Corrections in "Topological central extensions of semi-simple groups over local fields" by Gopal Prasad and M. S. Raghunathan, Ann. Math. 119 (1984)

1. At the begining of 7.14 add the following sentence:

"We assume that the K-root system of G has roots of unequal lengths."

In this paragraph delete the sentence " Moreover, it is easy to see that $\mathsf{C}\cdot\mathsf{Z} = \mathsf{C}^*$."

2. In the remark on p. 211, replace "In both the cases, $\mathsf{C}^* = \mathsf{C}$ and" by "If G is of type A, $\mathsf{C}^* = \mathsf{C}$. In case G/K is of outer type D_n, C^* is diconnected and C is of index 2 in it. In both the cases"

3. In the first line of the fourth paragraph on p. 218, replace "with kernel" by "which is an isomorphism if n is odd, and in case n is even its kernel is".

4. In the statement and the proof of Lemma 7.27, replace F with $\mathfrak{f}$ every where.

5. Replace the proof of Proposition 7.28 with the following.

Proof. We note that in all cases $\mathfrak{L}_1(\mathfrak{f})$ is an irreducible $\mathbf{Z}\mathsf{M}(\mathfrak{f})$-module. Now Lemma 7.27 implies the proposition.

6. Delete the first paragraph of 7.36 add the following sentence after the second sentence of the second paragraph.

Now in case G splits over K and its K-root system has roots of unequal lengths, define $G^{\cdot}$ to be the algebraic subgroup generated by the root groups, $U_{\dot{\omega}}$, $\omega \in \Omega$.

After the third sentence of the third paragraph of 7.36 add the following sentence.

Let $G^{\cdot}$ be the subgroup generated by $U_{\dot{\omega}}$, $\omega \in \Omega$, and $U_{\dot{\beta}}$.

7. Replace the statement and the proof of Lemma 7.37 with the following.

7.37 Lemma. *Assume that G does not split over K and $p = 2$. Then the following short exact sequence*

$$1 \to \mathscr{P}_2/(\mathscr{P}_1, \mathscr{P}_1) \to \mathscr{P}_1/(\mathscr{P}_1, \mathscr{P}_1) \to \mathscr{P}_1/\mathscr{P}_2 \to 1$$

does not admit an $\mathsf{M}(\mathfrak{f})$-equivariant splitting if either (i) $\#\mathfrak{f} > 2$, or (ii) $\#\mathfrak{f} = 2$ and the K-root system of G is of type B_{n+1} for $n \geqslant 2$.

If the K-root system of G is of type C_{n+1}, $n \geqslant 1$ and $\#\mathfrak{f} = 2$, then the above short exact sequence does admit an $\mathsf{M}(\mathfrak{f})$-equivariant splitting.

Proof. We shall identify $\mathscr{P}_1/\mathscr{P}_2$, and $\mathscr{P}_2/(\mathscr{P}_1, \mathscr{P}_1)$, with $\mathfrak{L}_1(\mathfrak{f})$, and ▪$\mathfrak{L}_2(\mathfrak{f})$ respectively (cf. 7.34 and 7.35). There is a natural $\mathbf{Z}[\mathsf{T}(\mathfrak{f})]$-module identification of $\mathscr{P}_1/(\mathscr{P}_1, \mathscr{P}_1)$ with $\mathfrak{L}_1(\mathfrak{f}) \oplus {}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$. For an affine root ψ, let u_ψ be the image in $\mathscr{P}/(\mathscr{P}_1, \mathscr{P}_1)$ of the root group of $\mathscr{P}$ corresponding to ψ.

Assume, if possible, that there is a $\mathsf{M}(\mathfrak{f})$-equivariant splitting $\sigma : \mathfrak{L}_1(\mathfrak{f}) = \mathscr{P}_1/\mathscr{P}_2 \to \mathscr{P}_1/(\mathscr{P}_1, \mathscr{P}_1)$. We first take up the case where the K-root system of G is of type B_{n+1}, for $n \geqslant 2$. Let $\Omega = \{\omega, \omega'\}$ and let $\beta = \sum_{\alpha\in\Delta-\Omega} \alpha$. Then $\delta = \omega+\beta+\omega'$. By a direct computation we see that since the gradients of $\beta + 2\omega$, and $\beta + 2\omega'$ are respectively $\dot{\omega} - \dot{\omega}'$ and $\dot{\omega}' - \dot{\omega}$, for arbitrary $\mathfrak{f}$, the subspace of ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ consisting of vectors fixed under the kernel in $\mathsf{T}(\mathfrak{f})$ of $\dot{\omega}$ and $\dot{\omega}'$ is precisely $\mathfrak{L}_{\beta+2\omega} \oplus \mathfrak{L}_{\beta+2\omega'}$.

As the intersection of $\mathscr{P}_2/(\mathscr{P}_1, \mathscr{P}_1) = {}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ with the image of σ is trivial, from the observations in the preceding paragraph we infer that for all t,

$$\sigma(u_\omega(t)\, u_{\omega'}(\bar{t})) = u_\omega(t)\, u_{\omega'}(\bar{t}) f(t),$$

where $f(t) \in (\mathfrak{L}_{2\omega+\beta} \oplus \mathfrak{L}_{2\omega'+\beta})(\mathfrak{f})\,(\subset \mathscr{P}_2/(\mathscr{P}_1, \mathscr{P}_1) = {}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f}))$. Let γ (resp. γ') be the affine root adjacent to ω (resp. ω') in the Dynkin diagram. These affine roots are long and conjugate to each other under the Galois group of K/k. Now we apply σ to the following commutator, for $s, t \in \mathsf{F}$:

$$(u_\gamma(s)\, u_{\gamma'}(\bar{s})) \cdot (u_\omega(t)\, u_{\omega'}(\bar{t})) \cdot (u_\gamma(s)\, u_{\gamma'}(\bar{s}))^{-1} \cdot (u_\omega(t)\, u_{\omega'}(\bar{t}))^{-1}$$

and use the $\mathsf{M}(\mathfrak{f})$ equivariance of σ, we obtain that (note that $(u_\gamma(s)\, u_{\gamma'}(\bar{s})) \in \mathsf{M}(\mathfrak{f})$ and it commutes with $f(t)$).

$$\begin{aligned}
&(u_\gamma(s)\, u_{\gamma'}(\bar{s})) \cdot \big(u_\omega(t)\, u_{\omega'}(\bar{t}) f(t)\big) \cdot (u_\gamma(s)\, u_{\gamma'}(\bar{s}))^{-1} \cdot \big(u_\omega(t)\, u_{\omega'}(\bar{t}) f(t)\big)^{-1} \\
&= (u_\gamma(s)\, u_{\gamma'}(\bar{s})) \cdot \big(u_\omega(t)\, u_{\omega'}(\bar{t})\big) \cdot (u_\gamma(s)\, u_{\gamma'}(\bar{s}))^{-1} \cdot \big(u_\omega(t)\, u_{\omega'}(\bar{t})\big)^{-1} \\
&= \big(u_{\omega+\gamma}(st)\, u_{\omega'+\gamma'}(\bar{s}\bar{t})\big) \cdot \big(u_{2\omega+\gamma}(st^2)\, u_{2\omega'+\gamma'}(\bar{s}\bar{t}^2)\big) \quad \text{in } \mathscr{P}_1/(\mathscr{P}_1, \mathscr{P}_1).
\end{aligned}$$

Taking $x = st$, we see that

$$(u_{\omega+\gamma}(x)\, u_{\omega'+\gamma'}(\bar{x})) \cdot (u_{2\omega+\gamma}(x^2/s)\, u_{2\omega'+\gamma'}(\bar{x}^2/\bar{s}))$$

lies in the image of σ for all $s, x \in \mathsf{F}$. Now fixing x and varying s over $\mathsf{F}^\times$, we see that a nonzero element of ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ lies in the image of σ (note that $u_{2\omega+\gamma}(y)\, u_{2\omega'+\gamma'}(\bar{y})) \in {}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ for every $y \in \mathsf{F}$). We have thus arrived at a contradiction.

We will now consider the case where the K-root system of G is of type C_{n+1}. In this case ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ is isomorphic to F with the trivial action of D (see 7.24(i)). In this case, all the simple affine roots, except the ones in Ω, are fixed under the Galois group of K/k, which forces us to assume that $\#\mathfrak{f} > 2$ to prove that the short exact sequence can not split.

Let $\Omega = \{\omega, \omega'\}$, and α_0 be the long simple affine root. Let $\beta = \sum_{\alpha \in \Delta - (\Omega \cup \{\alpha_0\})} \alpha$. Then $\delta = \omega + \omega' + 2\beta + \alpha_0$. Hence the gradient of $2\omega + 2\beta + \alpha_0$ is $\dot{\omega} - \dot{\omega}'$ and that of $2\omega' + 2\beta + \alpha_0$ is $\dot{\omega}' - \dot{\omega}$.

For an affine root ψ of length 1 with respect to Ω, we will denote its conjugate by ψ' and let $\sigma(u_\psi(t)\, u_{\psi'}(\bar{t})) = u_\psi(t)\, u_{\psi'}(\bar{t})\, f_\psi(t)$, with $f_\psi(t) \in {}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f}) = \mathsf{F}$.

We observe that given an affine root ψ of length 1, there is an affine root η of length 1 and a long root γ of the group D (i.e., the subroot system spanned by $\Delta - \Omega$) such that $\psi = \eta + \gamma$ and $2\eta + \gamma$ equals either $2\omega + 2\beta + \alpha_0$ or $2\omega' + 2\beta + \alpha_0$. In fact, if ω appears in the expression for ψ in terms of simple affine roots, then $\gamma = 2\psi - (2\omega + 2\beta + \alpha_0)$ and if ω' appares in the expression for ψ, then $\gamma = 2\psi - (2\omega' + 2\beta + \alpha_0)$, and $\eta = \psi - \gamma$. In the sequel, without any loss of generality, we assume that $2\eta + \gamma = 2\omega + 2\beta + \alpha_0$. Consider the commutator $c := u_\gamma(1)\,(u_\eta(t)\, u_{\eta'}(\bar{t}))\, u_\gamma(1)^{-1}\,(u_\eta(t)\, u_{\eta'}(\bar{t}))^{-1}$. This commutator equals

$$x := u_\psi(t)\, u_{\psi'}(\bar{t})\, u_{2\omega+2\beta+\alpha_0}(t^2)\, u_{2\omega'+2\beta+\alpha_0}(\bar{t}^2)$$

in $\mathscr{P}_1/(\mathscr{P}_1, \mathscr{P}_1)$, so it equals $u_\psi(t)\, u_{\psi'}(\bar{t})$ in $\mathscr{P}_1/\mathscr{P}_2$. Therefore,

$$\begin{aligned} \sigma(c) &= u_\gamma(1)\,(u_\eta(t)\, u_{\eta'}(\bar{t})\, f_\eta(t))\, u_\gamma(1)^{-1}\ (u_\eta(t)\, u_{\eta'}(\bar{t})\, f_\eta(t))^{-1}. \\ &= u_\gamma(1)\,(u_\eta(t)\, u_{\eta'}(\bar{t}))\, u_\gamma(1)^{-1}\,(u_\eta(t)\, u_{\eta'}(\bar{t}))^{-1} \\ &= x = (u_\psi(t)\, u_{\psi'}(\bar{t}))\,(u_{2\omega+2\beta+\alpha_0}(t^2)\, u_{2\omega'+2\beta+\alpha_0}(\bar{t}^2)). \end{aligned}$$

On the other hand, as c equals $u_\psi(t)\, u_{\psi'}(\bar{t})$ in $\mathscr{P}_1/\mathscr{P}_2$, we obtain $\sigma(c) = u_\psi(t)\, u_{\psi'}(\bar{t}) f_\psi(t)$. Comparing the above two values of $\sigma(c)$ we see that $f_\psi(t) = u_{2\omega+2\beta+\alpha_0}(t^2)\, u_{2\omega'+2\beta+\alpha_0}(\bar{t}^2)$. Thus

$$\sigma(u_\psi(t)\, u_{\psi'}(\bar{t})) = (u_\psi(t)\, u_{\psi'}(\bar{t})) \cdot (u_{2\omega+2\beta+\alpha_0}(t^2)\, u_{2\omega'+2\beta+\alpha_0}(\bar{t}^2)). \quad (1)$$

In case $\#\mathfrak{f} = 2$, we can verify that σ defined by (1) provides an $\mathsf{M}(\mathfrak{f})$-equivariant splitting of the exact sequence of the lemma.

Now we assume that $\#\mathfrak{f} > 2$. We take $\psi = \omega + \beta$ in the above (then $\eta = \omega + \beta + \alpha_0$ and $\gamma = -\alpha_0$). Equation (1) gives the following

$$\sigma(u_{\omega+\beta}(t)\, u_{\omega'+\beta}(\bar{t})) = (u_\psi(t)\, u_{\psi'}(\bar{t})) \cdot (u_{2\omega+2\beta+\alpha_0}(t^2)\, u_{2\omega'+2\beta+\alpha_0}(\bar{t}^2)). \quad (2)$$

It is easily seen, that in the kernel of $\dot{\omega}$ and $\dot{\omega}'$ in $\mathsf{T}(\mathfrak{f})$, there is an element z such that $\beta(z) =: \lambda \neq 1$. considering the conjugates of both the sides of the last equation under z we get

$$\sigma(u_{\omega+\beta}(\lambda t)\, u_{\omega'+\beta}(\lambda\bar{t})) = (u_{\omega+\beta}(\lambda t)\, u_{\omega'+\beta}(\lambda\bar{t})) \cdot (u_{2\omega+2\beta+\alpha_0}(t^2)\, u_{2\omega'+2\beta+\alpha_0}(\bar{t}^2)). \quad (3)$$

Replacing λt with t in the previous equation we obtain

$$\sigma(u_{\omega+\beta}(t)\, u_{\omega'+\beta}(\bar{t})) = (u_{\omega+\beta}(t)\, u_{\omega'+\beta}(\bar{t})) \cdot (u_{2\omega+2\beta+\alpha_0}(t^2/\lambda^2)\, u_{2\omega+2\beta+\alpha_0}(\bar{t}^2/\lambda^2)). \quad (4)$$

From equations (2) and (4) we see that the image of σ contains a nontrivial element of ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$. This is a contradiction, and hence in case $\#\mathfrak{f} > 2$ and the K-root system of G is of type C_{n+1}, with $n > 2$, the short exact sequence of the lemma does not split.

Finally we treat the case where the K-root system of G is of type C_2 ($=B_2$). Let $\Omega = \{\omega, \omega'\}$ and α be the unique long affine root. Assume that the short exact sequence of the lemma admits an $\mathsf{M}(\mathfrak{f})$-equivariant splitting σ. The affine roots of length 1 are ω, ω', $\omega + \alpha$ and $\omega' + \alpha$. It is obvious that given one of these roots ψ, there is a $\gamma \in \{\pm a_0\}$ such that $\eta := \psi - \gamma$ is a root and $\psi + \gamma = \eta + 2\gamma$ equals either $2\omega + \alpha$ or $2\omega' + \alpha$. For definiteness we will assume that $\eta + 2\gamma = 2\omega + a$. Arguing as above, in the case C_{n+1}, $n \geqslant 2$, we see that

$$\sigma(u_\psi(t)\, u_{\psi'}(\bar{t})) = (u_\psi(t)\, u_{\psi'}(\bar{t}))\,(u_{2\omega+\alpha}(t^2)\, u_{2\omega'+\alpha}(\bar{t}^2)). \quad (5)$$

It can be checked that σ described by (5) is an $\mathsf{M}(\mathfrak{f})$-equivariant splitting of the exact sequence of the lemma if $\#\mathfrak{f} = 2$,

Now let us assume that $\#\mathfrak{f} > 2$. We will now show that σ is not a $\mathsf{T}(\mathfrak{f})$-equivariant splitting. For this purpose, assume to the contrary and let $z \in \mathsf{F}$. Then there is a $x \in \mathsf{T}(\mathfrak{f})$ such that $\omega(x) = z^2$, $\omega'(x) = \bar{z}^2$ and $\alpha(x) = (z\bar{z})^{-2}$. Now taking the conjugate by x of both the sides of (5), for $\psi = \omega$, we obtain

$$\sigma(u_\omega(tz^2)\, u_{\omega'}(\bar{t}\bar{z}^2)) = (u_\omega(tz^2)\, u_{\omega'}(\bar{t}\bar{z}^2))\,(u_{2\omega+\alpha}(t^2z^4(z\bar{z})^{-2})\, u_{2\omega'+\alpha}(\bar{t}^2z^4(z\bar{z})^{-2})).$$

Replacing tz^2 by t in the above, we obtain

$$\sigma(u_\omega(t)\, u_{\omega'}(\bar{t})) = (u_\omega(t)\, u_{\omega'}(\bar{t}))\,(u_{2\omega+\alpha}(t^2(z\bar{z})^{-2})\, u_{2\omega'+\alpha}(\bar{t}^2(z\bar{z})^{-2})). \quad (6)$$

As $\#\mathfrak{f} > 2$, there is a z such that $z\bar{z} \neq 1$, using such a z, and also $z = 1$, we infer from (6) that the image of σ contains a nontrivial element of ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f}) = \mathsf{F}$. This implies that σ is not a splitting.

8. Replace the statement and the proof of Proposition 7.38 with the following.

7.38 Proposition *The natural homomorphism*:

$$\mathrm{Hom}_{\mathbf{Z}[\mathsf{M}(\mathfrak{f})]}(\mathfrak{L}_1(\mathfrak{f}), \hat{\mathfrak{L}}_s(\mathfrak{f})) \to \mathrm{Hom}_{\mathbf{Z}[\mathsf{M}(\mathfrak{f})]}(\mathscr{P}_1/(\mathscr{P}_1, \mathscr{P}_1), \hat{\mathfrak{L}}_s(\mathfrak{f}))$$

is an isomorphism except where (i) $s \equiv 2 \pmod 4$, (ii) *G does not split over K and its K-root system is of type C_{n+1}, with $n \geqslant 1$, and* (iii) $\#\mathfrak{f} = 2$.

Except in the exceptional cases mentioned above, if $s \not\equiv -1 \pmod m$, there is no nontrivial $\mathbf{Z}[\mathsf{M}(\mathfrak{f})]$-module homomorphism from $\mathscr{P}_1/(\mathscr{P}_1, \mathscr{P}_1)$ into $\hat{\mathfrak{L}}_s(\mathfrak{f})$.

Proof. If $(\mathscr{P}_1, \mathscr{P}_1) = \mathscr{P}_2$, then $(\mathscr{P}_1, \mathscr{P}_1)/\mathscr{P}_2 = \mathfrak{L}_1(\mathfrak{f})$ and the first assertion of the proposition is obvious. Once the first assertion is established in genral, the second assertion will follow from Proposition 7.25. So we assume that $(\mathscr{P}_1, \mathscr{P}_1) \neq \mathscr{P}_2$. Then $p = 2$, G does not split over K, $m = 2$, and there is an identification of $\mathscr{P}_2/(\mathscr{P}_1, \mathscr{P}_1)$ with ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ (7.34 and 7.35). We identify $\mathscr{P}_1/\mathscr{P}_2$ with $\mathfrak{L}_1(\mathfrak{f})$. Then we have the following short exact sequence of $\mathsf{M}(\mathfrak{f})$-modules:

$$\{0\} \to {}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f}) \to \mathscr{P}_1/(\mathscr{P}_1, \mathscr{P}_1) \to \mathfrak{L}_1(\mathfrak{f}) \to \{0\}. \quad (1)$$

Let $\lambda : \mathscr{P}_1/(\mathscr{P}_1, \mathscr{P}_1) \to \hat{\mathfrak{L}}_s(\mathfrak{f})$ be a $\mathbf{Z}[\mathsf{M}(\mathfrak{f})]$-module homomorphism and $\mathfrak{K}$ be its kernel. We assume first that s is odd. Proposition 7.25 implies that the restriction of λ to ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ is trivial and hence $\mathfrak{K}$ contains ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$. This implies that λ factors through $\mathscr{P}_1/\mathscr{P}_2$ which proves the first assertion. If s is a multiple of 4, then $\mathfrak{L}_s$ is isomorphic to the Lie algebra of M, $\mathsf{C}(\mathfrak{f})$ acts trivially on it, whereas ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ does not contain any nonzero $\mathsf{C}(\mathfrak{f})$-invariants, so the restriction of λ to ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ is trivial and hence $\mathfrak{K}$ contains ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ which again implies that λ factors through $\mathscr{P}_1/\mathscr{P}_2$.

Finally, we consider the case $s \equiv 2 \pmod 4$. If $\mathfrak{K} \cap {}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f}) \neq \{0\}$, then irreducibility of ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ implies that $\mathfrak{K}$ contains ${}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f})$ and hence, as before, λ factors through $\mathfrak{L}_1(\mathfrak{f})$. So let us assume that $\mathfrak{K} \cap {}^{\blacksquare}\mathfrak{L}_2(\mathfrak{f}) = \{0\}$. In this case, irreducibility of $\hat{\mathfrak{L}}_s(\mathfrak{f})$ as a $\mathsf{M}(\mathfrak{f})$-module implies that $\lambda(\mathfrak{K}) = \hat{\mathfrak{L}}_s(\mathfrak{f})$ and hence $\mathfrak{K}$ provides a $\mathbf{Z}[\mathsf{M}(\mathfrak{f})]$-module splitting of the short exact sequence (1). But Lemma 7.37 proves that a splitting can (and does) exist only in the exceptional case.

9. Add the following at the end of section 7.

If G does not split over K and its K-root system is of type C_{n+1}, then it is of the form $\mathrm{SU}(h)$, where h is a hermitian form in $n+2$ variables defined in terms of a ramified quadratic Galois extension.

10 In view of the exceptional cases in Lemma 7.37 and Proposition 7.38, in the rest of the paper we will need to excude these cases for now.

11. Replace the first line on page 233 with the following:

"and let the induced automorphism of K be σ."

12. In the second and the third lines of 8.17 replace "*if G is not of type C, $\mathfrak{x}$ restricts to zero on $G^{\cdot}(k)$; if G is of type C, then it restricts to zero on $G^{*}(k)$*" with "*if G is of type C, $\mathfrak{x}$ restricts to zero on $G^{\cdot}(k)$; if G is not of type C, then it restricts to zero on $G^{*}(k)$*".

13. *At the end of the third line (from the top) on page 254 add the following:*

" *(note that $\lambda_m(\mathscr{P}^*_m \times \mathscr{P}^*_{t-m+1}) = \{0\}$)*"

In the second line (from the bottom) on page 254, the first mathematical expression should be $\sum_{\alpha\in\Delta-\Omega} m_i(\alpha)\,\alpha$ and the last mathematical expression on this line should be $\beta \in \langle \Delta - \Omega \rangle$

14. *In the second line (from the top) on page 256, replace* $\mathscr{P}_t^{\bullet}/\mathscr{P}_{t+2}^{\bullet}$ *with* $\mathscr{P}_t^{\bullet}/\mathscr{P}_{t+1}^{\bullet}$.

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