

# THE RANK OF THE FUNDAMENTAL GROUP OF THOSE HYPERBOLIC 3-MANIFOLDS FIBERING OVER THE CIRCLE

JUAN SOUTO

ABSTRACT. We determine the rank of the fundamental group of those hyperbolic 3-manifolds fibering over the circle whose monodromy is a sufficiently high power of a pseudo-Anosov map. Moreover, we show that any two generating sets with minimal cardinality are Nielsen equivalent.

## 1. INTRODUCTION

Probably the most basic invariant of a finitely generated group is its *rank*, i.e. the minimal number of elements needed to generate it. While in general the rank is not computable [8], recently Kapovich-Weidmann [6] gave an algorithm determining  $\text{rank}(\pi_1(M))$  when  $M$  is a 3-manifold. However, it is not possible to give a priori bounds on the complexity of this algorithm and hence it seems difficult to use it to obtain precise results in concrete situations. The goal of this note is to determine the rank of the fundamental group of a particularly nice class of 3-manifolds.

Let  $\Sigma_g$  be the closed (orientable) surface of genus  $g \geq 2$ . It is well-known that the group  $\pi_1(\Sigma_g)$  has rank  $2g$  and admits a presentation as follows:

$$(1.1) \quad \pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] \rangle$$

Given a mapping class  $F : \Sigma_g \rightarrow \Sigma_g$  we consider the mapping torus

$$M(F) = \Sigma_g \times [0, 1] / (x, 1) \simeq (F(x), 0)$$

By construction we have  $\pi_1(M(F)) = \pi_1(\Sigma) *_{\mathbb{Z}}$  and hence, considering generating sets of  $\pi_1(\Sigma_g)$  as in (1.1) and adding a further element corresponding to the HNN-extension we obtain generating sets of  $\pi_1(M(F))$  with  $2g + 1$  elements. We will say that the so-obtained generating sets are *standard*. In this note we prove:

**Theorem 1.** *Let  $\Sigma_g$  be the closed surface of genus  $g \geq 2$ ,  $F \in \text{Map}(\Sigma_g)$  a pseudo-Anosov mapping class and  $M(F^n)$  the mapping torus of  $F^n$ . There is  $n_F$  such that for all  $n \geq n_F$*

$$\text{rank}(\pi_1(M(F^n))) = 2g + 1$$

*Moreover for any such  $n$  any generating set of  $\pi_1(M(F^n))$  with minimal cardinality is Nielsen equivalent to a standard generating set.*

Recall that two (ordered) generating sets  $\mathcal{S} = (g_1, \dots, g_r)$  and  $\mathcal{S}' = (g'_1, \dots, g'_r)$  are *Nielsen equivalent* if they belong to the same class of the equivalence relation generated by the following three moves:

$$\begin{array}{ll} \text{Inversion of } g_i & \left\{ \begin{array}{l} g'_i = g_i^{-1} \\ g'_k = g_k \quad k \neq i \end{array} \right. \\ \text{Permutation of } g_i \text{ and } g_j \text{ with } i \neq j & \left\{ \begin{array}{l} g'_i = g_j \\ g'_j = g_i \\ g'_k = g_k \quad k \neq i, j \end{array} \right. \\ \text{Twist of } g_i \text{ by } g_j \text{ with } i \neq j & \left\{ \begin{array}{l} g'_i = g_i g_j \\ g'_k = g_k \quad k \neq i \end{array} \right. \end{array}$$

In section 2 we recall the relation between Nielsen equivalence classes of generating sets of the fundamental group of a manifold  $M$  and free homotopy classes of graphs in  $M$ . Choosing such a graph with minimal length we obtain a link between the algebraic problem on the rank of  $\pi_1(M)$  and the geometry of the manifold. In section 3 we prove Proposition 1 which is essentially a generalization of the fact that paths in hyperbolic space  $\mathbb{H}^3$  which consist of large geodesic segments meeting at large angles are quasi-geodesic. Hyperbolic geometry comes into the picture through a theorem of Thurston who proved that the mapping torus  $M(F)$  of a pseudo-Anosov mapping class admits a metric of constant negative curvature; equivalently, there is a discrete torsion-free subgroup  $\Gamma \subset \text{PSL}_2 \mathbb{C} = \text{Isom}_+(\mathbb{H}^3)$  with  $M(F)$  homeomorphic to  $\mathbb{H}^3/\Gamma$ . The geometry of the manifolds  $M(F^n)$  is well understood and in section 4 we review very briefly some facts needed in section 5 to prove Theorem 1.

The method of proof of Theorem 1 is suggested by the proof of a result of White [11] who proved that the rank of the fundamental group of a hyperbolic 3-manifold yields an upper bound for the injectivity radius. Similar ideas appear also in the work of Delzant [5] on subgroups of hyperbolic groups with two generators, in the proof of a recent result of Ian Agol relating rank and Heegaard genus of some 3-manifolds and in the work of Kapovich-Weidmann [6]. See also [7] and [9].

I would like to thank to Ian Agol, Michel Boileau, Yo'av Moriah and Richard Weidmann for many very helpful and motivating conversations. I also thank Ian Biringer for useful comments enhancing the exposition.

This paper was written while the author was a member of the Laboratoire de mathématiques Emile Picard at the Université Paul Sabatier.

## 2. NIELSEN EQUIVALENCE OF GENERATING SETS AND CARRIER GRAPHS

Let  $M$  be a hyperbolic 3-manifold.

**Definition.** *A map  $f : X \rightarrow M$  of a connected graph  $X$  into  $M$  is a carrier graph if the homomorphism  $f_* : \pi_1(X) \rightarrow \pi_1(M)$  is surjective. Two carrier graphs  $f : X \rightarrow M$  and  $g : Y \rightarrow M$  are equivalent if there is a homotopy equivalence  $h : X \rightarrow Y$  such that  $f$  and  $g \circ h$  are free homotopic.*

To every generating set  $\mathcal{S} = (g_1, \dots, g_r)$  of  $\pi_1(M)$  one can associate an equivalence class of carrier graphs as follows: Let  $\mathbb{F}_{\mathcal{S}}$  be the free non-abelian group generated by the set  $\mathcal{S}$ ,  $\phi_{\mathcal{S}} : \mathbb{F}_{\mathcal{S}} \rightarrow \pi_1(M)$  the homomorphism given by mapping the free basis  $\mathcal{S} \subset \mathbb{F}_{\mathcal{S}}$  to the generating set  $\mathcal{S} \subset \pi_1(M)$  and  $X_{\mathcal{S}}$  a graph with  $\pi_1(X_{\mathcal{S}}) = \mathbb{F}_{\mathcal{S}}$ . The homomorphism  $\phi_{\mathcal{S}} : \mathbb{F}_{\mathcal{S}} \rightarrow \pi_1(M)$  determines a free homotopy class of maps  $f_{\mathcal{S}} : X_{\mathcal{S}} \rightarrow M$ , i.e. a carrier graph, and any two carrier graphs obtained in this way are equivalent. The so determined equivalence class is said to be the *equivalence class of carrier graphs associated to  $\mathcal{S}$* .

**Lemma 1.** *Let  $\mathcal{S}$  and  $\mathcal{S}'$  be finite generating sets of  $\pi_1(M)$  with the same cardinality. Then the following are equivalent:*

- (1)  $\mathcal{S}$  and  $\mathcal{S}'$  are Nielsen equivalent.
- (2) There is a free basis  $\bar{\mathcal{S}}$  of  $\mathbb{F}_{\mathcal{S}'}$  with  $\mathcal{S} = \phi_{\mathcal{S}'}(\bar{\mathcal{S}})$ .
- (3) There is an isomorphism  $\psi : \mathbb{F}_{\mathcal{S}} \rightarrow \mathbb{F}_{\mathcal{S}'}$  with  $\phi_{\mathcal{S}} = \phi_{\mathcal{S}'} \circ \psi$ .
- (4)  $\mathcal{S}$  and  $\mathcal{S}'$  have the same associated equivalence classes of carrier graphs.

*Sketch of the proof.* The implications (1)  $\Rightarrow$  (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4) are almost tautological. The implication (2)  $\Rightarrow$  (1) follows from a theorem of Nielsen, who proved that any two free basis of a free group are Nielsen equivalent (see for example [4]).  $\square$

The natural bijection given by Lemma 1 between the set of Nielsen equivalence classes of generating sets of  $\pi_1(M)$  and the set of equivalence classes of carrier graphs  $f : X \rightarrow M$  plays a central role in the proof of Theorem 1.

**Convention:** From now on we will only consider generating sets of minimal cardinality. Equivalently, we consider only carrier graphs  $f : X \rightarrow M$  with  $\text{rank}(\pi_1(X)) = \text{rank}(\pi_1(M))$ .

Given a carrier graph  $f : X \rightarrow M$  and a path  $I$  in  $X$  we say that its length is the length, with respect to the hyperbolic metric, of the path  $f(I)$  in  $M$ . Measuring the minimal length of a path joining two points in  $X$  we obtain a semi-distance  $d_{f:X \rightarrow M}$  on  $X$  and we define the *length*  $l_{f:X \rightarrow M}(X)$  of the carrier graph  $f : X \rightarrow M$  as the sum of the lengths of the edges of  $X$  with respect to  $d_{f:X \rightarrow M}$ . The semi-distance  $d_{f:X \rightarrow M}$  induced on  $X$  is not always a distance since there may be some edges of length 0 but minimality of the generating set ensures that by collapsing these edges we obtain an equivalent carrier graph on which the induced semi-distance is in fact a distance. Moreover, this collapsing process does not change the length of the carrier graph. From now on we will assume without further remark that the semi-distance  $d_{f:X \rightarrow M}$  is in fact a distance.

**Definition.** A carrier graph  $f : X \rightarrow M$  has minimal length if

$$l_{f:X \rightarrow M}(X) \leq l_{f':X' \rightarrow M}(X')$$

for every equivalent carrier graph  $f' : X' \rightarrow M$ .

If  $M$  is closed then it follows from Arzela-Ascoli's theorem that every equivalence class of carrier graphs contains a carrier graph with minimal length:

**Lemma 2.** *If  $M$  is a closed hyperbolic 3-manifold, then every equivalence class of carrier graphs contains a carrier graph with minimal length. Moreover, every such minimal length carrier graph is trivalent, hence it has  $3(\text{rank}(\pi_1(M)) - 1)$  edges, the image in  $M$  of its edges are geodesic segments, the angle between any two adjacent edges is  $\frac{2\pi}{3}$  and every simple closed path in  $X$  represents a non-trivial element in  $\pi_1(M)$ .*  $\square$

See White [11, Section 2] for a proof of Lemma 2.

### 3. QUASI-CONVEX SUBGRAPHS

Recall that a map  $\phi : X_1 \rightarrow X_2$  between two metric spaces is a  $(L, A)$ -quasi-isometric embedding if

$$\frac{1}{L}d_{X_1}(x, y) - A \leq d_{X_2}(\phi(x), \phi(y)) \leq Ld_{X_1}(x, y) + A$$

for all  $x, y \in X_1$ . A  $(L, A)$ -quasi-isometric embedding  $\phi : \mathbb{R} \rightarrow X$  is said to be a quasi-geodesic. Before going further, we state here and for further reference the following well-known fact:

**Lemma 3.** *There are constants  $l_0, A > 0$  such that for all  $L \geq l_0$  the following holds:*

- *Every path in hyperbolic space  $\mathbb{H}^3$  which consists of geodesic segments of at least length  $L$  and such that all the angles are at least  $\frac{\pi}{4}$  is a  $A$ -bi-Lipschitz embedding.*
- *If  $K \subset \mathbb{H}^3$  is convex then every geodesic ray  $\gamma : [0, \infty) \rightarrow \mathbb{H}^3$  with  $\gamma(0) \in K$  meets the boundary  $\partial\mathcal{N}_L(K)$  of the neighborhood  $\mathcal{N}_L(K)$  of radius  $L$  around  $K$  with at least angle  $\frac{\pi}{4}$ .  $\square$*

If  $f : X \rightarrow M$  is a carrier graph in a hyperbolic 3-manifold  $M$  we denote by  $\tilde{f} : \tilde{X} \rightarrow \mathbb{H}^3$  the lift of  $f$  to a map between the universal covers of  $X$  and  $M$ . We will be mainly interested in manifolds whose fundamental group is not free; in this case, the map  $\tilde{f}$  cannot be an embedding. However, subgraphs of  $X$  may well be quasi-isometrically embedded.

**Definition.** *A connected subgraph  $Y \subset X$  of a carrier graph  $f : X \rightarrow M$  is  $A$ -quasi-convex for some  $A > 0$  if:*

- *The restriction  $\tilde{f}|_{\tilde{Y}} : \tilde{Y} \rightarrow \mathbb{H}^3$  of the map  $\tilde{f}$  to the universal cover  $\tilde{Y}$  of  $Y$  is an  $(A, A)$ -quasi-isometric embedding.*
- *Every point in  $\tilde{Y}$  is at most at distance  $A$  from the axis of some element of  $\pi_1(Y)$ .*
- *The translation length of every element  $f_*(\gamma)$  in  $\mathbb{H}^3$  is at least  $\frac{1}{A}$  for every  $\gamma \in \pi_1(Y)$ .*

Recall that a discrete subgroup  $G$  of  $\mathrm{PSL}_2\mathbb{C}$  is *convex-cocompact* if there is a convex  $G$ -invariant subset  $C \subset \mathbb{H}^3$  of hyperbolic space with  $C/G$  compact. The smallest such convex subset of  $\mathbb{H}^3$  is the *convex-hull*  $CH(G)$  of  $G$  and it is well-known that  $CH(G)$  is the closure of the union of all axis of elements in  $G$ .

If  $Y$  is a graph and  $g : Y \rightarrow M$  is a map whose lift  $\tilde{g} : \tilde{Y} \rightarrow \mathbb{H}^3$  is a quasi-isometric embedding then the image  $g_*(\pi_1(Y))$  is a free convex-cocompact subgroup. Intuitively, considering  $A$ -quasi-convex graphs amounts to considering uniformly convex-cocompact free subgroups. More precisely, if  $Y \subset X$  is  $A$ -quasi-convex and  $\gamma \in \pi_1(Y)$  is non-trivial then the image  $\tilde{f}(\mathrm{Axis}(\gamma))$  is an  $(A, A)$ -quasi-geodesic and hence it is at uniformly bounded distance of the axis  $\mathrm{Axis}(f_*(\gamma))$  of  $f_*(\gamma)$ . In particular, there is a  $d$  depending only on  $A$  with

$$\tilde{f}(\tilde{Y}) \subset \mathcal{N}_d(CH(f_*(\pi_1(Y)))) \subset \mathcal{N}_{2d}(\tilde{f}(\tilde{Y}))$$

This fact, together with the last condition in the definition of  $A$ -quasi-convex, implies:

**Lemma 4.** *For all  $A$  there is  $d$  such that for every hyperbolic manifold  $M$  and every  $A$ -quasi-convex subgraph  $Y$  of a minimal length carrier graph  $f : X \rightarrow M$  there is a  $f_*(\pi_1(Y))$ -invariant convex-subset  $\bar{C}(Y)$  with*

$$\tilde{f}(\tilde{Y}) \subset \bar{C}(Y) \subset \mathcal{N}_d(\tilde{f}(\tilde{Y})),$$

*and such that  $d_{\mathbb{H}^3}(x, \gamma x) \geq l_0$  for all  $x \in \partial \bar{C}(Y)$  and  $\gamma \in f_*(\pi_1(Y))$ . Here  $l_0$  is the constant provided by Lemma 3.  $\square$*

The following result is the main technical point of the proof of Theorem 1.

**Proposition 1.** *For all  $A, s > 0$  there is  $L$  such that whenever  $M$  is a hyperbolic 3-manifold,  $f : X \rightarrow M$  is a minimal length carrier graph with  $s$  edges,  $Y_1, \dots, Y_k$  are disjoint connected  $A$ -quasi-convex subgraphs of  $X$  then either*

- *$\tilde{f} : \tilde{X} \rightarrow \mathbb{H}^3$  is a quasi-isometric embedding and hence  $\pi_1(M)$  is free, or*
- *the graph  $X \setminus \cup_i Y_i$  contains an edge of at most length  $L$ .*

The author suggests to the reader that he or she proves this proposition him or herself. In fact, a proof by picture takes two not particularly complicated drawings and this is clearly much more economic than the proof written below.

*Proof.* Let  $l_0$  and  $d$  be the constants provided by Lemma 3 and Lemma 4. We are going to show that  $\tilde{f} : \tilde{X} \rightarrow \mathbb{H}^3$  is a quasi-isometric embedding whenever every edge in  $X \setminus \cup Y_i$  has at least length  $6l_0 + 4d$ . Seeking a contradiction, assume that this is not the case. Then there is an infinite geodesic ray  $\gamma : [0, \infty) \rightarrow \tilde{X}$  whose image  $\tilde{f}(\gamma)$  is not a quasi-geodesic. If there is some  $t \in (0, \infty)$  such that  $\gamma(t, \infty)$  is disjoint from the union of the preimages of the graphs  $Y_i$ , then  $\tilde{f}(\gamma(t, \infty))$  consists of a perhaps short starting segment and geodesic segments of length at least  $6l_0 + 4d$  meeting with angle  $\frac{2\pi}{3}$ ; Lemma 3 implies that  $\tilde{f}(\gamma(t, \infty))$ , and hence  $\tilde{f}(\gamma)$ , is a quasi-geodesic ray, contradicting our assumption. Similarly, if there is  $t \in (0, \infty)$  such that  $\gamma(t, \infty)$  is contained in a preimage  $\tilde{Y}_i$  of some  $Y_i$  then the assumption that  $\tilde{f}|_{\tilde{Y}_i}$  is a quasi-isometric embedding implies again that  $\tilde{f}(\gamma(t, \infty))$  is quasi-geodesic, contradicting again our assumption. This implies that the curve  $\gamma$  has to enter and leave the union of the preimages of  $Y_i$  infinitely often.

Let  $a_1 < b_1 < a_2 < b_2 < a_3 < \dots$  be such that  $\gamma(a_j, b_j)$  is contained in and  $\gamma(b_j, a_{j+1})$  is disjoint from the preimage of  $\cup_i Y_i$  for all  $j \geq 1$ . Let also  $Z_j$  be the component of the preimage of  $\cup_i Y_i$  containing

$\gamma(a_j, b_j)$ . For every  $j$ , the path  $\gamma(b_j, a_{j+1})$  consists of edges which by assumption have length at least  $6l_0 + 4d$ . Let  $\gamma(b_j, c_j)$  be the first edge of this path. We claim that most of the length of  $\gamma(b_j, c_j)$  is outside of  $\mathcal{N}_{2l_0}(\bar{C}(\tilde{f}(Z_j)))$ . In fact, by Lemma 4, every point in the boundary of  $\mathcal{N}_{2l_0}(\bar{C}(\tilde{f}(Z_j)))$  is at at most distance  $2l_0 + d$  from  $\tilde{f}(Z_j)$ ; the assumption that  $f : X \rightarrow M$  is a minimal length graph implies that  $\tilde{f}(\gamma(b_j, c_j))$  spends at most  $2l_0 + d$  time within  $\mathcal{N}_{2l_0}(\bar{C}(\tilde{f}(Z_j)))$ . Let  $b_j^+$  be the exit time. Then  $\tilde{f}(\gamma(b_j^+, c_j))$  is a geodesic segment of at least length  $4l_0 + 3d$  which, by lemma 3 has at least angle  $\frac{\pi}{4}$  with the boundary of  $\mathcal{N}_{2l_0}(\bar{C}(\tilde{f}(Z_j)))$ . A similar discussion applies not when exiting but when entering  $\mathcal{N}_{2l_0}(\bar{C}(\tilde{f}(Z_{j+1})))$ ; let  $a_{j+1}^-$  be the entry time. Setting  $I_1 = \gamma(a_1, b_1^+)$ ,  $J_1 = \gamma(b_1^+, a_2^-)$ ,  $I_2 = \gamma(a_2^-, b_2^+)$ ,  $J_2 = \gamma(b_2^+, a_3^-)$ , ... we obtain a decomposition of  $\gamma(a_1, \infty)$  in segments with the following properties:

- $\tilde{f}(I_j) \subset \mathcal{N}_{2l_0}(\bar{C}(Z_j))$  and is  $A'$ -quasi-geodesic for some  $A'$  and all  $j$ .
- For all  $j$ ,  $\tilde{f}(J_j)$  is a path consisting of geodesic segments of at least length  $2l_0$ , with at least angle  $\frac{2\pi}{3}$  at the vertices, with endpoints in the boundaries of  $\mathcal{N}_{2l_0}(\bar{C}(Z_j))$  and  $\mathcal{N}_{2l_0}(\bar{C}(Z_{j+1}))$  and such that the angles with these boundaries at the endpoints are at least  $\frac{\pi}{4}$ .

Before going further we observe that for all  $j$  we have  $\tilde{f}(a_j^-) \neq \tilde{f}(b_j^+)$  because the homomorphism  $f_*|_{\pi_1(Y_i)}$  is injective for all  $i$ . Assume now that the distance of  $\tilde{f}(a_j^-)$  and  $\tilde{f}(b_j^+)$  is less than  $l_0$ . Then, by Lemma 4 we have that the images in  $M$  of  $\tilde{f}(a_j^-)$  and  $\tilde{f}(b_j^+)$ , and hence the images of the segments  $\tilde{f}(a_j^-, a_j)$  and  $\tilde{f}(b_j, b_j^+)$ , are different. This implies that we can replace equivariantly the segment  $\tilde{f}(a_j^-, a_j)$  by the geodesic segment  $[\tilde{f}(a_j^-), \tilde{f}(b_j^+)]$ , getting a new carrier graph  $f' : X' \rightarrow M$  with length

$$\begin{aligned} l_{f':X' \rightarrow M'}(X') &\leq l_{f:X \rightarrow M}(X) - l(\tilde{f}(a_j^-, a_j)) + l([\tilde{f}(a_j^-), \tilde{f}(b_j^+)]) \\ &\leq l_{f:X \rightarrow M}(X) - 2l_0 + l_0 < l_{f:X \rightarrow M}(X) \end{aligned}$$

This contradicts the minimality of  $l_{f:X \rightarrow M}(X)$  and proves that the distance between the points  $\tilde{f}(a_j^-)$  and  $\tilde{f}(b_j^+)$  of  $I_j$  is less than  $l_0$ . Let  $I'_j$  be the geodesic segment joining the endpoints of  $I_j$ ; observe that the length of this homotopy is bounded by some constant  $A''$  because  $I_j$  is  $A'$ -quasi-geodesic for all  $j$ . Then the path  $\gamma$  is properly homotopic to the path  $\gamma'$  obtained as the juxtaposition of the segments  $I'_1 \cup J_1 \cup I'_2 \cup J_2 \cup \dots$ . This path consists now of geodesic segments of

at least length  $l_0$  and meeting with angles at least  $\frac{\pi}{4}$ . Lemma 3 implies that  $\gamma'$  is a quasi-geodesic. Then the same holds for  $\gamma$  because the homotopy from  $\gamma$  to  $\gamma'$  has at most length  $A''$ . This yields the desired contradiction.  $\square$

#### 4. SOME FACTS ON THE GEOMETRY OF MAPPING TORI

As mentioned in the introduction, the following is the starting point of our considerations:

**Theorem** (Thurston [10]). *Let  $\Sigma_g$  be the closed surface of genus  $g \geq 2$  and  $F \in \text{Map}(\Sigma_g)$  a pseudo-Anosov mapping class. Then the mapping torus  $M(F) = \Sigma_g \times [0, 1]/(x, 1) \simeq (F(x), 0)$  admits a hyperbolic metric.*

The manifold  $M(F)$  fibers over the circle with fiber  $\Sigma_g$  and monodromy  $F$ . Let  $\pi : \pi_1(M(F)) \rightarrow \mathbb{Z}$  be the homomorphism given by this fibering and observe that  $M(F^n)$  is homeomorphic, and hence isometric by Mostow's rigidity theorem, to the cover of  $M(F)$  corresponding to the kernel of the composition of  $\pi$  and the canonical homomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ . Let  $M'$  be the infinite cover of  $M(F)$  corresponding to the kernel of  $\pi$ ; in the sequel we will always consider  $M'$  with the unique hyperbolic metric such that the covering  $M' \rightarrow M(F)$  is Riemannian. Before going further we observe the following fact that we state here for further reference:

**Lemma 5.** *For every  $D$  there is  $n_D$  such that the following holds for all  $n \geq n_D$ : Every subset  $K \subset M(F^n)$  of diameter at most  $D$  lifts homeomorphically to  $M'$ .*  $\square$

Many of the arguments used in the present paper rely on properties of finitely generated subgroups of the fundamental group of  $M'$ .

**Proposition 2.** *Every proper subgroup  $G$  of  $\pi_1(M')$  of rank at most  $2g$  is free and convex-cocompact.*

*Sketch of the proof.* The manifold  $M'$  is homeomorphic to  $\Sigma_g \times \mathbb{R}$ . In particular, every proper subgroup of  $\pi_1(M') \simeq \pi_1(\Sigma_g)$  is either free or isomorphic to the fundamental group of a closed surface which covers  $\Sigma_g$  with at least degree 2. Any such surface has genus greater than  $g$  and hence its fundamental group has rank greater than  $2g$ . We have proved that the group  $G$  is free. A result due to Thurston in this case and to Agol [1] and Calegari-Gabai [2] in much more generality asserts that  $\mathbb{H}^3/G$  is homeomorphic to the interior of a handlebody. Now, Canary's generalization of Thurston's covering theorem [3] implies that  $G$  is convex-cocompact.  $\square$

## 5. PROOF OF THEOREM 1

As the kind reader may have deduced from the title of this section, we prove here Theorem 1. But first, as a warm-up, we show the result of White mentioned in the introduction:

**Theorem** (White [11]). *For all  $r$  there is  $R$  such that every closed hyperbolic 3-manifold  $M$  with  $\text{rank}(\pi_1(M)) \leq r$  has  $\text{inj}(M) \leq R$ .*

*Proof.* Let  $f : X \rightarrow M$  be a minimal length carrier graph in the class of a minimal generating set of  $\pi_1(M)$ ; observe that  $X$  has at most  $s = 3(r - 1)$  edges. Denote by  $X^{<t}$  the (possibly empty) subgraph of  $X$  consisting of the union of all the edges with length less than  $t$ . Every simple closed circuit in  $X^{<t}$  represents a non-trivial element in  $\pi_1(M)$  by Lemma 2 and has at most length  $3t(r - 1)$ . In particular, it suffices to show that there is  $t_r$  depending only on  $r$  such that some component  $Y$  of  $X^{<t_r}$  is not a tree.

Let  $l_0$  be the constant provided by Lemma 3. Since  $M$  is closed we have that  $\pi_1(M)$  is not free and in particular  $\tilde{f} : \tilde{X} \rightarrow \mathbb{H}^3$  cannot be a quasi-isometric embedding. In particular,  $X^{<l_0}$  is not empty by Lemma 2 and Lemma 3. If every component  $Y$  of  $X^{<l_0}$  is a tree then  $\text{diam}(\tilde{Y}) = \text{diam}(Y) \leq 3(r - 1)l_0$  and hence the map

$$\tilde{f}|_{\tilde{Y}} : \tilde{Y} \rightarrow \mathbb{H}^3$$

is a  $(3(r - 1)l_0, 3(r - 1)l_0)$ -quasi-isometric embedding. We obtain from Proposition 1 a constant  $l_1 = l_1(r)$  depending only on  $r$  such that  $X^{<l_0}$  is a proper subgraph of  $X^{<l_1}$ . If again every connected component of  $X^{<l_1}$  is tree then we get  $l_2 = l_2(r)$  depending only on  $r$  such that  $X^{<l_1}$  is a proper subgraph of  $X^{<l_2}$ . This process can be repeated at most  $3(r - 1)$  times since this is the number of edges in  $X$ ; this concludes the proof of White's theorem.  $\square$

As we see, the proof of White's theorem yields in fact that every generating set  $(g_1, \dots, g_r)$  is Nielsen-equivalent to a generating set  $(g'_1, \dots, g'_r)$  such that the translation length of  $g'_1$  is uniformly bounded. The idea of the proof of Theorem 1 is to show that every generating set of  $\pi_1(M(F^n))$  is Nielsen equivalent to a generating set such that the translation lengths of all elements but 1 are uniformly bounded.

**Theorem 1.** *Let  $\Sigma_g$  be the closed surface of genus  $g \geq 2$ ,  $F \in \text{Map}(\Sigma_g)$  a pseudo-Anosov mapping class and  $M(F^n)$  the mapping torus of  $F^n$ . There is  $n_F$  such that for all  $n \geq n_F$*

$$\text{rank}(\pi_1(M(F^n))) = 2g + 1$$

Moreover for any such  $n$  any generating set of  $\pi_1(M(F^n))$  with minimal cardinality is Nielsen equivalent to an standard generating set.

*Proof.* For all  $n$  let  $\mathcal{S}_n$  be a generating set of  $\pi_1(M(F^n))$  with minimal cardinality and  $f_n : X_n \rightarrow M(F^n)$  a minimal length carrier graph in the equivalence class determined by  $\mathcal{S}_n$ . As remarked in the introduction

$$\text{rank}(\pi_1(M(F^n))) \leq 2g + 1$$

and hence  $X_n$  has at most  $6g$  edges. As in the proof of White's theorem, we denote by  $X_n^{<t}$  the subgraph of  $X_n$  consisting of all the edges of  $X_n$  of length less than  $t$ .

**Claim 1.** *For every  $D$  there are  $n_D$  and  $A_D$  such that for every subgraph  $Y_n$  of  $X_n$  of length less than  $D$  and such that the image of  $\pi_1(Y_n)$  is convex-cocompact one has:  $Y_n$  is  $A_D$ -quasi-convex for all  $n \geq n_D$ .*

*Proof of Claim 1.* To begin with observe that the injectivity radius of the manifold  $M(F^n)$  is bounded from below by  $\text{inj}(M(F))$  for all  $n$ . In particular, the last condition in the definition of  $A$ -quasi-convex is automatically satisfied for every  $A$  with  $A^{-1} \leq \text{inj}(M(F))$ .

Seeking a contradiction assume that for some  $D$  there are sequences  $A_i, n_i \rightarrow \infty$  such that for all  $i$  there is a subgraph  $Y_{n_i}$  of  $X_{n_i}$  which has length less than  $D$  and fails to be  $A_i$ -quasi-convex but such that  $(f_{n_i})_*(\pi_1(Y_{n_i}))$  is convex-cocompact. Composing the map  $f_{n_i} : X_{n_i} \rightarrow M(F^{n_i})$  with the covering  $M(F^{n_i}) \rightarrow M(F)$  we obtain from Arzela-Ascoli's theorem that, up to conjugacy in  $\pi_1(M(F))$  and passing to a subsequence, we may assume that  $(f_{n_i})_*(\pi_1(Y_{n_i})) = (f_{n_j})_*(\pi_1(Y_{n_j}))$  are conjugated for all  $i, j$ . In particular, the desired contradiction follows if we show that the map  $\pi_1(Y_{n_i}) \rightarrow f_*(\pi_1(Y_{n_i}))$  is an isomorphism.

By Lemma 5 there is  $i_D$  such that for all  $i \geq i_D$  the graph  $Y_{n_i}$  lifts to  $M'$ . In particular we obtain from Proposition 2 that  $(f_{n_i})_*(\pi_1(Y_{n_i}))$  is a free subgroup of  $\pi_1(M')$  which has in particular at most the same rank as  $\pi_1(Y_{n_i})$ . Minimality of the generating set ensures that

$$\text{rank}((f_{n_i})_*(\pi_1(Y))) = \text{rank}(\pi_1(Y_{n_i})).$$

We are done, since every surjective homomorphism between two free groups of the same rank is an isomorphism.  $\square$

We use now an argument similar to the one in the proof of White's theorem to show:

**Claim 2.** *There are  $n_1$  and  $t$  such that for all  $n \geq n_1$  there is a connected component  $Y_n$  of  $X_n^{<t}$  such that the image of  $\pi_1(Y_n)$  into  $\pi_1(M(F^n))$  is not convex-cocompact.*

*Proof of Claim 2.* As in the proof of White's theorem we obtain a first constant  $t_1$  such that for all  $n$  at least one of the components  $Y_{n,t_1}^1, \dots, Y_{n,t_1}^{k(n,t_1)}$  of  $X_n^{<t_1}$  is not a tree. If for all  $n$  the image of the fundamental group of one of these component fails to be convex-cocompact then are done with  $t = t_1$ . Assume that there is a subsequence  $(n_i)_i$  such that the image of  $\pi_1(Y_{n_i,t_1}^j)$  is convex-cocompact for all  $j$  and  $i$ . By claim 1 there is a constant  $A_1$  such that  $Y_{n_i,t_1}^j$  is  $A_1$ -quasi-convex for all  $i, j$ . In particular, we obtain from Proposition 1 a constant  $t_2$  such that  $X_{n_i}^{<t_1}$  is a proper subgraph of  $X_{n_i}^{<t_2}$  for all  $i$ . If again the image in of the fundamental group of every connected component of  $X_{n_i}^{<t_2}$  is convex-cocompact for infinitely many  $i$ , say for all  $i$ , then we can repeat the process. The bound on the number of edges of  $X_n$  ensures that after at most  $6g$  steps we find the desired subgroup.  $\square$

We can now conclude the proof of Theorem 1. The subgraph  $Y_n$  of  $X_n$  provided by claim 2 is contained in  $X_n^{<t}$  and therefore it has at most diameter  $6gt$ . By Lemma 5 there is  $n_1$  such that  $Y_n$  lifts to  $M'$  for all  $n \geq n_1$ ; in particular,  $\pi_1(Y_n)$  does not surject onto  $\pi_1(M(F^n))$  and hence one has

$$(5.1) \quad \text{rank}(\pi_1(Y_n)) \leq \text{rank}(\pi_1(M(F^n))) - 1 \leq 2g$$

Since the image of  $\pi_1(Y_n)$  into  $\pi_1(M')$  is not convex-cocompact we deduce from Proposition 2 that  $\pi_1(Y_n)$  surjects on  $\pi_1(M')$ ; thus

$$(5.2) \quad 2g = \text{rank}(\pi_1(M')) \leq \text{rank}(\pi_1(Y_n))$$

The first claim of Theorem 1 follows from (5.1) and (5.2). Choosing generators  $(a_1, \dots, a_{2g+1})$  of  $\pi_1(X_n)$  in such a way that  $(a_1, \dots, a_{2g})$  generate  $\pi_1(Y_n)$  we obtain, by Lemma 1, that  $\mathcal{S}_n$  is Nielsen equivalent to a generating set  $(a_1, \dots, a_{2g+1})$  of  $\pi_1(M(F^n))$  where  $(a_1, \dots, a_{2g})$  generate  $\pi_1(\Sigma_g) = \pi_1(M')$ . A theorem of Zieschang [12] ensures that any generating set of  $\pi_1(\Sigma_g)$  of cardinality  $2g$  is Nielsen equivalent to a generating set as in (1.1).  $\square$

## REFERENCES

- [1] I. Agol, *Tameness of hyperbolic 3-manifolds*, preprint (2004).
- [2] D. Calegari and D. Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, preprint (2004).
- [3] R.D. Canary, *A covering theorem for hyperbolic 3-manifolds and its applications*, Topology 3 (1996).
- [4] D.J. Collins, R.I. Grigorchuk, P.F. Kurchanov and H. Zieschang, *Combinatorial Group Theory and Applications to Geometry*, Algebra VII, Encyclopaedia of Mathematical Sciences 58, Springer (1993).
- [5] T. Delzant, *Sous-groupes à deux générateurs des groupes hyperboliques*, in *Group theory from a geometric view point*, World Scientific (1991).

- [6] I. Kapovich and R. Weidmann, *Kleinian groups and the rank problem*, preprint (2004).
- [7] H. Namazi and J. Souto, *Heegaard Splittings and pseudo-Anosov maps*, in preparation.
- [8] E. Rips, *Subgroups of small cancellation groups*, Bull. Lond. Math. Soc. 14 (1982).
- [9] J. Souto, *Rank and topology of hyperbolic 3-manifolds I*, preprint (2006).
- [10] W. Thurston, *The geometry and topology of 3-manifolds*, la palabra de dios.
- [11] M. White, *Injectivity radius and fundamental groups of hyperbolic 3-manifolds*, Comm. Anal. Geom. 10 (2002).
- [12] H. Zieschang, *Über die Nielsensche Kurzungsmethode in freien Produkten mit Amalgam*, Invent. Math., 10 (1970).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO.