ERRATA

CONTENTS

We give the numbering in both the arXiv version and the published version, in that order.

Just before statement of Fact 2.3.6/3.8

There is a gap in the argument. In order to prove that the representation

$$\text{c-Ind}_{S(F)G(F)_{y,\frac{1}{e}}}^{G(F)}\widehat{\chi}$$

is epipelagic, one would have to show that $1/e$ is the smallest positive value taken by an affine root at the point y. At this place it is not essential to prove this – if one omits the word "epipelagic" everything else is still fine. It does however become essential later in the paper, when the character identities are proved. Thankfully, the gap can be filled, see below.

Statement of Proposition 3.2.1(1)/4.3(1)

κ_α should be replaced by $\kappa_\alpha^{\text{coh}}$.

Statement of Lemma 3.4.1/4.8

$\text{tr}_{F_\alpha/F_{\pm\alpha}}$ should be replaced by $\text{tr}_{F_{\pm\alpha}/F}$.

Proof of Lemma 3.4.1/4.8

In the final sentence, $f_{(G,S)}(\alpha)$ should be replaced by $f_{(G,S)}(X_\alpha)$.

Statement of Corollary 3.5.2/4.11

$\text{tr}_{F_\alpha/F_{\pm\alpha}}$ should be replaced by $\text{tr}_{F_{\pm\alpha}/F}$.

Equation (5.1.1)/(6.1)

Here we encounter the above mentioned gap in the argument again. The Adler-Spice character formula implies this equation only under the assumption that the representation $\pi_{S,\chi}$ is epipelagic and its depth is witnessed by the point of the building associated to the maximal torus S. This is what implies that none of the roots of unity present in the general character formula appear: We already know that $1/m$ is the smallest positive break of the Moy-Prasad filtration for $S(F)$, because the splitting field of S has ramification degree m. This implies that the good product expansion of $\gamma = \gamma_{<1/m} \cdot \gamma_{\geq 1/m}$ coinsides the the topological Jordan decomposition. If in addition we knew that $1/m$ is the smallest positive value of any absolute affine root, then all the sets of roots occurring in the formulas for the roots of unity in the Adler-Spice character formula become empty.

Let us now give the argument that $1/m$ is indeed the smallest positive value of an absolute affine root at the point y associated to S. Since S is inertially elliptic, we may base change to the maximal unramified extension F^u, over which S is still elliptic and y is its associated point. This has the effect of making G quasi-split and further of allowing us to replace S by any torus stably conjugate to S over F, because such torus becomes rationally conjugate to S over F^u. We choose the stably conjugate torus in a manner similar to the proof of Proposition 3.3.1/4.4: By choosing a-data for $R(S,G)$. But we choose the a-data slightly differently then in that proof – we use the fact that inertia acts by a regular element, so that the extensions F^u_α/F^u are all equal to the unique extension E/F of degree $1/m$, and choose $a_\alpha \in E^\times$ to be of valuation $1/m$ for every $\alpha \in R(S,G)$, symmetric or not. Just as in the beginning of the proof of Proposition 3.3.1/4.4 we choose a pinning $(T_0, B_0, \{X_\alpha\})$ of G, an element $h \in G(F^u)$ s.t. $hT_0h^{-1} = S$ and transport the a-data to T_0 by $\mathrm{Ad}(h)$. We form $x_S(\sigma) = \prod_{\beta>0,(w_S(\sigma)\sigma)^{-1}\beta)} \beta^\vee(a_\beta)$ and $n_S(\sigma) = x_S(\sigma)n(w_S(\sigma))$, so that $n_S \in Z^1(\Gamma, N(T_0,G))$. Choose $g \in G(\bar{F})$ so that $g^{-1}\sigma(g) = n_S(\sigma)$ and now replace S by gT_0g^{-1}.

The values of absolute affine roots of S at the point y are the same as the values of absolute affine roots of T_0 at the unique point of $\mathcal{A}(T_0,E)$ fixed by $n_S(\sigma)$ for all $\sigma \in \mathrm{Gal}(E/F^u)$. Let $o \in \mathcal{A}(T_0,F^u)$ be the hyperspecial vertex corresponding to the fixed pinning, and let $v \in X_*(S_{\mathrm{ad}}) \otimes \mathbb{R}$ be such that $o+v$ is the unique fixed point. Then we have

$$o+v = n_S(\sigma)\sigma(o+v) = x_S(\sigma)n(w_S(\sigma))\sigma(o+v) = o + w_S(\sigma)\sigma(v) + t_{x_S(\sigma)},$$

where $t_{x_S(\sigma)} \in X_*(S)_{\mathrm{ad}} \otimes \mathbb{R}$ is the translation by which $x_S(\sigma)$ acts on $\mathcal{A}(T_0,E)$ and we have used that $n(w_S(\sigma))\sigma(o) = o$. From the above we get for every $\alpha \in R(T_0,G)$

$$\langle \alpha, v - w_S(\sigma)\sigma(v)\rangle = -\mathrm{ord}\langle \alpha, x_S(\sigma)\rangle,$$

where the right hand side is the definition of $\langle \alpha, t_{x_S(\sigma)}\rangle$. Recalling that $\mathrm{ord}(a_\beta) = 1/m$ for every $\beta \in R(T_0,G)$ we compute the right hand side as $-\sum_{\beta>0,\sigma^{-1}\beta<0}\langle \alpha, \beta^\vee\rangle$.

We know a-priori that there is a unique v that verifies this identity. Let's check that $v = \rho^\vee/m$ does verify it, where $\rho^\vee$ is half the sum of the B_0-positive coroots. Indeed, one then sees that $v - \sigma(v) = \frac{1}{m}\sum_{\beta>0,\sigma^{-1}\beta<0}\beta^\vee$. This is enough – all absolute affine roots of T_0 take integer values at o, so the smallest positive value of any absolute affine root at $o+v$ equals the smallest positive value of any absolute root at v, which is $1/m$.

Statement of Lemma 5.3.2/6.4

In the final sentence, $\pi_0(H^y)(F)$ should be replaced by $\pi_0(J^y)(F)$.

Proof of Theorem 5.4.1/6.6

- In the fourth displayed formula, the index of the second sum should be ξ, and not f, and the text following that formula should read "where ξ now runs over the set $\Xi(H_y, G^b_{\gamma_0^b})$", instad of "where f now runs over the set $\Xi(H_{\gamma_0^H}, G^b_{\gamma_0^b})$".

- In the chain of equations appearing below the sentence "As a first step, we undo the descent of the transfer factor", the subscripts for Δ should be W, b, ξ, and not $W, 1, \xi$, in the first 3 lines.

Proof of Proposition 6.2.5/7.7

The word before $2|n$ should be "when", and not "of".

Proposition 3.6

The proof of the surjectivity of the map a assumes that the algebraic group G is either abelian or connected, but the statement of the proposition allows arbitrary affine algebraic groups. Here is a direct argument in the generality of the statement: In the second paragraph of the proof a continuous set-theoretic section $s : \Gamma \to W$ is chosen and is used to define the 2-cocycle $\dot{\xi}$. This section induces the isomorphism $u \boxtimes_\xi \Gamma \to W$, $x \boxtimes \sigma \mapsto x \cdot s(\sigma)$. Let $z \in Z^1(\Gamma, \bar{G})$. Choose a lift $\dot{z} \in C^1(\Gamma, G)$. Then $\alpha(\sigma,\tau) = \dot{z}(\sigma)^\sigma \dot{z}(\tau)\dot{z}(\sigma\tau)^{-1}$ defines an element $\alpha \in Z^2(\Gamma, Z)$. The surjectivity of $\mathrm{Hom}(u,Z)^\Gamma \to H^2(\Gamma, Z)$ implies that there exists $\varphi \in \mathrm{Hom}(u,Z)^\Gamma$ so that, after possibly modifying the lift $\dot{z}$, the equality $\alpha = \varphi \circ \xi$ holds. Then $x \boxtimes \sigma \mapsto \varphi(x) \cdot \dot{z}(\sigma)$ belongs to $Z^1(u \to W, Z \to G)$ and lifts z.

Section 4.4

In the text before the statement of Lemma 4.4 are chosen for each k a section $s_k : \Gamma_{E_k/F} \to W_{E_k/F}$ of the projection $p_k : W_{E_k/F} \to \Gamma_{E_k/F}$ and $\zeta_k : \Gamma_{E_k/F} \to \Gamma_{E_{k+1}/F}$ of the natural projection $\pi_k^\Gamma : \Gamma_{E_{k+1}/F} \to \Gamma_{E_k/F}$. These sections are to satisfy the two properties

$$s_{k+1}(y\zeta_k(x)) = s_{k+1}(y)s_{k+1}(\zeta_k(x)) \qquad \text{and} \qquad s_k(x) = \pi_k^W(s_{k+1}(\zeta_k(x))).$$

While the text suggests that one is to choose inductively, for each k, first ζ_k and then s_{k+1} arbitrarily, Olivier Taïbi has brought to our attention that there exist choices of s_k and ζ_k for which no suitable s_{k+1} exists. In other words, we cannot choose ζ_k arbitrarily and expect to be able to choose s_{k+1}.

As he points out, given $x \in \Gamma_{E_k/F}$, the second of the two equations above requires that $s_{k+1}(\zeta_k(x))$ be a lift to $W_{E_{k+1}/F}$ of $s_k(x) \in W_{E_k/F}$. But it may well happen that no lift of $s_k(x)$ maps under p_{k+1} to $\zeta_k(x)$.

Taïbi has suggested two possible ways to overcome this. One way is again by induction on k. Assuming s_k, but not yet ζ_k, is given, one chooses a lift $y_x \in W_{E_{k+1}/F}$ of $s_k(x)$ for each $x \in \Gamma_{E_k/F}$, and defines $\zeta_k(x) = p_{k+1}(y_x)$ and $s_{k+1}(\zeta_k(x)) = y_x$. Then one further picks $s_{k+1}(y) \in W_{E_{k+1}/F}$ arbitrarily for $y \in \Gamma_{E_{k+1}/E_k}$ and sets $s_{k+1}(y\zeta_k(x)) = s_{k+1}(y)s_{k+1}(\zeta_k(x))$. This defines both ζ_k and s_{k+1}.

Another way is to fix sections $\eta_k : \Gamma_{E_k/E_{k-1}} \to W_{E_{k-1}}$ for all $k = 0, 1, \dots$, with $E_{-1} = F$. Then any $x \in \Gamma_{E_k/F}$ can be written uniquely as the projection to $\Gamma_{E_k/F}$ of $y = \eta_k(x_k)\eta_{k-1}(x_{k-1}) \dots \eta_0(x_0) \in W_F$. The projection of y to $W_{E_k/F}$ can then be taken as $s_k(x)$ and the projection of y to $\Gamma_{E_{k+1}/F}$ can be taken as $\zeta_k(x)$.

Formula (4.8)

In order to make sense of this formula, the homomorphism $p : u_{k+1} \to u_k$ has to be extended to a homomorphism $p : S_{E_{k+1}/F} \to S_{E_k/F}$, and the element $\delta_e \in Z^{-1}_{\text{Tate}}(\Gamma_{E_k/F}, \mathrm{Hom}(u_k, Z))$ has to be reinterpreted as an element of $Z^{-1}_{\text{Tate}}(\Gamma_{E_k/F}, \mathrm{Hom}(S_{E_k/F}, S))$. Both are given by the same formulas as before.

Proof of Lemma 4.7

There is a typo in the second to last displayed formula. It reads

$$\phi_{\bar{\lambda},k}(p(x)\alpha_k(x)) \cdot (l_k c_k \sqcup_{E_k/F} \delta_e)(\sigma)$$

but it should rather be

$$\phi_{\bar{\lambda},k}(p(x)\alpha_k(x)) \cdot (l_k c_k \sqcup_{E_k/F} n_k\bar{\lambda})(\sigma).$$

Moreover, in the last displayed formula, which consists of two equalities separated by a comma, in order to make sense of the right equality, the element $\phi_{\bar{\lambda},k} \in \mathrm{Hom}(u_k, Z)^\Gamma$ has to be reinterpreted as an element of $\mathrm{Hom}(S_{E_k}, S)^\Gamma$. It is given by the same definition as before. An explicit formula for it is

$$\phi_{\bar{\lambda},k}(\gamma) = \prod_{a \in \Gamma_{E_k/F}} (n_k a\bar{\lambda})(\gamma(a)),$$

where $\gamma \in \mathrm{Maps}(\Gamma_{E_k/F}, \bar{F}^\times) = R_{E_k/F}(\bar{F})$ represents an element of $S_{E_k/F}(\bar{F})$.

Section 6.2

In diagram (6.6), the groups $\widehat{B}^0(\Gamma, \widehat{Z})$ appear. While their appearance is not a mistake, it is useful to observe that they are in fact trivial, because $\widehat{Z}$ is finite.

Section 6.3

Just after equation (6.7) the following two sentences appear:

For this let n be large enough so that $x_{1,\text{rig}}, x_{2,\text{rig}} \in Z^1(u \to W, Z_n \to G)$. Then there exists $y \in Z^1(W, Z_n)$ with $x_{2,\text{rig}} = y \cdot x_{1,\text{rig}}$.

This is incorrect – the fact that n is sufficiently large so that $x_{1,\text{rig}}, x_{2,\text{rig}} \in Z^1(u \to W, Z_n \to G)$ does not by itself imply the existence of $y \in Z^1(W, Z_n)$ with $x_{2,\text{rig}} = y \cdot x_{1,\text{rig}}$. We are only assuming that the images in $H^1(\Gamma, G_{\text{ad}})$ of $x_{1,\text{rig}}$ and $x_{2,\text{rig}}$ are equal, but the existence of y is equivalent to the stronger assumption that their images in $H^1(\Gamma, G/Z_n)$ are equal. Thankfully, n can be enlarged further to ensure that the stronger assumption is true. Indeed, as discussed in §3.3 we have the identification $G/Z_n = G_{\text{ad}} \times Z(G_1)$ for all n, where the transition map $G/Z_n \to G/Z_m$ for $n|m$ is given by the m/n-power map of the torus $Z(G_1) = Z(G)/Z(G_{\text{der}}) = G/G_{\text{der}}$. The images of $x_{1,\text{rig}}$ and $x_{2,\text{rig}}$ in $H^1(\Gamma, G/Z_n) = H^1(\Gamma, G_{\text{ad}}) \times H^1(\Gamma, Z(G_1))$ have the same first coordinate. If we take m so that m/n is a multiple of the order of the finite group $H^1(\Gamma, Z(G_1))$, then the images of $x_{1,\text{rig}}$ and $x_{2,\text{rig}}$ in $H^1(\Gamma, G/Z_m)$ will be equal.

Assumption on p in Sections 3.6, 3.7, 4.10, 5.3

Charlotte Chan and Masao Oi have brought to my attention that I have not been sufficiently careful with the assumptions on the residual characteristic p of the ground field F that need to be imposed in order for the results of Section 3 to be valid. In summary, one needs to make in Sections 3.6, 3.7, 4.10, 5.3 the following assumptions on p:

1. p is not a bad prime for (any irreducible factor of) the (absolute) root system of G,
2. p does not divide the order of $\pi_1(G_{\text{der}})$,
3. p does not divide the order of $\pi_0(Z(G))$.

The argument of Section 3.7.4, where the second assumption above is claimed to be removed, is invalid.

We note that the second and third assumptions are implied by the first assumption unless the root system of G has a component of type A_n. Indeed, $|\pi_1(G_{\text{der}})|$ is a divisor of the connection index of the root system of G, and the primes dividing the connection index are always bad for G except when a component of type A_n is present. Moreover, $\pi_0(Z(G))$ is dual to $\pi_1(\widehat{G}_{\text{der}})$, and the set of bad primes is invariant under duality of root systems.

We now discuss the situation more precisely. In Section 3.6 the stated assumptions on p are that p is not a bad prime for the root system of G and does not divide $|\pi_1(G_{\text{der}})|$. The assumption that p does not divide $|\pi_0(Z(G))|$ is not stated, but turns out to be necessary as well. The culprit is Lemma 3.6.8, which invokes Lemma 8.1 of Jiu-Kang Yu's paper "Construction of tame supercuspidal representations". The latter needs the assumption that p is not a torsion prime for the dual root *datum* of G. This assumption is stronger than the assumption that p is not a torsion prime for the dual root *system* of G. In fact, one sees easily that p is not a torsion prime for the dual root datum of G if and only if p is not a torsion prime for the dual root system of G *and* p does not divide $|\pi_0(Z(G))|$. The assumption that p is not a torsion prime for the dual root system of G is implied by, and almost equivalent to, the assumption that p is not a bad prime for the root system of G. However, one must add to it the assumption that p does not divide $|\pi_0(Z(G))|$ in order to justify appealing to Lemma 8.1 of Yu's paper. Since Section 3.7.3 uses Section 3.6, the same assumption must be added there too.

In Section 3.7.4 an argument is given that is supposed to remove the condition that p does not divide $|\pi_1(G_{\text{der}})|$, by replacing G by a z-extension $\tilde{G}$. But even if p does not divide $|\pi_0(Z(G))|$, it is not clear that p will not divide $|\pi_0(Z(\tilde{G}))|$. Therefore, this reduction argument becomes invalid.

The upshot is that all parts of the paper that use Section 3.6 – Sections 3.6, 3.7, 4.10, 5.3 – must assume the above three conditions on p. The assumption that p does not divide $|\pi_0(Z(G))|$ is actually stated in Section 5, because it is needed for other arguments, but the assumption that p does not divide $|\pi_1(G_{\text{der}})|$ is not stated (due to the material of Section 3.7.4, which is now invalid), and must now be added.

Remarks on Section 3.6

We make a few remarks about the Howe factorization algorithm that we hope will help clarify some points. These remarks pertain to the proof of Proposition 3.6.7, as well as to Lemma 3.6.9 and Corollary 3.6.10, which are used in the proof of Proposition 3.6.7. At the moment I am not aware of any mistakes in section 3.6 beyond the condition on p already discussed, so these remarks are just for clarification.

First, we note that while by definition we have $R(S, G^i) = R_{r_{i-1}+}$, the latter is also equal to R_{r_i}, due to the fact that the real numbers r_i are the jumps of the filtration $r \mapsto R_r$. This justifies the application of Corollary 3.6.10 in the proof of Proposition 3.6.7.

Next, we will give a slight simplification of the exposition of Lemma 3.6.9 and Corollary 3.6.10. The proof of Corollary 3.6.10 is by inductive application of Lemma 3.6.9. This induction is however not necessary, and one can modify the proof of Lemma 3.6.9 so that it gives the statement of Corollary 3.6.10. More precisely, we have the following:

Lemma. *Let G be a connected reductive F-group, $S \subset G$ a tame maximal torus, and $\theta : S(F) \to \mathbb{C}^\times$ a character. Let r' be a non-negative real number such that*

$$\theta(N_{E/F}(\alpha^\vee(E^\times_{r'+}))) = 1$$

for all $\alpha \in R(S, G)$, where E/F is the tamely ramified splitting field of S. Then there exists a character $\phi : G(F) \to \mathbb{C}^\times$, trivial on $G_{der}(F)$, such that $\phi|_{S(F)_{r'+}} = \theta|_{S(F)_{r'+}}$. Moreover, $\theta' = \theta \cdot \phi^{-1}$ has depth at most r', and in fact precisely r' if $r' > 0$ and minimal with the above property.

Proof. The argument in the first half of the proof of Lemma 3.6.9 shows that θ restricts trivially to $S_{\text{der}}(F)_{r'+}$, where $S_{\text{der}} = S \cap G_{\text{der}}$. Following the second half of that proof, we let $D = G_{\text{ab}}$ and see that θ descends to a character of $D(F)_{r'+}$, which is of finite order, since it is trivial on some open, hence finite-index, subgroup of $D(F)_{r'+}$. By Pontryagin duality this character extends to a character of $D(F)$.

We let ϕ denote the pullback of this character to $G(F)$. Then $\phi|_{S(F)_{r'+}} = \theta|_{S(F)_{r'+}}$, so $\theta' = \theta \cdot \phi^{-1}$ is trivial on $S(F)_{r'+}$, so its depth is at most r'. If $r' > 0$ and minimal, then θ' is non-trivial on $N_{E/F}(\alpha^\vee(E^\times_{r'})) \subset S(F)_{r'}$, so that θ' has depth r'. □

We now comment on the proof of Proposition 3.6.7 and give a slight modification, which uses the above lemma in place of Corollary 3.6.10.

First, we note that the two special cases handled in the beginning of that proof must indeed be handled separately, because they fail the recursion hypothesis, the first because $r_0 = 0$, and the second because $i = d - 1 = 0$, but $R(S, G^i) = R(S, G^0) = \emptyset$.

Next we note that in all other cases the recursion hypothesis is satisfied. If $d > 1$ then we have $r_d \geq r_{d-1} > r_0 > 0$, and $R(S, G^0) \subsetneq R(S, G^{d-1})$, hence $R(S, G^{d-1}) \neq \emptyset$, so $R(S, G^i) \neq \emptyset$ for $i = d$ and $i = d - 1$. If $d = 1$, the fact that we are not in the second special case means that either $r_1 > r_0$, so $i = 1$ and we see $R(S, G) \supsetneq R(S, G^0)$, or $r_1 = r_0$ but $R(S, G^0) \neq \emptyset$.

In the recursion step we apply the above Lemma in place of Corollary 3.6.10, to the pair G^i and θ_i, and with r' being the smallest non-negative real number such that $\theta_i(N_{E/F}(\alpha^\vee(E^\times_{r'+}))) = 1$ for all $\alpha \in R(S, G^i)$. Since this equation holds for $r' = r_i$ by definition of r_i, as mentioned above, we know that $r' \leq r_i$. If $i > 0$ then $r' = r_{i-1}$ again by definition of r_{i-1} and the part of the identity $\theta_i(N_{E/F}(\alpha^\vee(E^\times_{r'}))) = \theta(N_{E/F}(\alpha^\vee(E^\times_{r'})))$ for all $\alpha \in R(S, G^i)$, which is part of the recursion hypothesis (in fact, this identity can be strengthened to $\theta_i \circ N_{E/F} \circ \alpha^\vee = \theta \circ N_{E/F} \circ \alpha^\vee$). Since $r_{i-1} > 0$ we see that the depth of $\theta_{i-1} = \theta_i \cdot \phi_i^{-1}$ equals r_{i-1}. If $i = 0$ then $r' = 0$, because r_0 is the smallest positive break of the filtration R_r. Then θ_{-1} has depth at most 0. The remainder of the proof of Proposition 3.6.7 proceeds the same way.

Section 4.11

In this section we consider discrete series representations of a real reductive group. It is implicit in the discussion that the discrete series representation is *sufficiently regular* in the following sense. The infinitesimal character $\mu \in X^*(S) \otimes \mathbb{C}$ of any discrete series representation, when projected to $X^*(S_{\text{sc}}) \otimes \mathbb{C}$, lies in $X^*(S_{\text{sc}}) \otimes \mathbb{Q}$ and is a regular element there. If ρ is one half the sum of the the corresponding positive roots, then $\mu - \rho$ is still dominant, but may not be regular. The representation is sufficiently regular if $\mu - \rho$ is still regular. One can think of sufficiently regular discrete series representations as the $\mathbb{R}$-analog of regular supercuspidal representations.

In terms of the character θ, one has $d\theta = \mu - \rho$. The regularity of $d\theta$ is stated implicitly in the assertion $\langle \alpha^\vee, d\theta \rangle < 0$ for all negative roots.

For $G = \mathrm{SL}_2/\mathbb{R}$, the discrete series representations are π_k^+ and π_k^- for positive integers k. They are related to modular forms of weight $k + 1$. The sufficiently regular discrete series representations are those for $k > 1$, i.e. related to modular forms of weight > 2.

Lemma 6.2.1

There is a typo in the character formula: the superscript ${}^{jg}X_*$ of the orbital integral should read ${}^{gj}X_*$.