times} : k^\times/k^{\times,2} \to \{\pm 1\}$. We want to evaluate this composition on subgroups $\mathsf{T}(k_F)$ for any tame maximal torus $T \subset G$ with $x \in \mathcal{B}^{\mathrm{enl}}(T,F)$.

Proposition 5.2.1. *The restriction of this character to $\mathsf{T}(k_F)$ is equal to the product*

$$ {}_T\epsilon_0^{G/T} \cdot \epsilon_{f,r} \cdot \epsilon_{\varphi,r}^{1,G/T}, $$

where ${}_T\epsilon_0^{G/T} : T(F)_b \to \{\pm 1\}$ is relative to the tame Levi subgroup $T \subset G$, and $\epsilon_{\varphi,r}^{1,G/T} : T(F)_b \to \{\pm 1\}$ is defined by

$$ \epsilon_{\varphi,r}^{1,G/T}(t) = \prod_{\substack{\alpha \in R(T,G)_{sym.ram}/\Gamma \\ \alpha(t) \in -1+\mathfrak{p}_\alpha}} sgn_{k_\alpha^\times}(\varphi(H_\alpha))(-1)^{[k_\alpha:k]+1}, $$

and φ is the invariant quadratic form on G used for the spinor norm.

Proof. Let E/F be the splitting extension of T and let F'/F be the maximal unramified subextension. We write H and S for the base changes of G and T to E. We have the disconnected reductive group H_x defined over $k_E = k_{F'}$. In fact we can give it structure over k_F by choosing a Frobenius element in $\phi \in \Gamma_{E/F}$. The action of $\Gamma_{E/F'}$ on H_x is via an algebraic automorphism θ and $\mathsf{G}_x = \mathsf{H}_x^\theta$. We have $\phi\theta\phi^{-1} = \theta^q$. Let S denote the special fiber of the connected Neron model of $S \times E$; it is a maximal torus of H_x° defined over k_F. We have $\mathsf{S}^\theta(k_F) = \mathsf{T}(k_F) = T(F)_b/T(F)_{0+}$ and $\mathsf{H}_x^\theta(k_F) = \mathsf{G}_x(k_F) = G(F)_x/G(F)_{x,0+}$.

Note that while the k_F-structure on H_x and S depends on the choice of ϕ, the k_F-structures on H_x^θ and S^θ do not.

By assumption, the quadratic form φ descends to a non-degenerate H-invariant quadratic form on the Lie-algebra of H°. We can therefore apply Proposition 5.1.10 to H_x° and S. We have $R(\mathsf{S},\mathsf{H}_x^\circ)=\{\alpha\in R(T,G)|0\in\mathrm{ord}_x(\alpha)\}$. The first two products give

$$\prod_{\substack{\alpha\in R(T,G)_{\text{asym}}/\Sigma\\ 0\in\mathrm{ord}_x(\alpha)}}\mathrm{sgn}_{k_\alpha^\times}(\alpha(t))\cdot\prod_{\substack{\alpha\in R(T,G)_{\text{sym.unram}}/\Gamma\\ 0\in\mathrm{ord}_x(\alpha)}}\mathrm{sgn}_{k_\alpha^1}(\alpha(t)).$$

The third product runs over the set $\{\alpha\in R(T,G)_{\text{sym.ram}}/\Gamma|\alpha(t)\in -1+\mathfrak{p}_{F_\alpha}\}$. Note that the condition $0\in\mathrm{ord}_x(\alpha)$ is automatic by [Kal19, Proposition 4.5.1]. Consider the contribution of a fixed α. It is

$$\mathrm{sgn}_{k_\alpha^\times}([X_\alpha,\tau(X_\alpha)]/H_\alpha)\cdot\mathrm{sgn}_{k_\alpha^\times}(\varphi(H_\alpha))\cdot(-1)^{[k_\alpha:k]+1}.$$

Here X_α is a chosen non-zero element of $\mathfrak{g}_\alpha(F_\alpha)_{x,0}\smallsetminus\mathfrak{g}_\alpha(F_\alpha)_{x,0+}$ and τ is the generator of $\mathrm{Gal}(F_\alpha/F_{\pm\alpha})$. Since both $[X_\alpha,\tau(X_\alpha)]$ and H_α are elements of $\mathfrak{s}(F_\alpha)_0\smallsetminus\mathfrak{s}(F_\alpha)_{0+}$ we see that $[X_\alpha,\tau(X_\alpha)]/H_\alpha\in O_{F_{\pm\alpha}}^\times$. Now all the maps below are isomorphisms

$$k_\alpha^\times/k_\alpha^{\times,2}\leftarrow O_{F_{\pm}\alpha}^\times/N_{F_\alpha/F_{\pm\alpha}}(O_{F_\alpha}^\times)\to F_{\pm\alpha}^\times/N_{F_\alpha/F_{\pm\alpha}}(F_\alpha^\times)\to\{\pm1\}$$

so their composition is the sign character on $k_\alpha^\times$. It follows that

$$\mathrm{sgn}_{k_\alpha^\times}([X_\alpha,\tau(X_\alpha)]/H_\alpha)=f_{(G,T)}(\alpha).$$

□

5.3 Construction and properties of ϵ_0

Define ${}_M\epsilon_0$ to be the character $\epsilon_{\mathsf{G}_x}/\epsilon_{\mathsf{M}_x}$ of §5.1. By Proposition 5.2.1 the restriction to $\mathsf{T}(k_F)$ equals ${}_T\epsilon_0\cdot\epsilon_{f,r}\cdot\epsilon_{\varphi,r}^1$ for any tame maximal torus $T\subset M$ with $x\in\mathcal{B}^{\text{enl}}(T,F)$. Here ${}_T\epsilon_0$ is relative to the tame Levi subgroup $M\subset G$, and $\epsilon_{\varphi,r}^1$ is defined by

$$\epsilon_{\varphi,r}^1(t)=\prod_{\substack{\alpha\in R(T,G/M)_{\text{sym.ram}}/\Gamma\\ \alpha(t)\in-1+\mathfrak{p}_\alpha}}\mathrm{sgn}_{k_\alpha^\times}(\varphi(H_\alpha))(-1)^{[k_\alpha:k]+1},$$

6 The extension of $\epsilon_{\text{SYM,RAM}}$

Consider

$$V=\bigoplus_{O\in R(Z_M,G)_{\text{sym,ram}}/\Gamma}\bigoplus_{t\in(0,s)}\mathfrak{g}_O(F)_{x,t:t+}.$$

This is a k_F-vector space with action of $M(F)_x$. Let

$${}_M\epsilon_{\text{sym.ram}}(m)=\mathrm{sgn}_{k^\times}(\det(m|V)).$$

Proposition 6.0.1. *For any tamely ramified maximal torus $T\subset M$ with $x\in\mathcal{B}(T,F)$ we have*

$${}_M\epsilon_{\textit{sym.ram}}|_{T(F)_c}={}_T\epsilon_{\textit{sym.ram}}\cdot\epsilon_r^2,$$

where

$$\epsilon_r^2(t) = \prod_{\substack{\alpha\in R(T,G/M)_{\textit{sym.ram}}/\Gamma\\ \alpha(t)\in -1+\mathfrak{p}_\alpha}} \mathit{sgn}_{k_\alpha^\times}(-1)^{e_\alpha s-\frac{1}{2}},$$

noting that $e_\alpha s \in \frac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z}$.

Proof. From the definition of V we have

$${}_M\epsilon_{\text{sym.ram}}(t) = \prod_{\substack{\alpha\in R(T,G/M)/\Gamma\\ \alpha_0\in R(Z_M,G)_{\text{sym.ram}}}} \text{sgn}_{k^\times}(N_{k_\alpha/k}(\alpha(t)))^{|\text{ord}_x(\alpha)\cap(0,s)|}.$$

Of course $\text{sgn}_{k^\times}(N_{k_\alpha/k}(\alpha(t))) = \text{sgn}_{k_\alpha^\times}(\alpha(t))$. Consider first the contribution of a pair of asymmetric Γ-orbits in $R(T,G/M)$. Since $\text{sgn}_{k_\alpha^\times}(\alpha(t))$ is insensitive to replacing α by $-\alpha$ this pair contributes

$$\text{sgn}_{k_\alpha^\times}(\alpha(t))^{|\text{ord}_x(\alpha)\cap((-s,0)\cup(0,s))|}.$$

We compute the size of the intersection $\text{ord}_x(\alpha)\cap((-s,0)\cup(0,s))$, which is the same as the size of the intersection $e_\alpha\text{ord}_x(\alpha)\cap((-se_\alpha,0)\cup(0,se_\alpha))$.

If $0\in\text{ord}_x(\alpha)$ then $e_\alpha\text{ord}_x(\alpha)=\mathbb{Z}$. If $s\in\text{ord}_x(\alpha)$ then $e_\alpha\text{ord}_x(\alpha)=\mathbb{Z}+\frac{1}{2}$ if re_α is odd and $e_\alpha\text{ord}_x(\alpha)=\mathbb{Z}$ is re_α is even. Either way, the above intersection is a finite set on which the action of the group $\{\pm 1\}$ by multiplication has no fixed points, so has even cardinality.

If on the other hand $0\notin\text{ord}_x(\alpha)\not\ni s$, the set $e_\alpha\text{ord}_x(\alpha)$ is a $\mathbb{Z}$-torsor that is not symmetric around 0. If $2\nmid e(\alpha/\alpha_0)$ then by Fact 3.0.3 the quantity se_α is a half-integer that is not an integer, and therefore the above intersection has odd cardinality. If on the other hand $2|e(\alpha/\alpha_0)$ then se_α is an integer and the above intersection has even cardinality.

We conclude that the contribution of the asymmetric roots is

$$\prod_{\substack{\alpha\in R(T,G/M)_{\text{asym}}/\Sigma\\ \alpha_0\in R(Z_M,G)_{\text{sym.ram}}\\ 0\notin\text{ord}_x(\alpha)\not\ni s\\ 2\nmid e(\alpha/\alpha_0)}} \text{sgn}_{k_\alpha^\times}(\alpha(t)).$$

Now note that any $\alpha\in R(T,G/M)$ with $\alpha_0\in R(Z_M,G)_{\text{sym.ram}}$, $0\notin\text{ord}_x(\alpha)\not\ni s$, and $2\nmid e(\alpha/\alpha_0)$, is automatically asymmetric, because $e_\alpha\text{ord}_x(\alpha)$ is a $\mathbb{Z}$-torsor containing neither 0 nor the non-integer half-integer se_α, and hence $\text{ord}_x(\alpha)\neq\text{ord}_x(-\alpha)$. Therefore the above product equals ${}_T\epsilon_{\text{sym.ram}}(t)$.

Next consider an unramified symmetric α. Its contribution is trivial, because $\alpha(t)\in k_\alpha^\times$ has trivial norm to $k_{\pm\alpha}^\times$, hence also trivial norm to $k^\times$.

Finally consider a ramified symmetric α. Then by [Kal19, Proposition 4.5.1] we have $0\in\text{ord}_x(\alpha)$, while by Fact 3.0.3 we have $e_\alpha s\in\frac{1}{2}\mathbb{Z}\smallsetminus\mathbb{Z}$. Therefore $e_\alpha\text{ord}_x(\alpha)=\mathbb{Z}$ and the size of the intersection $e_\alpha\text{ord}_x(\alpha)\cap(0,e_\alpha s)$ is $e_\alpha s-\frac{1}{2}$. $\square$

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