Appendix A. Depth-zero supercuspidal $L$-packets on general inner forms

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Let $F$ be a $p$-adic field with ring of integers $O_F$ and residue field $k_F$ of cardinality $q$ and characteristic $p$. We fix an algebraic closure $\bar{F}$ of $F$ and let $\Gamma$ be the absolute Galois group of $F/F$ and $W$ the absolute Weil group. Let $G$ be a connected reductive algebraic group defined over $F$ and unramified, i.e. quasi-split and split over the maximal unramified extension $F^u$ of $F$. Let $(\xi, G)$ be an inner twist of $G$: That is, $G'$ is a connected reductive algebraic group defined over $F$, and $\xi : G \times \bar{F} \to G' \times \bar{F}$ is an isomorphism with the property that for all $\sigma \in \Gamma$, the automorphism $\xi^{-1} \sigma \xi$ of $G \times \bar{F}$ is inner.

Let $^L G$ be the Weil-form of the Langlands dual group of $G$. Since we are identifying the dual groups of $G$ and $G'$ via $\xi$, $^L G$ is also the Langlands dual group of $G'$. Let $\phi : W \to ^L G$ be a TRSELP parameter [DR09, §4.1]. If the inner twist $(G', \xi')$ comes from a pure inner twist $(G', \xi, z)$ (see [Kal11, §2] for the terminology), an $L$-packet $\Pi_\phi$ consisting of irreducible supercuspidal depth-zero representations of $G'(F)$ was defined in [DR09]. This was later generalized to the cases when $(G', \xi)$ comes from an extended pure inner twist in [Kal11b]. Moreover, the packets constructed in [DR09] and [Kal11b] were shown to have a natural parameterization in terms of the centralizer of $\phi$ in the dual group $\hat{G}$ of $G$, and satisfy stability and endoscopic transfer.

In this appendix, we are going to repeat some of the arguments of [Kal11b] in order to obtain a special case of the results of that paper in a more general setting. Namely, we will first construct the $L$-packet $\Pi_{\phi}'$ on $G'$ corresponding to $\phi$ without assuming any condition on $(G', \xi')$. In the cases where an $L$-packet on $G'$ has already been constructed, we will obtain the same set of representations. What we lose in this more general setting is the ability to provide a natural parameterization of the set $\Pi_{\phi}'$ in terms of $S_\phi = \text{Cent}(\phi, \hat{G})$. Instead, the following much weaker statement is all we can show: Let $\zeta_{G'}$ be any character of $Z(\hat{G}_{sc})$ whose restriction to $Z(\hat{G}_{sc})^F$ corresponds to the class of $(\xi, G)$ in $H^1(F, G_{ad})$ under the Kottwitz isomorphism. Let $A_\phi$ be the inverse image in $\hat{G}_{sc}$ of the image in $G_{ad}$ of $S_\phi$. Then we can show that there exists a non-canonical bijection between $\Pi_{\phi}'$ and the set $\text{Irr}(A_\phi, \zeta_{G'})$ of irreducible representations of $A_\phi$ which transform under $Z(\hat{G}_{sc})$ by $\zeta_{G'}$.

Second, we will prove the following character identity: Let $\Pi_{\phi}'$ be the $L$-packet on $G$, and $\Pi_{\phi}'$ that on $G'$. Then for any strongly-regular semi-simple elements $\gamma \in G(F)$ and $\gamma' \in G'(F)$ whose stable classes correspond to each other, we have

$$\sum_{\pi \in \Pi_{\phi}'} \Theta_\pi(\gamma) = e(G') \sum_{\pi' \in \Pi_{\phi}'} \Theta_{\pi'}(\gamma').$$

Recall that the stable classes of $\gamma$ and $\gamma'$ correspond to each other if there exists $g \in G(\bar{F})$ such that $\xi \circ \text{Ad}(g) \gamma = \gamma'$.

This character identity is a special case of the endoscopic character identities for depth-zero supercuspidal $L$-packets established in [Kal11] and [Kal11b], and the necessary argument – a slight generalization of the stability calculations of [DR09] – is very similar to the proof of [Kal11b, Thm. 4.3.1].
A.1. Construction of \( L \)-packets. Given a TRSELP parameter \( \phi : W \to \mathcal{L}G \) (see [DR09, §4.1] or [Kal11b, §3.2]), we apply ”Step 1” in [Kal11b, §3.3] and obtain a triple \((S_0, [a], [L^j])\) where \( S_0 \) is an elliptic unramified maximal torus of \( G \) whose unique fixed point in \( \mathcal{B}^{\text{red}}(G, F) \) is a vertex, \([a]\) is an equivalence class of Langlands parameters for \( S \), and \([L^j]\) is a \( G \)-conjugacy class of \( L \)-embeddings \( LS \to \mathcal{L}G \), with the property that \( \phi \) belongs to the \( G \)-conjugacy class \([L^j \circ a]\). The construction of this triple involves choices of \( \chi \)-data and hyperspecial vertex \( o \in \mathcal{B}^{\text{red}}(G, F) \). However, using [Kal11b, Lemma 3.4.1] we see that the \( G \)-conjugacy class of this triple involves choices of \( \chi \)-data and hyperspecial vertex \( o \in \mathcal{B}^{\text{red}}(G, F) \). The construction of this triple is independent of all choices. We let \( \theta_0 : S_0(F) \to \mathbb{C}^\times \) be the character corresponding to \([a]\).

Next we consider admissible embeddings of \( S_0 \) into \( G' \). We recall that an embedding \( j : S_0 \to G' \) is called admissible if it is defined over \( F \) and is of the form \( \xi \circ \text{Ad}(g) \) for some \( g \in G(F) \). The set of admissible embeddings forms a unique stable conjugacy class by definition. Given such an embedding \( j \), we obtain the maximal torus \( S_j := j(S_0) \) of \( G' \) and the character \( \theta_j : S_j(F) \to \mathbb{C}^\times \) given by \( \theta_j = j \circ \theta_0 \circ j^{-1} \). Thus, \( j \) gives rise to the representation \( \pi_{G',S_j,\theta_j} \) discussed in [Kal11b, §3.1]. We set
\[
\Pi^G_{\phi} = \{ \pi_j \},
\]
where \( j \) runs over the set of \( G'(F) \)-conjugacy classes inside the stable class of admissible embeddings \( j : S_0 \to G' \).

**Lemma A.1.** The group \( \mathcal{A}_\phi \) is abelian, and there exists a bijection between \( \Pi^G_{\phi} \) and \( \text{Irr}(\mathcal{A}_\phi, \zeta_{G'}) \).

**Proof.** Choose arbitrarily one admissible embedding \( j_a : S_0 \to G' \). Then set of \( G'(F) \)-conjugacy classes of admissible embeddings \( S_0 \to G' \) is in canonical bijection with the kernel of the map
\[
H^1(j_a) : H^1(F, S_0) \to H^1(F, G').
\]
Using [Kot86, Thm 1.2], this kernel is in bijection with the kernel of
\[
\pi_0(S_0^\Gamma)^D \to \pi_0(Z(\hat{G})^\Gamma)^D.
\]
Since \( S_0 \) is elliptic, we know that \([S_0^\Gamma]^D = [Z(\hat{G})^\Gamma]^\circ\), which implies that the kernel of the last displayed map is equal to \([S_0^\Gamma/Z(\hat{G})^\Gamma]^D\).

On the other hand, the map \( L^j \) provides an isomorphism \( \hat{S}_0^\Gamma \to \text{Cent}(\phi, \hat{G}) \), from which we obtain an isomorphism between \( \hat{S}_0^\Gamma/Z(\hat{G})^\Gamma \) and the image of \( \text{Cent}(\phi, \hat{G}) \) in \( \hat{G}_\text{ad} \). The pre-image \( \mathcal{A}_\phi \) of \( \hat{S}_0^\Gamma/Z(\hat{G})^\Gamma \) in \( \hat{G}_\text{ad} \) is certainly contained in \([\hat{S}_0]_\text{sc}\), and is thus abelian. It follows that \( \text{Irr}(\mathcal{A}_\phi, \zeta_{G'}) \) is a torsor under the abelian group \( \text{Irr}(\mathcal{A}_\phi, 1) \). The latter is by definition is equal to \([S_0^\Gamma/Z(\hat{G})^\Gamma]^D\). \( \square \)

A.2. Stability and transfer to inner forms. Given that \( \mathcal{A}_\phi \) is abelian, we define the stable character of the packet \( \Pi^G_{\phi} \) to be
\[
S \Theta^G_{\phi} = e(G') \sum_{\pi \in \Pi^G_{\phi}} \Theta_\pi,
\]
where \( e(G') \) is the Kottwitz sign of \( G' \) [Kot83]. We recall that
\[
e(G') = (-1)^{rk_F(G) - rk_F(G')}.
\]
More generally, we define for any two reductive groups $G_1, G_2$ the sign

$$e(G_1, G_2) = (-1)^{rk_F(G_1) - rk_F(G_2)}.$$ 

Clearly $e(G') = e(G, G')$.

Before discussing stability, we need to give a formula for $S\Theta^G_{G'}$. For this, we will closely follow the proof of [Kal11b, Prop 4.2.2]. In order to obtain the result, we need to assume that the residual characteristic of $F$ is large enough. More precisely, we assume that the cardinality of the residue field $k_F$ is large as the number of positive roots of $G$, and that both $G$ and $G'$ have faithful rational representations of dimension not exceeding $p/(2 + e)$, where $p = \text{char}(k_F)$ and $e$ is the ramification index of $F/\mathbb{Q}_p$.

Under these conditions, DeBacker and Reeder have shown [DR09, Lemma 12.4.1] that the following statements hold for any group $H$ which is the connected component of the centralizer of a topologically semi-simple element of $G(F)$ or $G'(F)$:

- Let $x \in \mathcal{B}^\text{red}(H, F)$ be a point and $H_x^\text{red}$ be the reductive quotient of the special fiber of the corresponding parahoric group scheme. Then the Lie-algebra of any maximal torus of $H_x^\text{red}$ defined over $k_F$ contains a strongly-regular semi-simple element defined over $k_F$.
- There exists an $H(F)$-equivariant bijection $\log : H(F)_{0+} \to \text{Lie}(H)(F)_{0+}$ which restricts, for each $x \in \mathcal{B}^\text{red}(H, F)$, to an $H_x^\text{red}(k_F)$-equivariant bijection from the set of unipotent elements of $H_x^\text{red}(k_F)$ to the set of nilpotent elements of $\text{Lie}(H_x^\text{red})(k_F)$.

We fix a character $\psi : F \to \mathbb{C}^\times$ whose restriction to $O_F$ factors through a non-trivial character of $k_F$. Furthermore, we fix a good bilinear form $B$ [DR09, §A] on the Lie-algebra $\mathfrak{h}$ of $H$. Reductive groups and their Lie-algebras will be endowed with the canonical Haar measures described in [DR09, §5.1]. Then, for any regular semi-simple element $X \in \text{Lie}(H)(F)$, the distribution on $\mathfrak{h}(F)$ defined by

$$f \mapsto \int_{H(F)/H_X(F)} \int_{\mathfrak{h}(F)} f(Y)\psi(B(\text{Ad}(g)X, Y))dYdg$$

is represented by a function, which we call $\hat{\mu}_X^H$. We also define

$$S\hat{\mu}_X^H = \sum_{X'} \hat{\mu}_X^{H'} ,$$

where the sum runs over a set of representatives for the $H(F)$-conjugacy classes inside the stable class of $X$.

We now let $Q_0 \in \text{Lie}(S_0)(O_F)$ be an element with strongly-regular reduction. This element exists due to the work of DeBacker and Reeder recalled above. Of course, $Q_0$ is automatically a regular semi-simple element of $\text{Lie}(G)(F)$, and hence

$$j \mapsto d_j(Q_0)$$

is a bijection between the set of $G'(F)$-conjugacy classes of admissible embeddings $j : S_0 \to G'$ and the set of $G'(F)$-conjugacy classes inside the set of elements of $\text{Lie}(G')(F)$ which are stably-conjugate to $Q_0$. We will denote the inverse of $j \mapsto d_j(Q_0)$ by $P \mapsto \phi_{Q_0, P}$.

The inner twist $\xi : G \to G'$ restricts to an isomorphism $Z_G \to Z_{G'}$ defined over $F$. We will identify these two groups from now on.
Lemma A.2. The function $S\Theta_\phi^{G'}$ is supported on the subset $G'_0(F)\mathcal{Z}_G(F)$. Moreover, for any $\gamma' \in G'_0(F)\mathcal{Z}_G(F)$ and $z \in \mathcal{Z}_G(F)$, the value of $S\Theta_\phi^{G'}(z\gamma')$ is given by

$$e(G, H)\theta_0(z) \sum_{P'} [\phi_{Q_0, P'}] \ast \theta_0(\gamma'_s) \mu_{Q'}^H(\log(\gamma'_s)),$$

where $\gamma' = \gamma'_s \gamma'_u$ is the topological Jordan decomposition of $\gamma'$, the sum over $P'$ runs over a set of representatives for the $G'_0\cdot$-stable classes of elements of $\text{Lie}(G'_0)\setminus(F)$ which are $G'$-stably conjugate to $Q_0$, and $H$ is equal to $[G'_0\gamma']^\circ$.

Proof. The vanishing result follows at once from [DR09, Lemma 9.3.1]. We turn to the formula. Let $j : S_0 \to G'$ be an admissible embedding, and $\pi_j \in \Pi_{G'}$ the corresponding representation. According to [Kal11b, Lemma 3.1.2] and [DR09, Lemma 12.4.3], we have

$$\Theta_{\pi_j} = e(G', A_G)\theta_j(z) \sum_{Q}[\phi_{dj(Q_0), Q}] \ast \theta_j(\gamma'_s) e(H, A_H)\mu_Q^H(\log(\gamma'_s)),$$

where $Q$ runs over a set of representatives for the $H(F)$-conjugacy classes inside the intersection of the $G'(F)$-conjugacy class of $dj(Q_0)$ with $\text{Lie}(H)(F)$. We recall that $A_G$ (resp. $A_H$) is the maximal split torus in the center of $G$ (resp. $H$).

Under the identification of $Z(G)$ with $Z(G')$ via $\xi$, the characters $\theta_0$ and $\theta_j$ have the same restriction to $Z(G)(F)$. Moreover, $\phi_{dj(Q_0), Q} \circ j = \phi_{Q_0, Q}$. Finally, we have $A_H = A_G$, since the centralizer of $Q$ in $H$ is a maximal torus of $H$ which is also an elliptic maximal torus of $G'$ (being stably-conjugate to the elliptic torus $S_0$). We conclude

$$\Theta_{\pi_j} = e(G', H)\theta_j(z) \sum_{Q}[\phi_{Q_0, Q}] \ast \theta_0(\gamma'_s) \mu_Q^H(\log(\gamma'_s)).$$

To obtain $S\Theta_\phi^{G'}$, we sum these formulas over the set of $G'(F)$-conjugacy classes of admissible embeddings $j : S_0 \to G'$. Using the bijection $j \leftrightarrow dj(Q_0)$ and its inverse $\phi_{Q_0, P} \leftrightarrow P$, we obtain

$$S\Theta_\phi^{G'} = e(G') e(G', H)\theta_0(z) \sum_{P} \sum_{Q}[\phi_{Q_0, Q}] \ast \theta_0(\gamma'_s) \mu_Q^H(\log(\gamma'_s)),$$

where now the sum over $P$ runs over the set of $G'(F)$-conjugacy classes inside the set of elements of $\text{Lie}(G')(F)$ which are stably conjugate to $Q_0$, while $Q$ runs over a set of representatives for the $H(F)$-conjugacy classes inside the intersection of the $G'(F)$-conjugacy class of $P$ with $\text{Lie}(H)(F)$. The double sum over $P, Q$ can be replaced by a single sum running over the $H(F)$-conjugacy classes inside the set of elements of $\text{Lie}(G')(F)$ which are stably-conjugate to $Q_0$ and belong to $\text{Lie}(H)(F)$. This single sum can then be written again as a double sum, where we first sum over a set of representatives $P'$ for the sets of $H$-stable classes in $\text{Lie}(H)(F)$ of elements which are $G'$-stably-conjugate to $Q_0$, and then sum over a set of representatives $Q'$ for the sets of $H(F)$-conjugacy classes inside the $H$-stable class of $P'$. With this re-indexing, we obtain

$$S\Theta_\phi^{G'} = e(G') e(G', H)\theta_0(z) \sum_{P'} \sum_{Q'}[\phi_{Q_0, Q'}] \ast \theta_0(\gamma'_s) \mu_{Q'}^H(\log(\gamma'_s)),$$

Since $P'$ and $Q'$ are elements of $\text{Lie}(H)(F)$ which are $H$-stably-conjugate, we have

$$[\phi_{Q_0, Q'}] \ast \theta_0(\gamma'_s) = [\phi_{Q_0, P'}] \ast \theta_0(\gamma'_s)$$
Combining this with \( e(G') e(G', H) = e(G, H) \) we obtain

\[
S\Theta^G_\phi = e(G, H) \theta_0(z) \sum_{p'} [\phi_{Q_0, p'}]_* \theta_0(\gamma'_s) \sum_{Q'} \hat{\mu}_p^H (\log(\gamma'_u)).
\]

and the statement follows. \( \square \)

**Proposition A.3.** Let \( \gamma \in G(F) \) and \( \gamma' \in G(F) \) be strongly-regular semi-simple elements whose stable classes correspond. Then

\[
S\Theta^G_\phi (\gamma) = S\Theta^{G'}_\phi (\gamma').
\]

**Proof.** We apply Lemma A.2 to both groups \( G \) and \( G' \). Since \( \gamma \in G_{ss}(F)_0 Z_G(F) \) if and only if \( \gamma' \in G'_{ss}(F)_0 Z_G(F) \), we see that we have to show

\[
e(G, H) \sum_p [\phi_{Q_0, p'}]_* \theta_0(\gamma_s) S\hat{\mu}_p^H (\log(\gamma_u)) = e(G, H') \sum_p [\phi_{Q_0, p'}]_* \theta_0(\gamma'_s) S\hat{\mu}_p^{H'} (\log(\gamma'_u)),
\]

when \( \gamma \in G_{ss}(F)_0 \) and \( \gamma' \in G'_{ss}(F)_0 \). Applying [Kal11b, Lemma 4.2.3] to the trivial ep twist \( (G, 1, 1) \) and \( \gamma \in G(F) \) we may assume that \( \gamma_s \in S_0(F) \). According to [Kot86, Prop 7.1], the group \( H \) is then unramified. Choose \( g \in G(\bar{F}) \) such that \( \xi \text{Ad}(g)\gamma = \gamma' \). Then

\[
\xi \circ \text{Ad}(g) : H \rightarrow H'
\]

is an inner twist. On the one hand, this implies via [Kal11b, Lemma 4.1.1] that

\[
e(G, H) e(G, H') = e(H, H') = \gamma_\psi(B) \gamma'_\psi(B')^{-1}
\]

where \( B \) and \( B' \) are compatible good bilinear forms on \( \mathfrak{h}(F) \) and \( \mathfrak{h}'(F) \), and \( \gamma \) are the corresponding Weil constants [Wal95, VIII]. On the other hand, the inner twist \( \xi \circ \text{Ad}(g) \) provides a bijection between the stable classes of regular semi-simple elliptic elements in \( \mathfrak{h}(F) \) and \( \mathfrak{h}'(F) \). This bijection restricts to a bijection \( P \leftrightarrow P' \) between the summation sets \( \{P\} \) and \( \{P'\} \) which has the property that whenever \( P \leftrightarrow P' \), we have

\[
[\phi_{Q_0, p'}]_* \theta_0(\gamma_s) = [\phi_{Q_0, p'}]_* \theta_0(\gamma'_s).
\]

In order to complete the proof, it will be enough to show

\[
\gamma_\psi(B) \hat{\mu}_p^H (\log(\gamma_u)) = \gamma'_\psi(B') \hat{\mu}_p^{H'} (\log(\gamma'_u))
\]

whenever \( P \leftrightarrow P' \). This is the main result of [Wal97], which is now unconditional thanks to [Ngo10] and [Wal06]. \( \square \)

**Corollary A.4.** The function \( S\Theta^{G'}_\phi \) is stable.
REFERENCES FOR APPENDIX A


