ENUMERATION OF POINTS, LINES, PLANES, ETC.

JUNE HUH AND BOTONG WANG

1. Introduction

One of the earliest results in enumerative combinatorial geometry is the following theorem of de Bruijn and Erdős [dBE48]:

> *Every finite set of points E in a projective plane determines at least $|E|$ lines, unless E is contained in a line.*

See [dW66, dW75] for an interesting account of its history and a survey of known proofs.

The following more general statement, conjectured by Motzkin in [Mot36], was subsequently proved by many in various settings:

> *Every finite set of points E in a projective space determines at least $|E|$ hyperplanes, unless E is contained in a hyperplane.*

Motzkin proved the above for E in real projective spaces [Mot51]. Basterfield and Kelly [BK68] proved the statement in general, and Greene [Gre70] strengthened the result by showing that there is a *matching* from E to the set of hyperplanes determined by E, unless E is contained in a hyperplane:

> *For every point in E one can choose a hyperplane containing the point in such a way that no hyperplane is chosen twice.*

Mason [Mas72] and Heron [Her73] obtained similar results by different methods.

Let $\mathbb{P}$ be the projectivization of an r-dimensional vector space over a field, $E \subseteq \mathbb{P}$ be a finite subset not contained in any hyperplane, and $\mathscr{L}$ be the poset of subspaces spanned by the subsets of E. The poset $\mathscr{L}$ is a graded lattice, and its rank function satisfies the submodular inequality

$$\text{rank}(F_1) + \text{rank}(F_2) \geq \text{rank}(F_1 \vee F_2) + \text{rank}(F_1 \wedge F_2) \text{ for all } F_1, F_2 \in \mathscr{L}.$$

Write $\mathscr{L}^p$ for the set of rank p elements in the lattice. Thus $\mathscr{L}^1$ is the set of points in E, $\mathscr{L}^2$ is the set of lines joining points in E, and $\mathscr{L}^r$ is the set with one element $\mathbb{P}$. Posets obtained in this way are standard examples of *geometric lattices* [Wel76]. These include the lattice of all subsets of a finite set (Boolean lattices), the lattice of all partitions of a finite set (partition lattices), and the lattice of all subspaces of a finite vector space (projective geometries). In [DW75], Dowling and Wilson further generalized the above results for geometric lattices:

For every nonnegative integer p less than $\frac{r}{2}$, there is a matching from the set of rank at most p elements of $\mathscr{L}$ to the set of corank at most p elements of $\mathscr{L}$.

The case $p \leq 1$ covers the results introduced before. Another proof of the same result is given by Kung from the point of view of Radon transformations [Kun79, Kun86].

In [DW74, DW75], Dowling and Wilson stated the following "top-heavy" conjecture.

Conjecture 1. Let $\mathscr{L}$ be a geometric lattice of rank r.

(1) For every nonnegative integer p less than $\frac{r}{2}$,

$$|\mathscr{L}^p| \leq |\mathscr{L}^{r-p}|.$$

In fact, there is an injective map $\iota : \mathscr{L}^p \to \mathscr{L}^{r-p}$ satisfying $x \leq \iota(x)$ for all x.

(2) For every nonnegative integer p less than $\frac{r}{2}$,

$$|\mathscr{L}^p| \leq |\mathscr{L}^{p+1}|.$$

In fact, there is an injective map $\iota : \mathscr{L}^p \to \mathscr{L}^{p+1}$ satisfying $x \leq \iota(x)$ for all x.

The conjecture was reproduced in [Sta12, Exercise 3.37] and [KRY09, Exercise 3.5.7]. For an overview and related results, see [Aig87]. When $\mathscr{L}$ is a Boolean lattice or a projective geometry, the validity of Conjecture 1 is a classical result [Kan72]. We refer to [HM+13] and [Sta13] for recent expositions. In these cases, the result implies that $\mathscr{L}$ has the *Sperner property*:

The maximal number of incomparable elements in $\mathscr{L}$ is the maximum of $|\mathscr{L}^p|$ over p.

Kung proved the second part of Conjecture 1 for partition lattices in [Kun93]. Later he proved the second part of Conjecture 1 for $p \leq 2$ when every line contains the same number of points [Kun00].

We now state our main result. As before, we write $\mathbb{P}$ for the projectivization of an r-dimensional vector space over a field.

Theorem 2. Let $E \subseteq \mathbb{P}$ be a finite subset not contained in any hyperplane, and $\mathscr{L}$ be the poset of subspaces spanned by subsets of E.

(1) For all nonnegative integers $p \leq q$ satisfying $p + q \leq r$,

$$|\mathscr{L}^p| \leq |\mathscr{L}^{r-q}|.$$

In fact, there is an injective map $\iota : \mathscr{L}^p \to \mathscr{L}^{r-q}$ satisfying $x \leq \iota(x)$ for all x.

(2) For every positive integer p less than $\frac{r}{2}$,

$$0 \leq |\mathscr{L}^{p+1}| - |\mathscr{L}^p| \leq \Big(|\mathscr{L}^p| - |\mathscr{L}^{p-1}|\Big)^{\langle p \rangle}.$$

Equivalently, $|\mathscr{L}^0|, |\mathscr{L}^1| - |\mathscr{L}^0|, \ldots, |\mathscr{L}^{p+1}| - |\mathscr{L}^p|$ is the h-vector of a shellable simplicial complex.

For undefined notions in the second statement, we refer to [Sta96, Chapter II].

This settles Conjecture 1 for all $\mathscr{L}$ realizable over some field. We believe this to be a good demonstration of the power of the main ingredient in the proof, the decomposition theorem package for intersection complexes [BBD82].

Example 3. Modular geometric lattices, such as Boolean lattices or finite projective geometries, satisfies a stronger matching property:

For every p, there is an injective or surjective map $\iota : \mathscr{L}^p \to \mathscr{L}^{p+1}$ satisfying $x \leq \iota(x)$.

As noted before, this implies that modular geometric lattices have the Sperner property.

Dilworth and Greene constructed in [DG71] a configuration of 21 points in any 10-dimensional projective space with the property that there is no injective or surjective map

$$\iota : \mathscr{L}^6 \to \mathscr{L}^7, \qquad x \leq \iota(x).$$

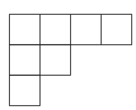
Canfield [Can78] discovered that such "no matching" successive rank level sets as above can be found in partition lattices with sufficiently many elements (exceeding $10^{10^{20}}$). These geometric lattices fail to satisfy the Sperner property.

It has been conjectured by Rota that the sizes of the rank level sets of a geometric lattice form a unimodal sequence [Rot71, RH71]:

$$|\mathscr{L}^0| \leq \cdots \leq |\mathscr{L}^{p-1}| \leq |\mathscr{L}^p| \geq |\mathscr{L}^{p+1}| \geq \cdots \geq |\mathscr{L}^r| \text{ for some } p.$$

Stronger versions of the above conjecture were proposed by Mason [Mas72]. The unimodality conjecture for the "upper half" remains as an outstanding open problem.

Example 4. Let λ be a partition of a positive integer, which we view as a Young diagram [Ful97]. For example, the partition $(4, 2, 1)$ of 7 corresponds to the Young diagram



Young's lattice associated to λ is the graded lattice $\mathscr{L}_\lambda$ of all partitions whose Young diagram fit inside λ. Björner and Ekedahl [BE09] showed that $\mathscr{L}_\lambda$ satisfies both conclusions of Conjecture 1 when r is the number of boxes in λ:

(1) For p less than $\frac{r}{2}$, there is an injective map $\iota : \mathscr{L}^p \to \mathscr{L}^{r-p}$ satisfying $x \leq \iota(x)$ for all x.

(2) For p less than $\frac{r}{2}$, there is an injective map $\iota : \mathscr{L}^p \to \mathscr{L}^{p+1}$ satisfying $x \leq \iota(x)$ for all x.

According to Stanton [Sta90], Young's lattice for the partition $(8, 8, 4, 4)$ defines a nonunimodal sequence

$$|\mathscr{L}^*_\lambda| = 1,\ 1,\ 2,\ 3,\ 5,\ 6,\ 9,\ 11,\ 15,\ 17,\ 21,\ 23,\ 27,\ 28,\ 31,\ 30,\ 31,\ 27,\ 24,\ 18,\ 14,\ 8,\ 5,\ 2,\ 1.$$

Face lattices of simplicial polytopes behaves similarly, starting from dimension 20 [BL80, Bjö81].

2. THE GRADED MÖBIUS ALGEBRA

2.1. We use the language of matroids, and use [Wel76] and [Oxl11] as basic references. Let r and n be positive integers, and let M be a rank r simple matroid on the ground set

$$E = \{1, \ldots, n\}.$$

Write $\mathscr{L}$ for the lattice of flats of M. We define a graded analogue of the Möbius algebra for $\mathscr{L}$.

Definition 5. Introduce symbols y_F, one for each flat F of M, and construct vector spaces

$$B^p(\mathrm{M}) = \bigoplus_{F \in \mathscr{L}^p} \mathbb{Q}\, y_F, \quad B^*(\mathrm{M}) = \bigoplus_{F \in \mathscr{L}} \mathbb{Q}\, y_F.$$

We equip $B^*(\mathrm{M})$ with the structure of a commutative graded algebra over $\mathbb{Q}$ by setting

$$y_{F_1} y_{F_2} = \begin{cases} y_{F_1 \vee F_2} & \text{if } \operatorname{rank}(F_1) + \operatorname{rank}(F_2) = \operatorname{rank}(F_1 \vee F_2), \\ 0 & \text{if } \operatorname{rank}(F_1) + \operatorname{rank}(F_2) > \operatorname{rank}(F_1 \vee F_2). \end{cases}$$

For simplicity, we write $y_1, \ldots, y_n$ instead of $y_{\{1\}}, \ldots, y_{\{n\}}$.

This algebra was introduced by Maeno and Numata in a slightly different form in [MN12], who used it to show that modular geometric lattices have the Sperner property. Note that $B^*(\mathrm{M})$ is generated by $B^1(\mathrm{M})$ as an algebra: If I_F is any basis of a flat F of M, then

$$y_F = \prod_{i \in I_F} y_i.$$

Unlike its ungraded counterpart, which is isomorphic to the product of $\mathbb{Q}$'s as a $\mathbb{Q}$-algebra [Sol67], the graded Möbius algebra $B^*(\mathrm{M})$ has a nontrivial algebra structure. Define

$$L = \sum_{i \in E} y_i.$$

We deduce Theorem 2 from the following algebraic statement. Similar injectivity properties have appeared in the context of Kac-Moody Schubert varieties [BE09] and toric hyperkähler varieties [Hau05].

Theorem 6. For nonnegative integer p less than $\frac{r}{2}$, the multiplication map

$$B^p(\mathrm{M}) \longrightarrow B^{r-p}(\mathrm{M}), \qquad \xi \longmapsto L^{r-2p}\, \xi$$

is injective, when M is realizable over some field.

It follows that, for nonnegative integers $p \leq q$ satisfying $p + q \leq r$, the multiplication map

$$B^p(\mathrm{M}) \longrightarrow B^{r-q}(\mathrm{M}), \qquad \xi \longmapsto L^{r-p-q}\, \xi$$

is injective, when M is realizable over some field. To deduce Theorem 2 from this, consider the matrix of the multiplication map with respect to the standard bases of the source and the target. Entries of this matrix are labelled by pairs of elements of $\mathscr{L}$, and all the entries corresponding to incomparable pairs are zero. The matrix has full rank, so there is a maximal square submatrix with nonzero determinant. In the standard expansion of this determinant, there must be a nonzero term, and the permutation corresponding to this term produces the injective map ι. The second part of Theorem 2 is obtained from Macaulay's theorem applied to the quotient of $B^*(\mathrm{M})$ by the ideal generated by L, see [Sta96, Chapter II].

We conjecture that Theorem 6 holds without the assumption of realizability.

2.2. Let M be as before, and let $\overline{\mathrm{M}}$ be a simple matroid on the ground set

$$\overline{E} = \{0, 1, \ldots, n\}.$$

Let $\overline{\mathscr{L}}$ be the lattice of flats of $\overline{\mathrm{M}}$. We suppose that $\mathrm{M} = \overline{\mathrm{M}}/0$, that is, M is obtained from $\overline{\mathrm{M}}$ by contracting the element 0.

Definition 7. Introduce variables $x_{\overline{F}}$, one for each nonempty proper flat $\overline{F}$ of $\overline{\mathrm{M}}$, and set

$$S_{\overline{\mathrm{M}}} = \mathbb{Q}[x_{\overline{F}}]_{\overline{F} \neq \varnothing, \overline{F} \neq \overline{E}, \overline{F} \in \overline{\mathscr{L}}}.$$

The *Chow ring* $A^*(\overline{\mathrm{M}})$ is the quotient of $S_{\overline{\mathrm{M}}}$ by the ideal generated by the linear forms

$$\sum_{i_1 \in \overline{F}} x_{\overline{F}} - \sum_{i_2 \in \overline{F}} x_{\overline{F}},$$

one for each pair of distinct elements i_1 and i_2 of $\overline{E}$, and the quadratic monomials

$$x_{\overline{F}_1} x_{\overline{F}_2},$$

one for each pair of incomparable nonempty proper flats of $\overline{\mathrm{M}}$.

The algebra $A^*(\overline{\mathrm{M}})$ and its generalizations were studied by Feichtner and Yuzvinsky in [FY04]. For every i in E, we define an element of $A^1(\overline{\mathrm{M}})$ by setting

$$\beta_i = \sum_{\overline{F}} x_{\overline{F}},$$

where the sum is over all flats $\overline{F}$ of $\overline{\mathrm{M}}$ that contain 0 and do not contain i. The linear relations show that we may equivalently define

$$\beta_i = \sum_{\overline{F}} x_{\overline{F}},$$

where the sum is over all flats $\overline{F}$ of $\overline{\mathrm{M}}$ that contain i and do not contain 0. We record here three basic implications of the defining relations of $A^*(\overline{\mathrm{M}})$:

(R1) When $\overline{F}$ is a nonempty proper flat of $\overline{\mathrm{M}}$ containing exactly one of i and 0,

$$\beta_i \cdot x_{\overline{F}} = 0.$$

This follows from the quadratic monomial relations.

(R2) For every element i in E,

$$\beta_i \cdot \beta_i = 0.$$

This follows from the previous statement.

(R3) For any two maximal chains of nonempty proper flats of $\overline{\mathrm{M}}$, say $\{\overline{F}_k\}_{1\leq k}$ and $\{\overline{G}_k\}_{1\leq k}$,

$$\prod_{k=1}^{r} x_{\overline{F}_k} = \prod_{k=1}^{r} x_{\overline{G}_k} \neq 0.$$

The proofs of (R1) and (R2) are straightforward. The proof of (R3) can be found in [AHK, Section 5].

Proposition 8. There is a unique injective graded $\mathbb{Q}$-algebra homomorphism

$$\varphi : B^*(\mathrm{M}) \longrightarrow A^*(\overline{\mathrm{M}}), \qquad y_i \longmapsto \beta_i.$$

Proof. First, we show that there is a well-defined $\mathbb{Q}$-linear map

$$\varphi : B^*(\mathrm{M}) \longrightarrow A^*(\overline{\mathrm{M}}), \qquad y_F \longmapsto \prod_{i\in I_F} \beta_i,$$

where I_F is any basis of a flat F of M. In other words, if J_F is any other basis of F, then

$$\prod_{i\in I_F} \beta_i = \prod_{i\in J_F} \beta_i.$$

Since any basis of F can be obtained from any other basis of F by a sequence of elementary exchanges, it is enough to check the equality in the special case when $I_F \setminus J_F = \{1\}$ and $J_F \setminus I_F = \{2\}$. Assuming that this is the case, we write the left-hand side of the claimed equality by

$$\Big(\prod_{i\in I_F\cap J_F} \beta_i\Big)\Big(\sum_{\overline{F}} x_{\overline{F}}\Big),$$

where the sum is over all nonempty proper flats $\overline{F}$ of $\overline{\mathrm{M}}$ that contain 0 and does not contain 1. The relation (R1) shows that we may take the sum only over those $\overline{F}$ satisfying

$$0 \in \overline{F},\ \ 1 \notin \overline{F},\ \ \text{and}\ \ I_F\cap J_F \subseteq \overline{F}.$$

Since $I_F \cup \{0\}$ and $J_F \cup \{0\}$ are bases of the same flat of $\overline{\mathrm{M}}$, the above condition is equivalent to

$$0 \in \overline{F},\ \ 2 \notin \overline{F},\ \ \text{and}\ \ I_F\cap J_F \subseteq \overline{F}.$$

This proves the claimed equality, which shows that φ is a well-defined linear map.

Second, we show that φ is a ring homomorphism. Given flats F_1 and F_2 of M, we show

$$\Big(\prod_{i\in I_{F_1}} \beta_i\Big)\Big(\prod_{i\in I_{F_2}} \beta_i\Big) = 0\ \ \text{when the rank of } F_1\vee F_2 \text{ is less than } |I_{F_1}| + |I_{F_2}|.$$

If the independent sets I_{F_1} and I_{F_2} intersect, this follows from the relation (R2). If otherwise, the condition on the rank of $F_1 \vee F_2$ implies that there are two distinct bases of $F_1 \vee F_2$ contained in $I_{F_1} \cup I_{F_2}$, say

$$I_{F_1\vee F_2} \subseteq I_{F_1} \cup I_{F_2}\ \ \text{and}\ \ J_{F_1\vee F_2} \subseteq I_{F_1} \cup I_{F_2}.$$

Using the first part of the proof, once again from the relation (R2) we deduce that

$$\Big(\prod_{i\in I_{F_1}}\beta_i\Big)\Big(\prod_{i\in I_{F_2}}\beta_i\Big)=\Big(\prod_{i\in I_{F_1\vee F_2}}\beta_i\Big)\Big(\prod_{i\in I_{F_1}\cup I_{F_2}\setminus I_{F_1\vee F_2}}\beta_i\Big)$$
$$=\Big(\prod_{i\in J_{F_1\vee F_2}}\beta_i\Big)\Big(\prod_{i\in I_{F_1}\cup I_{F_2}\setminus I_{F_1\vee F_2}}\beta_i\Big)=0.$$

This completes the proof that φ is a ring homomorphism.

Third, we show that φ is injective in degree r. Choose any ordered basis $\{i_1,\ldots,i_r\}$ of M, and set

$$\overline{G}_q=\text{the closure of }\Big\{0,i_1,\ldots,i_{q-1}\Big\}\text{ in }\overline{\mathrm{M}},\quad\text{for }q=1,\ldots,r.$$

We deduce from the relation (R1) that

$$\Big(\beta_{i_1}\cdots\beta_{i_{r-1}}\Big)\beta_{i_r}=\Big(\beta_{i_1}\cdots\beta_{i_{r-1}}\Big)x_{\overline{G}_r}.$$

Similarly, for any positive integer $q\le r$,

$$\Big(\beta_{i_1}\cdots\beta_{i_{q-2}}\beta_{i_{q-1}}\Big)x_{\overline{G}_q}=\Big(\beta_{i_1}\cdots\beta_{i_{q-2}}\Big)x_{\overline{G}_{q-1}}x_{\overline{G}_q}=x_{\overline{G}_1}\cdots x_{\overline{G}_{q-1}}x_{\overline{G}_q},$$

since $\overline{G}_{q-1}$ is the only flat of $\overline{\mathrm{M}}$ containing $\overline{G}_{q-1}$, comparable to $\overline{G}_q$, and not containing i_{q-1}. Combining the above formulas, we deduce from the relation (R3) that

$$\beta_{i_1}\cdots\beta_{i_r}=x_{\overline{G}_1}\cdots x_{\overline{G}_r}\neq 0.$$

This proves that φ is injective in degree r.

Last, we show that φ is injective in any degree q less than r. For this we analyze the bilinear map given by the multiplication

$$\varphi\Big(B^q(\mathrm{M})\Big)\times\bigoplus_{\overline{G}}\mathbb{Q}\,x_{\overline{G}}\longrightarrow A^{q+1}(\overline{\mathrm{M}}),$$

where the sum is over all rank $q{+}1$ flats $\overline{G}$ of $\overline{\mathrm{M}}$ containing 0. For any independent set $\{i_1,\ldots,i_q\}$ of M, we claim that, for any $\overline{G}$ as above,

$$\Big(\beta_{i_1}\cdots\beta_{i_q}\Big)x_{\overline{G}}\neq 0\ \text{ if and only if }\ \overline{G}=\text{the closure of }\Big\{0,i_1,\ldots,i_q\Big\}\text{ in }\overline{\mathrm{M}}.$$

The "if" statement follows from the analysis made above. For the "only if" statement, suppose that the product is nonzero. Since $\overline{G}$ contains 0, it must contain $i_1,\ldots,i_q$ by the relation (R1). Since $\overline{G}$ and the closure both have the same rank, we have

$$\overline{G}=\text{the closure of }\Big\{0,i_1,\ldots,i_q\Big\}\text{ in }\overline{\mathrm{M}}.$$

It follows from the claim that the image of the basis $\{y_F\}$ of $B^q(\mathrm{M})$ under φ is a linearly independent in $A^q(\overline{\mathrm{M}})$. □

3. The simplex, the cube, and the permutohedron

3.1. In this section, we prove our main result, Theorem 6. For undefined terms in toric geometry and intersection theory, we refer to [Ful93] and [Ful98]. All the Chow groups and rings will have rational coefficients.

As in the previous section, we fix a positive integer n and work with the sets

$$E = \{1, \ldots, n\} \ \text{ and } \ \overline{E} = \{0, 1, \ldots, n\}.$$

Let $\mathbb{Z}^{\overline{E}}$ be the abelian group generated by the basis vectors $\mathbf{e}_i$ corresponding to $i \in \overline{E}$. For an arbitrary subset $\overline{I} \subseteq \overline{E}$, we define

$$\mathbf{e}_{\overline{I}} = \sum_{i \in \overline{I}} \mathbf{e}_i.$$

We associate to $\overline{E}$ the abelian group $N_{\overline{E}} = \mathbb{Z}^{\overline{E}} / \langle \mathbf{e}_{\overline{E}} \rangle$ and the vector space $N_{\overline{E},\mathbb{R}} = N_{\overline{E}} \otimes_{\mathbb{Z}} \mathbb{R}$.

(1) Let $\Sigma(\mathrm{S}_n)$ be the normal fan of the standard n-dimensional simplex

$$\mathrm{S}_n = \mathrm{conv}\Big\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n\Big\} \subseteq \mathbb{R}^{\overline{E}}.$$

There are $(n+1)$ maximal cones in the fan, one for each maximal proper subset $\overline{I}$ of $\overline{E}$:

$$\sigma_{\overline{I}} = \mathrm{cone}\Big\{\mathbf{e}_i \mid i \in \overline{I}\Big\} \subseteq N_{\overline{E},\mathbb{R}}.$$

The fan defines $\mathbb{P}^n$, whose homogeneous coordinates are labelled by $i \in \overline{E}$.

(2) Let $\Sigma(\mathrm{C}_n)$ be the normal fan of the standard n-dimensional cube

$$\mathrm{C}_n = \mathrm{conv}\Big\{\pm \mathbf{e}_1, \ldots, \pm \mathbf{e}_n\Big\} \subseteq \mathbb{R}^{E}.$$

There are 2^n maximal cones in the fan, one for each subset I of E:

$$\sigma_I = \mathrm{cone}\Big\{\mathbf{e}_i \mid i \in I\Big\} - \mathrm{cone}\Big\{\mathbf{e}_i \mid i \notin I\Big\} \subseteq \mathbb{R}^{E}.$$

The fan defines $(\mathbb{P}^1)^n$, whose multi-homogeneous coordinates are labelled by $i \in E$.

(3) Let $\Sigma(\mathrm{P}_n)$ be the normal fan of the standard n-dimensional permutohedron

$$\mathrm{P}_n = \mathrm{conv}\Big\{(x_0, \ldots, x_n) \mid x_0, \ldots, x_n \text{ is a permutation of } 0, \ldots, n\Big\} \subseteq \mathbb{R}^{\overline{E}}.$$

There are $(n+1)!$ maximal cones in the fan, one for each maximal chain $\mathscr{I}$ in $2^{\overline{E}}$:

$$\sigma_{\mathscr{I}} = \mathrm{cone}\Big\{\mathbf{e}_{\overline{I}} \mid \overline{I} \in \mathscr{I}\Big\} \subseteq N_{\overline{E},\mathbb{R}}.$$

The fan $\Sigma(\mathrm{P}_n)$ defines the permutohedral space X_{A_n}. See [BB11] for a detailed study of X_{A_n} and its analogues for other root systems.

The inclusion $\mathbb{Z}^E \subseteq \mathbb{Z}^{\overline{E}}$ induces an isomorphism

$$\psi^{-1} : \mathbb{R}^E \longrightarrow N_{\overline{E},\mathbb{R}}.$$

This identifies the underlying vector spaces of the normal fans $\Sigma(\mathrm{S}_n)$, $\Sigma(\mathrm{P}_n)$, $\Sigma(\mathrm{C}_n)$:

$$\begin{tikzcd} & N_{\overline{E},\mathbb{R}} \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\psi"] & \\ N_{\overline{E},\mathbb{R}} & & \mathbb{R}^E. \end{tikzcd}$$

We observe that id and ψ induce morphisms between the fans and their toric varieties

$$\begin{tikzcd} & \Sigma(\mathrm{P}_n) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \Sigma(\mathrm{S}_n) & & \Sigma(\mathrm{C}_n), \end{tikzcd} \qquad \begin{tikzcd} & X_{A_n} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^n & & (\mathbb{P}^1)^n. \end{tikzcd}$$

The morphism p_1 is the standard barycentric subdivision. We check that p_2 is a subdivision.

Proposition 9. The isomorphism ψ induces a morphism p_2.

In other words, the image of a cone in $\Sigma(\mathrm{P}_n)$ under ψ is contained in a cone in $\Sigma(\mathrm{C}_n)$.

Proof. For each $i \in E$, define ψ_i as the composition of ψ with the i-th projection

$$\psi_i = \mathrm{proj}_i \circ \psi, \qquad \mathrm{proj}_i : \mathbb{R}^E \longrightarrow \mathbb{R}^{\{i\}} \simeq \mathbb{R}.$$

For any subset $\overline{I} \subseteq \overline{E}$, we have

$$\psi_i(\mathbf{e}_{\overline{I}}) = \begin{cases} \mathbf{e}_i & \text{if } \overline{I} \text{ contains } i \text{ and does not contain } 0, \\ -\mathbf{e}_i & \text{if } \overline{I} \text{ contains } 0 \text{ and does not contain } i, \\ 0 & \text{if otherwise.} \end{cases}$$

It is enough to check that ψ_i induces a morphism $\Sigma(\mathrm{P}_n) \longrightarrow \Sigma(\mathrm{C}_1)$.

Recall that any nonzero cone in the normal fan of P_n is of the form

$$\sigma_{\mathscr{I}} = \mathrm{cone}\Big\{\mathbf{e}_{\overline{I}} \mid \overline{I} \in \mathscr{I}\Big\},$$

where $\mathscr{I}$ is a nonempty chain in $2^{\overline{E}}$. Viewing $\mathscr{I}$ as an ordered collection of sets, we see that

$$\begin{cases} \psi_i(\sigma_{\mathscr{I}}) \text{ is contained in the cone generated by } \mathbf{e}_i \text{ if } i \text{ appears before } 0 \text{ in } \mathscr{I}, \text{ and} \\ \psi_i(\sigma_{\mathscr{I}}) \text{ is contained in the cone generated by } -\mathbf{e}_i \text{ if } i \text{ appears after } 0 \text{ in } \mathscr{I}. \end{cases}$$

Thus the image of a cone in $\Sigma(\mathrm{P}_n)$ under ψ_i is contained in a cone in $\Sigma(\mathrm{C}_1)$, for each $i \in E$. $\square$

Geometrically, π_1 is the blowup of all the torus invariant points in $\mathbb{P}^n$, all the strict transforms of torus invariant $\mathbb{P}^1$'s in $\mathbb{P}^n$, all the strict transforms of torus invariant $\mathbb{P}^2$'s in $\mathbb{P}^n$, and so on. The map π_2 is the blowup of points 0^n and ∞^n, all the strict transforms of torus invariant $\mathbb{P}^1$'s in $(\mathbb{P}^1)^n$ containing 0^n or ∞^n, all the strict transforms of torus invariant $(\mathbb{P}^1)^2$'s in $(\mathbb{P}^1)^n$ containing 0^n or ∞^n, and so on.

Remark 10. For later use, we record here a combinatorial description of the pullback of piecewise linear functions under the linear map $\psi_i = \mathrm{proj}_i \circ \psi$:

> *Let α be the piecewise linear function on $\Sigma(\mathrm{C}_1)$ determined by its values*
> $$\alpha(\mathbf{e}_i) = 1 \ \textit{ and } \ \alpha(-\mathbf{e}_i) = 0.$$
> *Then $\psi_i^*(\alpha)$ is the piecewise linear function on $\Sigma(\mathrm{P}_n)$ determined by its values*
> $$\psi_i^*(\alpha)(\mathbf{e}_{\overline{I}}) = \begin{cases} 1 & \textit{if } \overline{I} \textit{ contains } i \textit{ and does not contain } 0, \\ 0 & \textit{if otherwise.} \end{cases}$$

Using the correspondence between piecewise linear functions on fans and torus invariant divisors on toric varieties [Ful93, Chapter 3], the above can be used to describe the pullback homomorphism between the Chow rings

$$\pi_2^* : A^*((\mathbb{P}^1)^n) \longrightarrow A^*(X_{A_n}).$$

Explicitly, writing y_i for the divisor of $\mathbf{e}_i$ in $\Sigma(\mathrm{C}_n)$ and $x_{\overline{I}}$ for the divisor of $\mathbf{e}_{\overline{I}}$ in $\Sigma(\mathrm{P}_n)$,

$$\pi_2^*(y_i) = \sum_{\overline{I}} x_{\overline{I}},$$

where the sum is over all subsets $\overline{I} \subseteq \overline{E}$ that contain i and do not contain 0.

3.2. Let M be a simple matroid on E, and let $\overline{\mathrm{M}}$ be a simple matroid on $\overline{E}$ with $\mathrm{M} = \overline{\mathrm{M}}/0$. For simplicity, we take $\overline{\mathrm{M}}$ to be the direct sum of M and the rank 1 matroid on $\{0\}$, so that M and $\overline{\mathrm{M}}$ share the same set of circuits.

Suppose that M is realizable over some field. Then M is realizable over some finite field, and hence over the algebraically closed field $\overline{\mathbb{F}}_p$ for some prime number p. The matroid $\overline{\mathrm{M}}$ is realizable over the same field, say by a spanning set of vectors

$$\overline{\mathscr{A}} = \{f_0, f_1, \ldots, f_n\} \subseteq \overline{\mathbb{F}}_p^{r+1}.$$

Dually, the realization $\overline{\mathscr{A}}$ of $\overline{\mathrm{M}}$ corresponds to an injective linear map between projective spaces

$$i_{\overline{\mathscr{A}}} : \mathbb{P}^r \longrightarrow \mathbb{P}^n, \qquad i_{\overline{\mathscr{A}}} = [f_0 : f_1 : \cdots : f_n].$$

The collection $\mathscr{A} = \{f_1, \ldots, f_n\}$ is a realization of the matroid M.

The restriction of the torus invariant hyperplanes of $\mathbb{P}^n$ to $\mathbb{P}^r$ defines an arrangement of hyperplanes in $\mathbb{P}^r$, which we denote by the same symbol $\overline{\mathscr{A}}$. We use $i_{\overline{\mathscr{A}}}$ to construct the commutative diagram

$$\begin{tikzcd}
 & X_{\overline{\mathscr{A}}} \arrow[r, hook, "j_{\overline{\mathscr{A}}}"] \arrow[dl, two heads, "\pi_1^{\overline{\mathscr{A}}}"'] \arrow[dr, two heads] & X_{A_n} \arrow[dl, two heads, "\pi_1"] \arrow[dr, two heads, "\pi_2"] & \\
\mathbb{P}^r \arrow[r, hook, "i_{\overline{\mathscr{A}}}"] & \mathbb{P}^n & Y_{\mathscr{A}} \arrow[r, hook] & (\mathbb{P}^1)^n,
\end{tikzcd}$$

where $X_{\overline{\mathscr{A}}}$ is the strict transform of $\mathbb{P}^r$ under π_1 and $Y_{\mathscr{A}}$ is the image of $X_{\overline{\mathscr{A}}}$ under π_2.

The induced map $\pi_1^{\overline{\mathscr{A}}}$ is the blowup of all the zero-dimensional flats of $\overline{\mathscr{A}}$, all the strict transforms of one-dimensional flats of $\overline{\mathscr{A}}$, all the strict transforms of two-dimensional flats of $\overline{\mathscr{A}}$, and so on. The variety $X_{\overline{\mathscr{A}}}$ is the *wonderful model* of $\overline{\mathscr{A}}$ corresponding to the maximal building set [dCP95]. The variety $Y_{\mathscr{A}}$ is studied in [AB16], and its affine part centered at ∞^n is the *reciprocal plane* in [EPW16, PS06]. All the varieties and the maps in the diagram are defined over $\overline{\mathbb{F}}_p$. More precisely, to apply the decomposition theorem of [BBD82], we notice that all varieties, maps, and sheaves under consideration may be defined over some finite extension of $\mathbb{F}_p$.

We know that the Chow ring of $X_{\overline{\mathscr{A}}}$ is determined by the matroid $\overline{\mathrm{M}}$ [dCP95, FY04]: There is an isomorphism of graded algebras

$$A^*(\overline{\mathrm{M}}) \simeq A^*(X_{\overline{\mathscr{A}}}),$$

where $x_{\overline{F}}$ is identified with the class of the strict transform of the exceptional divisor produced when blowing up the flat of $\overline{\mathscr{A}}$ corresponding to $\overline{F}$. When $\overline{\mathrm{M}}$ is the Boolean matroid $\overline{\mathrm{B}}$ on $\overline{E}$, this describes the Chow ring of the permutohedral space $A^*(X_{A_n})$. In general, the pullback homomorphism

$$A^*(\overline{\mathrm{B}}) \simeq A^*(X_{A_n}) \xrightarrow{j^*_{\overline{\mathscr{A}}}} A^*(X_{\overline{\mathscr{A}}}) \simeq A^*(\overline{\mathrm{M}})$$

is determined by the assignment, for nonempty proper subsets $\overline{I}$ of $\overline{E}$,

$$x_{\overline{I}} \longmapsto \begin{cases} x_{\overline{I}} & \text{if } \overline{I} \text{ is a flat of } \overline{\mathrm{M}}, \\ 0 & \text{if } \overline{I} \text{ is a not flat of } \overline{\mathrm{M}}. \end{cases}$$

Fix a prime number ℓ different from p, and consider the ℓ-adic étale cohomology ring and the ℓ-adic étale intersection cohomology group of the varieties in the diagram above. These are $\overline{\mathbb{Q}}_\ell$-vector spaces of the form

$$\mathrm{H}^*(X, \overline{\mathbb{Q}}_\ell) := \mathrm{H}^*(X, \overline{\mathbb{Q}}_{\ell,X}) \ \text{ and } \ \mathrm{IH}^*(X, \overline{\mathbb{Q}}_\ell) := \mathrm{H}^*(X, \mathrm{IC}_X),$$

where $\overline{\mathbb{Q}}_{\ell,X}$ and IC_X are constructible complexes of $\overline{\mathbb{Q}}_\ell$-sheaves on X as in [BBD82]. We note that the cycle class map of any smooth variety X appearing in the diagram above induces an isomorphism of commutative graded $\overline{\mathbb{Q}}_\ell$-algebras

$$A^*(X) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq \mathrm{H}^{2*}(X, \overline{\mathbb{Q}}_\ell),$$

see [Kee92, Appendix]. For the variety $Y_{\mathscr{A}}$, which may be singular, we show in Remark 12 that there is an isomorphism of graded $\overline{\mathbb{Q}}_\ell$-vector spaces

$$A_{r-*}(Y_{\mathscr{A}}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq \mathrm{H}^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell).$$

In general, the intersection cohomology $\mathrm{IH}^*(X, \overline{\mathbb{Q}}_\ell)$ is a module over the cohomology $\mathrm{H}^*(X, \overline{\mathbb{Q}}_\ell)$, satisfying the Poincaré duality and the hard Lefschetz theorems. See [dCM09] for an introduction.

We obtain Theorem 6 from the following general observation. Let f be a proper map from an r-dimensional smooth projective variety

$$f : X_1 \longrightarrow X_2.$$

Let L be an ample line bundle on X_2, and consider the pullback homomorphism in even degrees

$$\mathrm{H}^{2*}(X_2, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathrm{H}^{2*}(X_1, \overline{\mathbb{Q}}_\ell).$$

The image of the pullback is a commutative graded algebra over $\overline{\mathbb{Q}}_\ell$, denoted $B^*(f)_{\overline{\mathbb{Q}}_\ell}$:

$$B^*(f)_{\overline{\mathbb{Q}}_\ell} = \mathrm{im}\Big(\mathrm{H}^{2*}(X_2, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathrm{H}^{2*}(X_1, \overline{\mathbb{Q}}_\ell)\Big).$$

$B^*(f)_{\overline{\mathbb{Q}}_\ell}$ is the cyclic $\mathrm{H}^{2*}(X_2, \overline{\mathbb{Q}}_\ell)$-submodule of $\mathrm{H}^{2*}(X_1, \overline{\mathbb{Q}}_\ell)$ generated by the element 1.

Proposition 11. If f is birational onto its image, then the multiplication map

$$B^p(f)_{\overline{\mathbb{Q}}_\ell} \longrightarrow B^{r-p}(f)_{\overline{\mathbb{Q}}_\ell}, \qquad \xi \longmapsto L^{r-2p}\, \xi$$

is injective for every nonnegative integer p less than $\frac{r}{2}$.

Proof. We reduce to the case when f is surjective. For this consider the factorization

$$X_1 \xrightarrow{\ g\ } \!\!\!\!\rightarrow f(X_1) \xhookrightarrow{\ h\ } X_2, \qquad f = h \circ g.$$

Then $B^*(f)_{\overline{\mathbb{Q}}_\ell}$ is a subalgebra of $B^*(g)_{\overline{\mathbb{Q}}_\ell}$, and hence the statement (f, L) follows from (g, h^*L).

Suppose that f is surjective. The decomposition theorem [BBD82] says that the intersection complex of X_2 appears as a direct summand of the direct image of the constant shear $\overline{\mathbb{Q}}_\ell$ on X_1:

$$Rf_*\overline{\mathbb{Q}}_{\ell, X_1} \simeq \mathrm{IC}_{X_2} \oplus \mathscr{C}.$$

Taking cohomology of both sides, we obtain a splitting injection of $\mathrm{H}^*(X_2, \overline{\mathbb{Q}}_\ell)$-modules

$$\Phi : \mathrm{IH}^*(X_2, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathrm{H}^*(X_1, \overline{\mathbb{Q}}_\ell).$$

Since Φ is an isomorphism in degree 0, it restricts to an isomorphism of commutative algebras

$$\mathrm{im}\Big(\mathrm{H}^{2*}(X_2, \overline{\mathbb{Q}}_\ell) \longrightarrow \mathrm{IH}^{2*}(X_2, \overline{\mathbb{Q}}_\ell)\Big) \simeq B^*(f)_{\overline{\mathbb{Q}}_\ell}.$$

The conclusion follows from the hard Lefschetz theorem [BBD82] for L on $\mathrm{IH}^{2*}(X_2, \overline{\mathbb{Q}}_\ell)$. $\square$

Theorem 6 will be deduced from the case when f is the map $X_{\overline{\mathscr{A}}} \to (\mathbb{P}^1)^n$.

Proof of Theorem 6. For each $i \in E$, let f_i be the composition of f with the i-th projection

$$f_i = \mathrm{proj}_i \circ f, \qquad \mathrm{proj}_i : (\mathbb{P}^1)^n \longrightarrow \mathbb{P}^1.$$

As in Proposition 8, for each $i \in E$, let β_i be the sum of $x_{\overline{F}}$ over all flats $\overline{F}$ of $\overline{\mathrm{M}}$ that contain i and do not contain 0. Let Ψ be the composition of isomorphisms

$$\Psi : A^*(\overline{\mathrm{M}}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq A^*(X_{\overline{\mathscr{A}}}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \simeq \mathrm{H}^{2*}(X_{\overline{\mathscr{A}}}, \overline{\mathbb{Q}}_\ell),$$

which maps $x_{\overline{F}}$ to the class of the strict transform in $X_{\overline{\mathscr{A}}}$ of the exceptional divisor produced when blowing up the flat of $\overline{\mathscr{A}}$ in $\mathbb{P}^r$ corresponding to $\overline{F}$. We prove that Ψ restricts to Ψ' in the commutative diagram

$$\begin{tikzcd} A^*(\overline{\mathrm{M}}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \arrow[r, "\Psi"] & \mathrm{H}^{2*}(X_{\overline{\mathscr{A}}}, \overline{\mathbb{Q}}_\ell) \\ B^*(\mathrm{M}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell \arrow[r, "\Psi'"] \arrow[u, hook, "\varphi \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell"] & B^*(f)_{\overline{\mathbb{Q}}_\ell}, \arrow[u, hook] \end{tikzcd}$$

where φ is the injective ring homomorphism sending y_i to β_i. Proposition 11 applied to f will then finish the proof.

Claim: The element $\Psi(\beta_i)$ is the pullback of the class of a point in $\mathbb{P}^1$ under f_i.

This implies the assertion on Ψ, since the cohomology ring of $(\mathbb{P}^1)^n$ is generated by the pullbacks under f_i. To show the claim, we factor f into the composition

$$X_{\overline{\mathscr{A}}} \overset{j_{\overline{\mathscr{A}}}}{\hookrightarrow} X_{A_n} \overset{\pi_2}{\twoheadrightarrow} (\mathbb{P}^1)^n.$$

As noted before, the pullback map associated to the inclusion $j_{\overline{\mathscr{A}}}$ satisfies

$$x_{\overline{I}} \longmapsto \begin{cases} x_{\overline{I}} & \text{if } \overline{I} \text{ is a flat of } \overline{\mathrm{M}}, \\ 0 & \text{if } \overline{I} \text{ is a not flat of } \overline{\mathrm{M}}. \end{cases}$$

Thus it is enough to prove the claim when $X_{\overline{\mathscr{A}}} = X_{A_n}$. This is the case when $\overline{\mathrm{M}}$ is the Boolean matroid on $\overline{E}$, and the claim in this case was proved in Remark 10 at the level of Chow rings. $\square$

In what follows, we write $z_0, z_1, \ldots, z_n$ for the homogeneous coordinates of $\mathbb{P}^n$, and write $(z_1, w_1), \ldots, (z_n, w_n)$ for the multi-homogeneous coordinates of $(\mathbb{P}^1)^n$.

Remark 12. Recall that M and $\overline{\mathrm{M}}$ share the same set of circuits. For every circuit C of M, there are nonzero constants $a_c \in \overline{\mathbb{F}}_p$, one for each element $c \in C$, such that

$$\sum_{c \in C} a_c\, z_c = 0 \ \text{ on the image of } \ i_{\overline{\mathscr{A}}} : \mathbb{P}^r \longrightarrow \mathbb{P}^n.$$

The collection $(a_c)_{c \in C}$ is uniquely determined by the circuit C, up to a common multiple.

A defining set of multi-homogeneous equations of $Y_{\mathscr{A}}$ is explicitly described in [AB16]:

$$Y_{\mathscr{A}} = \left\{ \ \sum_{c \in C} a_c\, z_c \Big(\prod_{d \in C \setminus c} w_d \Big) = 0, \ \ C \text{ is a circuit of } \mathrm{M} \ \right\} \subseteq (\mathbb{P}^1)^n.$$

This can be used to show that $Y_{\mathscr{A}}$ has an *algebraic cell decomposition*

$$Y_{\mathscr{A}} = \coprod_F \mathbb{A}^{\mathrm{rank}(F)},$$

where the disjoint union is over all flats F of M, and $\mathbb{A}^{\mathrm{rank}(F)}$ is the intersection of $Y_{\mathscr{A}}$ with the affine space

$$\mathbb{A}^{|F|} = \Big\{ w_i = 0 \text{ if and only if } i \text{ is not in } F \Big\} \subseteq (\mathbb{P}^1)^n.$$

It follows that there is an isomorphism between graded vector spaces

$$\mathrm{H}^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell) \simeq A_{r-*}(Y_{\mathscr{A}}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell.$$

In addition, the cell decomposition shows that there is an isomorphism between graded algebras

$$\mathrm{H}^{2*}(Y_{\mathscr{A}}, \overline{\mathbb{Q}}_\ell) \simeq B^*(\mathrm{M}) \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}_\ell,$$

for the proof of Theorem 6 shows that the right-hand side is a quotient of the left-hand side, and the cell decomposition shows that both sides have the same graded dimensions. See [BE09, Section 5] for relevant techniques.

We remark that the cell decomposition can be used to prove the first part of Theorem 2 without using the results of Section 2.

Remark 13. Let M be a simple matroid on $E = \{1, \dots, n\}$ with rank $r \geq 2$ and $n \geq 3$, and let $\mathrm{HR}(\mathrm{M})$ be the symmetric $n \times n$ matrix with entries

$$\mathrm{HR}(\mathrm{M})_{ij} = \begin{cases} 0 & \text{if } i = j, \\ b_{ij}(\mathrm{M}) & \text{if } i \neq j, \end{cases}$$

where $b_{ij}(\mathrm{M})$ is the number of bases of M containing i and j. It is straightforward to check that the matrix $\mathrm{HR}(\mathrm{M})$ represents the Hodge-Riemann form

$$B^1(\mathrm{M}) \times B^1(\mathrm{M}) \longrightarrow \mathbb{Q}, \qquad (\xi_1, \xi_2) \longmapsto \deg(L^{r-2}\, \xi_1\, \xi_2).$$

Thus $\mathrm{HR}(\mathrm{M})$ has exactly one positive eigenvalue, at least when M is realizable over some field. We conjecture that $\mathrm{HR}(\mathrm{M})$ has exactly one positive eigenvalue for any matroid M of rank $r \geq 2$.

Consider the restriction of the Hodge-Riemann form to the three dimensional subspace of $B^1(\mathrm{M})$ spanned by y_i, y_j, and L. The one positive eigenvalue condition says that the determinant of the resulting symmetric 3×3 matrix is nonnegative, and this implies

$$2 > b(\mathrm{M}) b_{ij}(\mathrm{M}) / b_i(\mathrm{M}) b_j(\mathrm{M}),$$

where $b(\mathrm{M})$ is the number of bases of M and $b_i(\mathrm{M})$ is the number of bases of M containing i.

Question: How large can the ratio $b(\mathrm{M}) b_{ij}(\mathrm{M}) / b_i(\mathrm{M}) b_j(\mathrm{M})$ be?

For graphic matroids, the work of Kirchhoff on electric circuits shows that the ratio is bounded above by 1, see [FM92]. In other words, for a randomly chosen spanning tree of a graph, the presence of an edge can only make any other edge less likely. It was once conjectured that this is the case for all matroids, but Seymour and Welsh found an example with the ratio $\simeq 1.02$ [SW75].

Acknowledgements. We thank Petter Brändén, Jeff Kahn, Satoshi Murai, Yasuhide Numata, Nick Proudfoot, Dave Wagner, and Geordie Williamson for very helpful conversations. This research started while Botong Wang was visiting Korea Institute for Advanced Study in summer 2016. We thank KIAS for excellent working conditions. June Huh was supported by a Clay Research Fellowship and NSF Grant DMS-1128155.

References

[AHK] Karim Adiprasito, June Huh, and Eric Katz, *Hodge theory for combinatorial geometries*. `arXiv:1511.02888`.

[Aig87] Martin Aigner, *Whitney numbers*. Combinatorial geometries, 139–160, Encyclopedia Math. Appl. **29**, Cambridge Univ. Press, Cambridge, 1987.

[AB16] Federico Ardila and Adam Boocher, *The closure of a linear space in a product of lines*. J. Algebraic Combin. **43** (2016), 199–235.

[BK68] J. G. Basterfield and L. M. Kelly, *A characterization of sets of n points which determine n hyperplanes*. Proc. Cambridge Philos. Soc. **64** (1968), 585–588.

[BB11] Victor Batyrev and Mark Blume, *The functor of toric varieties associated with Weyl chambers and Losev-Manin moduli spaces*. Tohoku Math. J. (2) **63** (2011), 581–604.

[BBD82] Alexander Beilinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*. Astérisque **100**, Paris, Soc. Math. Fr. 1982.

[BL80] Louis Billera and Carl Lee, *Sufficiency of McMullen's conditions for f-vectors of simplicial polytopes*. Bulletin Amer. Math. Soc. **2** (1980), 181–185.

[Bjö81] Anders Björner, *The unimodality conjecture for convex polytopes*, Bulletin of the American Mathematical Society **4** (1981), 187–188.

[BE09] Anders Björner and Torsten Ekedahl, *On the shape of Bruhat intervals*. Ann. of Math. (2) **170** (2009), 799–817.

[Can78] Rodney Canfield, *On a problem of Rota*. Bull. Amer. Math. Soc. **84** (1978), 164.

[dBE48] Nicolaas de Bruijn and Paul Erdős, *On a combinatorial problem*. Indagationes Math. **10** (1948), 421–423.

[dCM09] Mark de Cataldo and Luca Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*. Bull. Amer. Math. Soc. (N.S.) **46** (2009), 535–633.

[dCP95] Corrado de Concini and Claudio Procesi, *Wonderful models of subspace arrangements*. Selecta Math. (N.S.) **1** (1995), 459–494.

[dW66] Paul de Witte, *Combinatorial properties of finite linear spaces. I.* Bull. Soc. Math. Belg. **18** (1966), 133–141.

[dW75] Paul de Witte, *Combinatorial properties of finite linear spaces. II.* Bull. Soc. Math. Belg. **27** (1975), 115–155.

[DG71] Robert Dilworth and Curtis Greene, *A counterexample to the generalization of Sperner's theorem*. J. Combinatorial Theory Ser. A **10** (1971), 18–21.

[DW74] Thomas Dowling and Richard Wilson, *The slimmest geometric lattices*. Trans. Amer. Math. Soc. **196** (1974), 203–215.

[DW75] Thomas Dowling and Richard Wilson, *Whitney number inequalities for geometric lattices.* Proc. Amer. Math. Soc. **47** (1975), 504–512.

[EPW16] Ben Elias, Nicholas Proudfoot, and Max Wakefield, *The Kazhdan-Lusztig polynomial of a matroid*. Adv. Math. **299** (2016), 36–70.

[FM92] Tomás Feder and Milena Mihail, *Balanced matroids*. Proceedings of the 24th ACM Symposium on Theory of Computing (1992), 26–38.

[FY04] Eva Maria Feichtner and Sergey Yuzvinsky, *Chow rings of toric varieties defined by atomic lattices*. Invent. Math. **155** (2004), 515–536.

[Ful93] William Fulton, *Introduction to toric varieties*. Annals of Mathematics Studies **131**. Princeton University Press, Princeton, NJ, 1993.

[Ful97] William Fulton, *Young tableaux*. London Mathematical Society Student Texts **35**, Cambridge University Press, 1997.

[Ful98] William Fulton, *Intersection theory*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics **2**, Springer-Verlag, Berlin, 1998.

[Gre70] Curtis Greene, *A rank inequality for finite geometric lattices*. J. Combinatorial Theory **9** (1970), 357–364.

[HM+13] Tadahito Harima, Toshiaki Maeno, Hideaki Morita, Yasuhide Numata, Akihito Wachi, and Junzo Watanabe, *The Lefschetz properties*. Lecture Notes in Mathematics **2080**. Springer, Heidelberg, 2013.

[Hau05] Tamás Hausel, *Quarternionic geometry of matroids*. Cent. Eur. J. Math. **3** (2005), 26–38.

[Her73] A.P. Heron, *A property of the hyperplanes of a matroid and an extension of Dilworth's theorem.* J. Math. Anal. Appl. **42** (1973), 119–131.

[Kan72] William Kantor, *On incidence matrices of finite projective and affine spaces.* Math. Z. **124** (1972), 315–318.

[Kee92] Sean Keel, *Intersection theory of moduli space of stable n-pointed curves of genus zero*. Trans. Amer. Math. Soc. **330** (1992), 545–574.

[Kun79] Joseph Kung, *The Radon transforms of a combinatorial geometry. I.* J. Combin. Theory Ser. A **26** (1979), 97–102.

[Kun86] Joseph Kung, *Radon transforms in combinatorics and lattice theory*. Combinatorics and ordered sets (Arcata, Calif., 1985), 33–74, Contemp. Math. **57** Amer. Math. Soc., Providence, RI, 1986.

[Kun93] Joseph Kung, *The Radon transforms of a combinatorial geometry. II. Partition lattices.* Adv. Math. **101** (1993), 114–132.

[Kun00] Joseph Kung, *On the lines-planes inequality for matroids*. In memory of Gian-Carlo Rota. J. Combin. Theory Ser. A **91** (2000), 363–368.

[KRY09] Joseph Kung, Gian-Carlo Rota, and Catherine Yan, *Combinatorics: the Rota way*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2009.

[MN12] Toshiaki Maeno and Yasuhide Numata, *Sperner property, matroids and finite-dimensional Gorenstein algebras*, Tropical geometry and integrable systems, Contemp. Math. **580** (2012), 73–84.

[Mas72] John Mason, *Matroids: unimodal conjectures and Motzkin's theorem*. Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972), 207–220. Inst. Math. Appl., Southend-on-Sea, 1972.

[Mot36] Theodore Motzkin, *Beiträge zur Theorie der linearen Ungleichungen*. Dissertation, Basel, Jerusalem, 1936.

[Mot51] Theodore Motzkin, *The lines and planes connecting the points of a finite set*. Trans. Amer. Math. Soc. **70** (1951), 451–464.

[Oxl11] James Oxley, *Matroid theory*. Second edition. Oxford Graduate Texts in Mathematics **21** Oxford University Press, Oxford, 2011.

[PS06] Nicholas Proudfoot and David Speyer, *A broken circuit ring*. Beiträge Algebra Geom. **47** (2006), 161–166.

[Rot71] Gian-Carlo Rota, *Combinatorial theory, old and new.* Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 3, pp. 229–233. Gauthier-Villars, Paris, 1971.

[RH71] Gian-Carlo Rota and Lawrence Harper, *Matching theory, an introduction*. Advances in Probability and Related Topics, Vol. 1 pp. 169–215 Dekker, New York, 1971.

[SW75] Paul Seymour and Dominic Welsh, *Combinatorial applications of an inequality from statistical mechanics*. Math. Proc. Cambridge Philos. Soc. **77** (1975), 485–495.

[Sol67] Louis Solomon, *The Burnside algebra of a finite group*. J. Combin. Theory **2** (1967), 603–615.

[Sta96] Richard Stanley, *Combinatorics and commutative algebra.* Second edition. Progress in Mathematics **41**. Birkhäuser, Boston, MA, 1996.

[Sta12] Richard Stanley, *Enumerative combinatorics. Volume 1.* Second edition. Cambridge Studies in Advanced Mathematics **49**. Cambridge University Press, Cambridge, 2012

[Sta13] Richard Stanley, *Algebraic combinatorics*. Undergraduate Texts in Mathematics, 2013.

[Sta90] Dennis Stanton, *Unimodality and Young's lattice.* J. Combin. Theory Ser. A **54** (1990), 41–53.

[Wel76] Dominic Welsh, *Matroid Theory*. London Mathematical Society Monographs, **8**, Academic Press, London-New York, 1976.

INSTITUTE FOR ADVANCED STUDY, FULD HALL, 1 EINSTEIN DRIVE, PRINCETON, NJ, USA.

E-mail address: huh@princeton.edu

UNIVERSITY OF WISCONSIN-MADISON, VAN VLECK HALL, 480 LINCOLN DRIVE, MADISON, WI, USA.

E-mail address: bwang274@wisc.edu