A FINITENESS THEOREM FOR HYPERBOLIC 3-MANIFOLDS

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Abstract. We prove that there are only finitely many closed hyperbolic 3-manifolds with injectivity radius and first eigenvalue of the Laplacian bounded below whose fundamental groups can be generated by a given number of elements.

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A common pursuit in differential geometry is to bound the number of homotopy types of closed n-manifolds that admit a Riemannian metric with controlled geometry: for instance, one often specifies constraints on diameter, curvature, volume or injectivity radius. If one considers only locally symmetric manifolds, these finiteness theorems combine with Mostow's rigidity theorem to yield much stronger results. For example, Wang's finiteness theorem [18] asserts that for every $d \geq 4$ and V positive, there are only finitely many isometry classes of hyperbolic d-manifolds M with volume $\mathrm{vol}(M) \leq V$. In dimension 3, Wang's finiteness theorem still holds if, in addition to the volume bound, one only considers manifolds M with injectivity radius $\mathrm{inj}(M) \geq \epsilon > 0$.

Our goal is to provide a finiteness result for hyperbolic 3-manifolds in terms of injectivity radius, first eigenvalue of the Laplacian and rank of the fundamental group. Here, the rank of a group is the minimal number of elements needed to generate it. We prove:

Theorem 1.1. *For every $\epsilon, \delta, k > 0$, there are only finitely many isometry classes of hyperbolic 3-manifolds M with injectivity radius $\mathrm{inj}(M) \geq \delta$, first eigenvalue of the Laplacian $\lambda_1(M) \geq \delta$ and $\mathrm{rank}(\pi_1(M)) \leq k$.*

In the final section of this note we sketch the proofs of stronger finiteness results under the assumption that the involved manifolds are arithmetic.

The idea of the proof of Theorem 1.1 is as follows. Assume that we have a sequence of pairwise distinct hyperbolic 3-manifolds (M_i)

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with $\mathrm{inj}(M_i) \geq \epsilon$, and suppose that each $\pi_1(M_i)$ can be generated by k elements. We will show that after passing to a subsequence, there are base points $p_i \in M_i$ such that the sequence of pointed manifolds (M_i, p_i) converges in the Gromov-Hausdorff topology to a pointed manifold (M_∞, p_∞) which has a degenerate end. In particular, the Cheeger constant of the manifolds M_i tends to 0. A result due to Buser [6] implies that the same is true for the sequence $(\lambda_1(M_i))$.

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Let M be a closed hyperbolic 3-manifold. Given a smooth function f and a smooth vector field X on M let ∇f and $\mathrm{div}(X)$ be their gradient and divergence respectively. The *Laplacian* Δf of a function $f \in C^\infty(\mathbb{H}^3)$ is then defined as

$$\Delta f = -\operatorname{div} \nabla f$$

The Laplacian extends to a self-adjoint linear operator with discrete spectrum on the Sobolev space $H_{1,2}(M)$. By the spectral theorem, there is a Hilbert basis of $H_{1,2}(M)$ consisting of eigenfunctions of Δ. Let

$$0 = \lambda_0(M) < \lambda_1(M) \leq \lambda_2(M) \leq \dots$$

be the eigenvalues of Δ in increasing order. In [11], Cheeger introduced the so-called Cheeger constant

$$h(M) = \inf_{U \subset M} \frac{\mathrm{vol}(\partial U)}{\min\{\mathrm{vol}(U), \mathrm{vol}(M \setminus U)\}}$$

where the infimum is taken over smooth 3-dimensional submanifolds with boundary inside M. Cheeger showed that $h(M)$ can be used to bound $\lambda_1(M)$ from below; later, Buser [6] showed that $\lambda_1(M)$ can be estimated from above using the Cheeger constant and lower-bounds on the Ricci-curvature. For hyperbolic 3-manifolds, their formulas combine as follows:

$$\frac{1}{4}h(M)^2 \leq \lambda_1(M) \leq 4h(M)^2 + 10h(M) \tag{2.1}$$

Recall that an end $\mathcal{E}$ of a non-compact hyperbolic 3-manifold M without cusps is *degenerate* if it has a neighborhood $\mathcal{U}$ homeomorphic to $\Sigma \times [0, \infty)$, where Σ is a closed surface, and there is a sequence of embedded surfaces S_i exiting the end $\mathcal{E}$, with bounded area and homotopic to $\Sigma \times \{0\}$ within $\mathcal{U}$. See [7, 8] for a description of the geometry of degenerate ends.

The next lemma follows directly from the definition.

Lemma 2.1. *Assume that a hyperbolic 3-manifold M without cusps has a degenerate end. For every η positive we can find two disjoint compact 3-dimensional submanifolds $U_1, U_2 \subset M$ with $\frac{\mathrm{vol}(\partial U_j)}{\mathrm{vol}(U_j)} < \eta$ for $j = 1, 2$.* □

To apply Lemma 2.1 in our work, we will need to find large pieces of degenerate ends in closed hyperbolic 3-manifolds with bounded rank and injectivity radius. This is the content of the following lemma, the proof of which we will give in the next section.

Lemma 2.2. *Assume that (M_i) is a sequence of pair-wise distinct hyperbolic 3-manifolds with $\mathrm{inj}(M_i) \geq \epsilon$ and $\mathrm{rank}(\pi_1(M_i)) \leq k$. Then there are base points $x_i \in M_i$ such that, up to passing to a subsequence, the pointed manifolds (M_i, x_i) converge in the pointed Gromov-Hausdorff topology to a pointed manifold (M_∞, x_∞) which has a degenerate end.*

Recall that the sequence (M_i, x_i) is said to converge to (M_∞, x_∞) in the pointed Gromov-Hausdorff topology if for all $\lambda > 1$ and $R > 0$ we have for large enough i that there is a λ-bi-Lipschitz embedding

$$\psi_i : (B_R(M_\infty, x_\infty), x_\infty) \to (M_i, x_i) \tag{2.2}$$

of the R-ball around $x_\infty \in M_\infty$ into M_i.

Assuming Lemma 2.2, let us give the proof of Theorem 1.1.

Theorem 1.1. *For every $\epsilon, \delta, k > 0$, there are only finitely many isometry classes of hyperbolic 3-manifolds M with $\mathrm{inj}(M) \geq \delta$, first eigenvalue of the Laplacian $\lambda_1(M) \geq \delta$ and $\mathrm{rank}(\pi_1(M)) \leq k$.*

Proof. Seeking a contradiction, assume that for some ϵ, δ and k positive, there is a sequence (M_i) of pairwise distinct hyperbolic 3-manifolds with $\mathrm{inj}(M_i) \geq \epsilon$, $\mathrm{rank}(\pi_1(M_i)) \leq k$ and $\lambda_1(M_i) \geq \delta$. Passing to a subsequence and choosing base points, assume that (M_i, x_i) converges to a manifold (M_∞, x_∞) as indicated in Lemma 2.2.

Since the manifold M_∞ has a degenerate end we have by Lemma 2.1 that for each $\eta > 0$, there are two disjoint compact submanifolds $U_1, U_2 \subset M$ with $\frac{\mathrm{vol}(\partial U_j)}{\mathrm{vol}(U_j)} < \eta$. Choose R with $U_1, U_2 \subset B_R(M_\infty, x_\infty)$ and let ψ_i be a sequence of bi-Lipschitz embeddings as in (2.2) with bi-Lipschitz constant closer and closer to 1. We have then

$$\lim_{i\to\infty} \frac{\mathrm{vol}(\partial(\psi_i(U_j)))}{\mathrm{vol}(\psi_i(U_j))} = \frac{\mathrm{vol}(\partial U_j)}{\mathrm{vol}(U_j)} < \eta$$

for $j = 1, 2$. Taking into account that $\psi_i(U_1) \cap \psi_i(U_2) = \emptyset$, we deduce that

$$\limsup_{i\to\infty} h(M_i) \leq \eta$$

Buser's inequality (2.1) implies that

$$\limsup_{i\to\infty} \lambda_1(M_i) \leq 4\eta^2 + 10\eta$$

Since $\eta > 0$ was arbitrary we obtain that $\lim_i \lambda_1(M_i) = 0$, contradicting the assumption that $\lambda_1(M_i) > \delta$ for all i. This concludes the proof of Theorem 1.1. □

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In this section we prove Lemma 2.2. Before doing so, we need the following preliminary result:

Lemma 3.1. *Assume that (M_i) is a sequence of pairwise distinct hyperbolic 3-manifolds with $\operatorname{inj}(M_i) \geq \epsilon$ and $\operatorname{rank}(\pi_1(M_i)) \leq k$. Then there are a constant L and a sequence (Y_i) of metric graphs with 1-Lipschitz maps $(f_i : Y_i \to M_i)$ such that*

1. $\operatorname{rank}(\pi_1(Y_i)) \leq k$,
2. $\operatorname{length}(Y_i) \leq L$, *and*
3. $\lim_{i\to\infty} \operatorname{diam}(CC(\mathbb{H}^3/(f_i)_*(\pi_1(Y_i)))) = \infty$.

Here $CC(\mathbb{H}^3/(f_i)_*(\pi_1(Y_i)))$ is the convex core of the cover of M_i corresponding to the image of the homomorphism $(f_i)_* : \pi_1(Y_i) \to \pi_1(M_i)$.

Recall that the convex core $CC(M)$ of a hyperbolic 3-manifold M is the smallest closed convex subset of M whose inclusion into M is a homotopy equivalence.

Before beginning the proof of Lemma 3.1, we recall some definitions and facts taken from [4, 16, 19]. Assume that $M = \mathbb{H}^3/\Gamma$ is a closed hyperbolic 3-manifold with $\operatorname{rank}(\pi_1(M)) = k$. A *carrier graph* is a 1-Lipschitz map

$$f : X \to M \tag{3.1}$$

from a metric graph X into M, such that the induced homomorphism $f_* : \pi_1(X) \to \pi_1(M)$ is surjective. We will consider here only carrier graphs with $\operatorname{rank}(\pi_1(X)) = \operatorname{rank}(\pi_1(M))$. If $Y \subset X$ is a subgraph, then $\operatorname{length}(Y)$ is defined to be the sum of the lengths of the edges of X contained in Y.

Using the Arzela-Ascoli theorem, it is not hard to see that the set of bounded length carrier graphs in a given hyperbolic 3-manifold is compact. In particular, there is one in each such 3-manifold which has minimal length. In [19], White proved that if $f : X \to M$ is a *minimal length carrier graph* then X has a circuit whose length is bounded only in terms of $\mathrm{rank}(\pi_1(M))$. In [4], the first author extended White's result as follows:

Proposition 3.2 (Chains of bounded length)**.** *Let M be a closed hyperbolic 3-manifold and $f : X \to M$ a minimal length carrier graph with* $\mathrm{rank}(\pi_1(X)) = \mathrm{rank}(\pi_1(M))$*. Then we have a sequence of possibly disconnected subgraphs*

$$\emptyset = Y_0 \subset Y_1 \subset \cdots \subset Y_n = X$$

such that the length of any edge in $Y_{i+1} \setminus Y_i$ is bounded from above by some constant depending only on $\mathrm{inj}(M)$*,* $\mathrm{rank}(\pi_1(M))$*,* $\mathrm{length}(Y_i)$ *and the diameters of the convex cores of the covers of M corresponding to $f_*(\pi_1(Y_i^j))$ where $Y_i^1, \ldots, Y_i^{n_i}$ are the connected components of Y_i. Moreover, the number n of subgraphs in the chain is bounded above by* $3(\mathrm{rank}(\pi_1(M)) - 1)$*.*

Lemma 3.1 follows from this proposition and some bookkeeping.

Proof of Lemma 3.1. To begin with fix ϵ, k and a sequence of hyperbolic 3-manifolds (M_i) as in the statement. For each i fix a minimal length carrier graph

$$f_i : X_i \to M_i$$

with $\mathrm{rank}(\pi_1(X_i)) = \mathrm{rank}(\pi_1(M_i)) = k$.

Assume for the moment that the sequence $(\mathrm{length}(X_i))$ is bounded from above by some positive number L. In other words, the graphs X_i themselves satisfy (1) and (2). On the other hand, we have by definition that $(f_i)_*(\pi_1(X_i)) = \pi_1(M_i)$ and hence

$$CC(\mathbb{H}^3/(f_i)_*(\pi_1(X_i))) = M_i$$

Since the sequence (M_i) consists of pairwise distinct manifolds with $\mathrm{inj}(M_i) \geq \epsilon$ we obtain, for example, from Wang's finiteness theorem that $\mathrm{diam}(M_i) \to \infty$. This means that the carrier graphs $f_i : X_i \to M_i$ themselves satisfy also (3). This concludes the proof if the sequence $(\mathrm{length}(X_i))$ is bounded.

We treat now the general case. In the light of the above, we may assume without loss of generality that $\mathrm{length}(X_i) \to \infty$. Consider for each i the chain

$$\emptyset = Y_0^i \subset Y_1^i \subset \cdots \subset Y_{n_i}^i = X_i \tag{3.2}$$

provided by Proposition 3.2. Since $\mathrm{length}(Y_0^i) = 0$, $\mathrm{length}(X_i) \to \infty$ and the length n_i of each chain is bounded independently of i, we can choose a sequence (m_i) with

(a) $0 \le m_i \le n_i - 1$,
(b) $\limsup_{i\to\infty} \mathrm{length}(Y_{m_i}^i) < \infty$, and
(c) $\lim_{i\to\infty} \mathrm{length}(Y_{m_i+1}^i) = \infty$.

Observe that by condition (b), any of the connected components $Z_1^i, \ldots, Z_{r_i}^i$ of $Y_{m_i}^i$ satisfies (1) and (2) for any $L < \infty$ with

$$\limsup_{i\to\infty} \mathrm{length}(Y_{m_i}^i) < L$$

By Proposition 3.2, $\mathrm{length}(Y_{m_i+1}^i)$ is bounded in terms of k, L and

$$\max_{j=1,\ldots,r_i} \{\mathrm{diam}(CC(\mathbb{H}^3/(f_i)_*(\pi_1(Z_j^i))))\}$$

Since $\mathrm{length}(Y_{m_i+1}^i)$ tends to ∞ by condition (c), we obtain that there is a sequence of component of $Y_{m_i}^i$, say Z_1^i, with

$$\lim_i \mathrm{diam}(CC(\mathbb{H}^3/(f_i)_*(\pi_1(Z_1^i)))) = \infty$$

In other words, the sequence of maps $f_i|_{Z_1^i} : Z_1^i \to M_i$ satisfies (3). This concludes the proof of Lemma 3.1 □

We are now ready to prove Lemma 2.2:

Lemma 2.2. *Assume that (M_i) is a sequence of pair-wise distinct hyperbolic 3-manifolds with $\mathrm{inj}(M_i) \ge \epsilon$ and $\mathrm{rank}(\pi_1(M_i)) \le k$. Then there are base points $x_i \in M_i$ such that, up to passing to a subsequence, the pointed manifolds (M_i, x_i) converge in the pointed Gromov-Hausdorff topology to a pointed manifold (M_∞, x_∞) which has a degenerate end.*

Proof. For each i, let $f_i : Y_i \to M_i$ be a sequence of graphs as provided by Lemma 3.1. We choose base points $y_i \in Y_i$ and set $x_i = f_i(y_i) \in M_i$; we also choose for all i an orthonormal frame of the tangent space $T_{x_i}M_i$ and, abusing notation, refer to it by x_i as well.

Choosing a base frame $x_{\mathbb{H}^3}$ of hyperbolic space we obtain for each i a unique discrete torsion-free subgroup $\Gamma_i \subset \mathrm{Isom}_+(\mathbb{H}^3)$ such that the hyperbolic manifolds M_i and $\mathbb{H}^3/\Gamma_i$ are isometric by an isometry mapping the base frame x_i to the projection of the base frame $x_{\mathbb{H}^3}$. From now on we identify $M_i = \mathbb{H}^3/\Gamma_i$.

The assumption that $M_i = \mathbb{H}^3/\Gamma_i$ has at least injectivity radius ϵ implies that the sequence of groups Γ_i contains a subsequence, say the whole sequence, which converges in the Chabauty topology to a discrete and torsion free group Γ_∞. It is well-known that this is equivalent to the convergence in the pointed Gromov-Hausdorff topology of the pointed manifolds (M_i, x_i) to the manifold $M_\infty = \mathbb{H}^3/\Gamma$ (see [3]).

The assumption that the manifolds M_i are pairwise distinct implies that M_∞ is not compact. In particular, in order to show that it has a degenerate end, it suffices by Canary's extension of Thurston's covering theorem [9] to find a manifold $\tilde{M}_\infty$ which has a degenerate end and covers M_∞. This is our goal.

Passing to a subsequence, we may assume that the group $\pi_1(Y_i, y_i)$ is isomorphic to the free group $\mathbb{F}_m$ of rank $m \leq k$ for each i. Moreover, since the subgraphs graphs Y_i have length bounded by some universal constant L, we may choose the identification $\mathbb{F}_m \simeq \pi_1(Y_i, y_i)$ in such a way that each element of the standard basis is represented by a loop of at most length $2L$. The composition of this identification, the homomorphism $(f_i)_* : \pi_1(Y_i, y_i) \to \pi_1(M_i, x_i)$ and the identification $\pi_1(M_i, x_i) \simeq \Gamma_i$ yields a representation

$$\rho_i : \mathbb{F}_m \to \mathrm{Isom}_+(\mathbb{H}^3)$$

in such a way that if $e_j \in \mathbb{F}_m$ is an element of the standard basis then for all i we have

$$d_{\mathbb{H}^3}((\rho_i(e_j))(x_{\mathbb{H}^3}), x_{\mathbb{H}^3}) \leq 2L$$

This implies that, up to passing to a further subsequence, the sequence of representations (ρ_i) converges to a representation ρ_∞ of $\mathbb{F}_m$ into $\mathrm{Isom}_+(\mathbb{H}^3)$, meaning that for each $\gamma \in \mathbb{F}_m$ we have $\lim_i \rho_i(\gamma) = \rho_\infty(\gamma)$. The image of ρ_∞ is a subgroup of Γ_∞; in particular, it is discrete and the manifold $\tilde{M}_\infty = \mathbb{H}^3/\rho_\infty(\mathbb{F}_m)$ covers M_∞. Observe that $\mathrm{inj}(\tilde{M}_\infty) \geq \epsilon$ and hence $\rho_\infty(\mathbb{F}^m)$ does not contain parabolic elements.

We claim that $\tilde{M}_\infty$ has a degenerate end. Otherwise, its convex-core $CC(\tilde{M}_\infty)$ is compact by the work of Agol [1], Calegari-Gabai [7] and Canary [8]. Marden's stability theorem [14] implies then that there are bi-Lipschitz maps (defined for large enough i)

$$\tilde{\phi}_i : \mathbb{H}^3/\rho_\infty(\mathbb{F}_m) \to \mathbb{H}^3/\rho_i(\mathbb{F}_m)$$

whose bi-Lipschitz constants tends to 1. This implies that

$$\lim_{i\to\infty} \mathrm{diam}(CC(\mathbb{H}^3/\rho_i(\mathbb{F}_m))) = \mathrm{diam}(CC(\mathbb{H}^3/\rho_\infty(\mathbb{F}_m))) < \infty$$

contradicting that by Lemma 3.1 we have

$$\lim_{i\to\infty} \mathrm{diam}(CC(\mathbb{H}^3/\rho_i(\mathbb{F}_m))) = \lim_{i\to\infty} \mathrm{diam}(CC(\mathbb{H}^3/(f_i)_*(\pi_1(Y_i)))) = \infty$$

We have proved that $\tilde{M}_\infty$ has at least a degenerate end $\tilde{\mathcal{E}}$. As remarked above, Thurston's covering theorem [9] implies that M_∞ has a degenerate end as well. We have proved Lemma 2.2. $\square$

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In this final section we sketch the proofs of some additional results which can be obtained if one considers only arithmetic hyperbolic 3-manifolds. See [15] for definitions.

Assume that M_i is a sequence of pairwise distinct arithmetic hyperbolic 3-manifolds with $\mathrm{inj}(M_i)$ and $\mathrm{rank}(\pi_1(M_i)) \leq k$. A deep result of Vigneras, combined with a lemma of Long-Reid (see Agol [2]), asserts that each of the arithmetic manifolds M_i covers some hyperbolic orbifold $\mathcal{O}_i$ with $\lambda_1(\mathcal{O}_i) \geq \frac{3}{4}$. By Lemma 2.2 we can find base points $x_i \in M_i$ such that, up to passing to a subsequence, the manifolds (M_i, x_i) converge in the pointed Gromov-Hausdorff topology to a manifold M_∞ which has a degenerate end. Denote by $\hat{x}_i$ the projection of the base point x_i under the covering

$$\tau_i : M_i \to \mathcal{O}_i$$

provided by Vigneras' theorem. Passing again to a subsequence we may assume that the orbifolds $(\mathcal{O}_i, \hat{x}_i)$ and the coverings τ_i converge in the Gromov-Hausdorff topology to an orbifold $\mathcal{O}_\infty$ and a covering

$$\tau_\infty : M_\infty \to \mathcal{O}_\infty$$

respectively. By Canary's extension of Thurston's covering theorem [10], we deduce that either $\mathcal{O}_\infty$ has a degenerate end or is compact. The former case is ruled out as in the proof of Theorem 1.1 using that $\lambda_1(\mathcal{O}_i) \geq \frac{3}{4}$ for all i. In particular, $\mathcal{O}_\infty$ is compact and hence there is i_0 with $\mathcal{O}_i = \mathcal{O}_\infty$ for all $i \geq i_0$. We have proved that, up to passing to a subsequence, all the manifolds M_i cover some fixed orbifold and in particular, they are commensurable.

Theorem 4.1. *For all ϵ and k positive, there are only finitely many commensurability classes of closed arithmetic hyperbolic 3-manifolds M with $\mathrm{inj}(M) \geq \epsilon$ and $\mathrm{rank}(\pi_1(M)) \leq k$.* □

Using the information given by Thurston's covering theorem more carefully, we obtain also that all but finitely many arithmetic hyperbolic 3-manifolds M with $\mathrm{inj}(M) \geq \epsilon$ and $\mathrm{rank}(\pi_1(M)) = k$ fiber over one of the two 1-dimensional orbifolds $\mathbb{S}^1$ and $\mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z})$ with fiber of bounded genus. Combining this observation with the main results of [4] it is not difficult to deduce that if k is odd then the fibering is over $\mathbb{S}^1$ and the fiber has genus $\frac{1}{2}(k-1)$, while if k is even then the fibering is over $\mathbb{S}^1/(\mathbb{Z}/2\mathbb{Z})$ and the regular fibers have genus $k-2$. In particular we obtain the two following consequences, the second of which is due to Agol in the case that $\mathrm{rank}(\pi_1(M)) = 2$.

Corollary 4.2. *For every ϵ and k there are only finitely many closed arithmetic hyperbolic 3-manifolds M with* $\mathrm{inj}(M) \geq \epsilon$ *and* $\mathrm{rank}(\pi_1(M)) \leq k$ *which have the same $\mathbb{Z}/2\mathbb{Z}$-homology as $\mathbb{S}^3$.* □

Corollary 4.3. *There are only finitely many closed arithmetic hyperbolic 3-manifolds M with* $\mathrm{inj}(M) \geq \epsilon$ *and* $\mathrm{rank}(\pi_1(M)) \leq 3$. □

Before we finish, recall that the geometric version of Lehmer's conjecture asserts that the injectivity radius of a closed arithmetic hyperbolic 3-manifold is bounded from below by some universal constant. A positive resolution of this conjecture would then remove all assumptions on injectivity radius from the theorems above.

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