TANGENCIES OF CIRCLE PACKINGS

LARS LOUDER & JUAN SOUTO

Under a circle packing we understand a finite collection $\{C_1, \ldots, C_n\}$ of (round) circles in the complex projective line $\mathbb{C}P^1$ which bound disjoint open disk $\{D_1, \ldots, D_n\}$ such that for every connected component T of $\mathbb{C}P^1 \setminus \cup_i \bar{D}_i$ the set of vertices of T is contained in some circle in $\mathbb{C}P^1$. Observe that any three points are always contained in some round circle; four points are contained in a round circle if and only if their cross-ratio is real. If $\mathcal{C} = \{C_1, \ldots, C_n\}$ is such a circle packing, then every point in $\mathbb{C}P^1$ belongs to at most two of the circles. The set of points which belong to two circles is the set of *tangency points* of $\mathcal{C}$; it is a finite set.

The action of $\mathrm{PSL}_2\,\mathbb{C}$ on $\mathbb{C}P^1$ maps circles to circles. Two circle packings which differ by an element of $\mathrm{PSL}_2\,\mathbb{C}$ are said to be *equivalent*. In this note we observe:

Theorem 1. *Every circle packing is equivalent to one such that all tangency points are algebraic.*

We say that a point z is *algebraic* if it belongs to the image of the map $\bar{\mathbb{Q}}P^1 \to \mathbb{C}P^1$. Equivalently, z belongs to $\bar{\mathbb{Q}} \cup \{\infty\}$ under the standard identification of $\mathbb{C}P^1$ with the Riemann sphere $\mathbb{C} \cup \{\infty\}$. Here $\bar{\mathbb{Q}}$ is the algebraic closure of $\mathbb{Q}$.

Let from now on $\mathcal{C} = \{C_1, \ldots, C_n\}$ be a circle packing and let D_i be the open disks associated to C_i for each i. Recall that the vertices of every connected component T of $\mathbb{C}P^1 \setminus \cup D_i$ are contained in a round circle C_T; let D_T be the disk with boundary C_T which contains T. For any two components T, T' of $\mathbb{C}P^1 \setminus \cup D_i$, the open disks D_T and $D_{T'}$ are disjoint. Moreover, every component τ of $\mathbb{C}P^1 \setminus \cup_T D_T$ is a curvilinear polygon whose vertices are contained in one of the circles in $\mathcal{C}$. In other words, the collection $\mathcal{C}'$ of all the circles C_T over all components T of $\mathbb{C}P^1 \setminus \cup D_i$ is also a circle packing. The circle packing $\mathcal{C}'$ is said to be the dual circle pack of $\mathcal{C}$. Observe that $\mathcal{C}$ is in turn the dual circle packing of $\mathcal{C}'$.

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From now on, we identify $\mathbb{C}P^1$ with the boundary at infinity of hyperbolic space $\mathbb{H}^3$. Recall that this identification is consistent with the identification of $\mathrm{PSL}_2\,\mathbb{C}$ with the group of orientation preserving isometries of $\mathbb{H}^3$. Every circle C in $\mathbb{C}P^1$ is now the boundary of a totally geodesic copy of $\mathbb{H}^2 \subset \mathbb{H}^3$. We denote by σ_C the reflection of $\mathbb{H}^3$ along this plane. The induced boundary map of σ_C is the inversion at the circle C in the sense of projective geometry.

Returning now to our circle packing $\mathcal{C}$ we denote by $\hat{\Gamma}_{\mathcal{C}} \subset \mathrm{Isom}(\mathbb{H}^3)$ the group generated by all the reflections σ_C over $C \in \mathcal{C} \cup \mathcal{C}'$. We denote by $\Gamma_{\mathcal{C}} \subset \mathrm{PSL}_2\,\mathbb{C}$ index 2 subgroup of $\hat{\Gamma}_{\mathcal{C}}$ consisting of orientation preserving isometries. It is well-known that $\hat{\Gamma}_{\mathcal{C}}$ is a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ whose quotient $\hat{\Gamma}_{\mathcal{C}}\backslash\mathbb{H}^3$ has finite volume; in particular, $\Gamma_{\mathcal{C}}$ is also discrete and $\Gamma_{\mathcal{C}}\backslash\mathbb{H}^3$ has also finite volume. It follows from the following proposition that $\Gamma_{\mathcal{C}}$ is conjugated, within $\mathrm{PSL}_2\,\mathbb{C}$, to a subgroup of $\mathrm{PSL}_2\,\bar{\mathbb{Q}}$.

Proposition 2. [2, Prop. 6.7.4] *If Γ is a discrete subgroup of $\mathrm{PSL}_2\,\mathbb{C}$ such that $\mathbb{H}^3/\Gamma$ has finite volume, then Γ is conjugate to a group of matrices whose entries are algebraic.*

Given $g \in \mathrm{PSL}_2\,\mathbb{C}$ such that $g\Gamma_{\mathcal{C}}g^{-1} \subset \mathrm{PSL}_2\,\bar{\mathbb{Q}}$ consider the circle packing $g\mathcal{C} = \{gC_1, \dots, gC_n\}$. Obviously, $\mathcal{C}$ and $g\mathcal{C}$ are equivalent and $\Gamma_{g\mathcal{C}} = g\Gamma_{\mathcal{C}}g^{-1}$. In other words, we may assume without loss of generality that $\Gamma_{\mathcal{C}} \subset \mathrm{PSL}_2\,\bar{\mathbb{Q}}$.

Discreteness of $\Gamma_{\mathcal{C}}$ implies that every infinite order element in $\gamma \in \Gamma_{\mathcal{C}}$ is either hyperbolic or parabolic. In particular, γ fixes either one or two points in $\mathbb{C}P^1$. Since the condition for a point to be fixed is algebraic, we obtain that any fixed point of γ is algebraic. In particular, the claim of Theorem 1 follows immediately from the following observation:

Lemma 3. *Every tangency point of $\mathcal{C}$ is fixed by some parabolic element in $\Gamma_{\mathcal{C}}$.*

Proof. The tangency point z belongs to two circles, say C_1, C_2 and hence is fixed by the reflections σ_{C_1} and σ_{C_2}. The product $\sigma_{C_1}\sigma_{C_2}$ is parabolic, belongs to $\Gamma_{\mathcal{C}}$ and fixes z. □

As mentioned above, this concludes the proof of Theorem 1. □

Before devoting our time and energy to more productive undertakings we would like to add a remark. Proposition 2 above is a particular case of a more general result of Raghunathan [1]. The idea of the proof in the case that Γ is cocompact is the following. Consider Γ as an abstract group and denote the embedding of Γ inside $\mathrm{PSL}_2\,\mathbb{C}$ by ρ_0. We consider ρ_0 as an element in $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$. The group $\mathrm{PSL}_2\,\mathbb{C}$ is an affine algebraic group defined over $\mathbb{Q}$. In particular, $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ is also an affine algebraic variety defined over $\mathbb{Q}$. In particular, the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$. The variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ has only finitely many irreducible components. Hence, each of them is invariant by a finite index subgroup in $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. This implies that each one of the irreducible components is defined over a finite extension of $\mathbb{Q}$ and hence defined over $\bar{\mathbb{Q}}$. Let V be the irreducible component of $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ containing ρ_0. Since V is defined over $\bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}}$ is algebraically closed, V contains an element ρ which is defined over $\bar{\mathbb{Q}}$. It follows from the Mostow rigidity theorem that ρ and ρ_0 are conjugated; this concludes the proof of Proposition 2 in the case that Γ is cocompact. In the general case that Γ, one has to replace the variety $\mathrm{Hom}(\Gamma, \mathrm{PSL}_2\,\mathbb{C})$ with the subvariety consisting of representations which map parabolic elements to elements whose trace squared is 4. The argument then is concluded using Prasad's extension of the Mostow rigidity theorem.

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References

[1] M. Raghunathan, *Discrete subgroups of Lie groups* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. Springer-Verlag, New York-Heidelberg, 1972.

[2] W. Thurston, *The Geometry and Topology of Three-Manifolds*, `http://www.msri.org/publications/books/gt3m/`