A slick proof of an identity on standard Young tableaux

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An introduction

In this note, we record a short, slick proof of the fundamental combinatorial identity

$$n! = \sum_{\lambda \vdash n} (f^\lambda)^2$$

Here, λ runs over the partitions of n and f^λ denotes the number of standard Young tableaux of shape λ. This proof was presented by John Stembridge in Math 631 at University of Michigan, and I liked it so much I wanted to write it down with some nice pictures. Traditionally, one proves the above identity via an explicit bijection between pairs of standard Young tableaux of size n and permutations on n letters; this is known as the Robinson-Schensted correspondence. Our nontraditional approach is more algebraic in flavor and, according to Stembridge, is an archetypical example of the power of such algebraic recastings in combinatorics. In particular, this approach could be seen as an upgrade of the classic combinatorial technique "count in two ways" to an algebraic "compute in two ways."

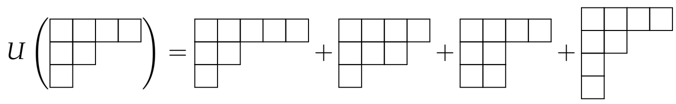
The proof

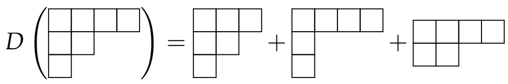
Step 1: Given a partition λ, we write D_λ to denote its Young diagram. Let $\mathcal{Y}$ be the $\mathbb{C}$-vector space spanned by all diagrams D_λ (coming from all partitions of all integers). We introduce the *up* operator $U : \mathcal{Y} \to \mathcal{Y}$ and *down* operator $D : \mathcal{Y} \to \mathcal{Y}$ on this space, given by

$$U(D_\lambda) = \sum_{\lambda \subset \mu, |\mu| = |\lambda|+1} D_\mu$$

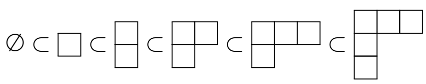
$$D(D_\lambda) = \sum_{\lambda \supset \mu, |\mu| = |\lambda|-1} D_\mu$$

In words, U takes in a Young diagram and adds up all the ways to append one box and obtain a valid Young diagram; D takes in a Young diagram and adds up all the ways to excise one box and obtain a valid Young diagram. For example, we may calculate





The key combinatorial observation is that standard Young tableau are in bijection with maximal chains of Young diagrams ordered by inclusion. For example, the chain



gives a recipe for filling the squares, namely to fill them in the order in which they are appended. Thus, the chain shown above corresponds to the standard Young tableau

1	3	4
2		
5		

(and conversely, given a standard Young tableau, one can easily reconstruct the corresponding maximal chain). In particular, this observation implies $U^n(D_\varnothing) = \sum_{\lambda \vdash n} f^\lambda D_\lambda$ and $D^n(D_\lambda) = f^\lambda D_\varnothing$. Applying these operations in turn, we obtain the promising result

$$D^n U^n (D_\varnothing) = \left(\sum_{\lambda \vdash n} (f^\lambda)^2 \right) D_\varnothing$$

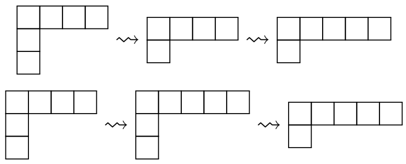
To show the desired identity, we will now compute this vector another way.

Step 2: Remarkably, the operators U and D satisfy the defining relation of the Weyl algebra, namely

$$\boxed{DU - UD = 1}$$

In this step, we show the relation holds. To do so, we should study and count the summands appearing in $UD(D_\lambda)$ and $DU(D_\lambda)$. The summands appearing in $UD(D_\lambda)$ are precisely the diagrams obtained from D_λ by excising a corner block and reattaching it somewhere that yields another valid Young diagram (possibly reattaching to the same position whence it came, yielding D_λ again). Similarly, the summands appearing in $DU(D_\lambda)$ are precisely the diagrams obtained from D_λ by attaching a block to produce a valid Young diagram, then excising a corner (possibly excising the same block that was attached, yielding D_λ again).

Call any summand not equal to D_λ a *proper* summand. Note that for each appearance of D_μ as a proper summand of $UD(D_\lambda)$, there is a corresponding appearance as a proper summand of $DU(D_\lambda)$ (and vice versa). To see this, just imagine doing the diagram surgeries in reverse, e.g.



Hence, when calculating $(DU - UD)(D_\lambda)$, all the proper summands cancel, leaving just

$$(DU - UD)(D_\lambda) = (\#\{\text{summands in } U(D_\lambda)\} - \#\{\text{summands in } D(D_\lambda)\})D_\lambda =: a_\lambda D_\lambda$$

To complete **Step 2**, we should show $a_\lambda = 1$ for all partitions λ. To attach a block to D_λ and obtain a valid Young diagram, we can append the block to the end of any row with length strictly less than that of the row above it, or we can put the block as a new row. To excise a block from D_λ and obtain a valid Young diagram, we can slice out the block at the end of any row whose length is strictly greater than that of the row below it. Indeed, this implies

$$a_\lambda = (\#\{\text{pairs of consec. rows with unequal lengths}\} + 1) - (\#\{\text{pairs of consec. rows with unequal lengths}\}) = 1$$

Step 3: It's smooth sailing from here. We just compute with the relation $DU - UD = 1$. A quick computation shows this relation implies $DU^n = nU^{n-1} + U^n D$ for all n, hence

$$D^n U^n (D_\varnothing) = nD^{n-1}U^{n-1}(D_\varnothing) + U^n D(D_\varnothing) = nD^{n-1}U^{n-1}(D_\varnothing)$$

By induction, this clearly implies $D^n U^n (D_\varnothing) = n! D_\varnothing$, thereby completing the proof.

A brief remark

Apparently, Young's lattice (the lattice of Young diagrams ordered by inclusion) is an example of what's called a **differential poset**, on account of the appearance of the Weyl algebra action that is analogous to the relation $[x, \partial_x] = 1$ for differential operators. There is basically one other interesting example of a differential poset, which is called the Young-Fibonacci lattice. Stanley conjectures that every differential poset must be squished between Young's lattice and the Young-Fibonacci lattice, in the sense that for any differential poset, the number $r(n)$ of vertices of rank n satisfies

$$p(n) \le r(n) \le F_n$$

where $p(n)$ is the number of partitions of n and F_n is the nth Fibonacci number. The upper bound was proven in 2012 and the lower bound remains open. I thought this was cool.