KNOTTED GEODESICS IN HYPERBOLIC 3-MANIFOLDS

KEN BROMBERG & JUAN SOUTO

Abstract. We prove that geodesics in hyperbolic manifolds homeomorphic to $\Sigma_g \times \mathbb{R}$ are knotted if they wrap around some very short geodesic.

1. Introduction

Throught this note let M_g be the interior of $\bar{M}_g = \Sigma_g \times I$, the trivial interval bundle over a closed orientable surface of genus g. Recall that a homotopically essential simple closed curve $\gamma \subset M_g$ is *unknotted* if it is isotopic to a simple closed curve on $\partial \bar{M}_g$. Equivalently, γ is contained in an embedded surface $S \subset M_g$ isotopic to one of the components of $\partial \bar{M}_g$. In [Ota95], Otal proved that for every g there is a constant ϵ_{Otal} smaller than the Margulis constant such that whenever ρ is a hyperbolic metric on the interior M_g of $\bar{M}_g$ and γ is primitive geodesic in (M_g, ρ) shorter than ϵ_{Otal} then γ is unknotted.

In this note we produce examples of knotted geodesics of moderate length. This answers a question of Mary Rees.

Theorem 1.1 (Knotted geodesics exist)**.** *There is a hyperbolic metric ρ on M_g such that the hyperbolic manifold (M_g, ρ) contains a geodesic which is homotopic but not isotopic to a simple curve in $\partial \bar{M}_g$.*

The statement of Theorem 1.1 is probably not surprising. What is perhaps remarkable is that the curves that we prove to be knotted have previously attracted a fair share of attention: We show that simple geodesics in pleated surfaces which wrap around a sufficiently short Margulis tube are knotted. In particular, it is not difficult to see that the following is also true: *There is some constant l (probably $l = 10$ suffices) and hyperbolic metrics ρ on M_g such that the hyperbolic manifold (M_g, ρ) contains infinitely many geodesics of at most length l which are homotopic but not isotopic to simple curves in $\partial \bar{M}_g$.* The only reason why we mention this is because it makes clear that knotted geodesics don't need to be extremely long.

In order to prove Theorem 1.1 we need to ensure that there are enough hyperbolic manifolds which contain simple geodesics which are homotopic to simple curves in $\partial\bar{M}_g$. We will actually prove the density, in a suitable sense, of the set of hyperbolic metrics on M_g such that every geodesic which is homotopic to a simple curve in $\partial\bar{M}_g$ is itself simple. This is the content of Proposition 2.1 below. We prove then Proposition 3.1, the main technical ingredient in the proof of Theorem 1.1. This result asserts that moderately long simple geodesics homotopic to simple curves in $\partial\bar{M}_g$ which are linked with the set of sufficiently short geodesics are knotted. In section 4 we put this two ingredients together with a well-known construction due to Kerckhoff and Thurston to prove Theorem 1.1.

2. Metrics with simple geodesics

Since we are investigating the knotting properties of geodesics, we care only about geodesics which are (1) simple and (2) homotopic to simple curves in $\partial\bar{M}_g$. The goal of this section is to prove that for a generic hyperbolic metric every geodesic which is homotopic to a simple curve in $\partial\bar{M}_g$ is itself simple.

Every hyperbolic 3-manifold homeomorphic to M_g is isometric to one of the manifolds $M_\sigma = \mathbb{H}^3/\sigma(\pi_1(\Sigma_g))$ where σ is faithful and discrete representation

$$\sigma : \pi_1(\Sigma_g) \to \mathrm{PSL}_2\,\mathbb{C} = \mathrm{Isom}_+(\mathbb{H}^3) \tag{2.1}$$

On the other hand, the work of Bonahon [Bon86] implies that for every discrete and faithful σ as in (2.1), the hyperbolic manifold M_σ is homeomorphic to M_g. Observe that under this identification, a curve $\gamma \subset M_\sigma$ is homotopic to a simple curve in $\partial\bar{M}_g$ if and only if its free homotopy class corresponds to the image under σ of the free homotopy class of a simple closed curve in Σ_g.

As we see, we can identify, up to certain equivalence relations which we don't discuss here, the set of metrics on M_g with the set AH_g of discrete and faithful of representations $\sigma : \pi_1(\Sigma_g) \to \mathrm{PSL}_2\,\mathbb{C}$. It follows from the work of Jorgensen that AH_g is a closed subset, with respect to the Hausdorff topology, of the complex algebraic variety $\mathrm{Hom}(\pi_1(\Sigma_g, \mathrm{PSL}_2\,\mathbb{C}))$.

Let $\mathcal{X}$ be the set of those $\sigma \in AH_g$ for which the hyperbolic manifold $M_\sigma = \mathbb{H}^3/\sigma(\pi_1(\Sigma_g))$ contains a non-simple geodesic which is homotopic to a simple closed curve in $\partial\bar{M}_g$. We prove:

Proposition 2.1. *The set $\mathcal{X}$ has empty interior in AH_g.*

Recall that a representation $\sigma \in AH_g$ is *quasi-fuchsian* if the hyperbolic manifold M_σ is convex-cocompact. Marden [Mar74] proved that the set QH_g of quasi-fuchsian representations $\sigma \in AH_g$ is open in $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2\,\mathbb{C})$ and it follows from the work of Ahlfors and Bers that QH_g is connected. On the other hand, Brock-Bromberg [BB04] proved that QH_g is dense in AH_g. In particular, in order to prove Proposition 2.1 it suffices to show that $\mathcal{X} \cap QH_g$ is the countable union of proper analytic sets.

For every $\gamma \in \pi_1(\Sigma_g)$ and $\sigma \in QH_g$ the element $\sigma(\gamma) \in \mathrm{PSL}_2\,\mathbb{C}$ is hyperbolic and hence has an axis $\mathrm{Axis}(\sigma(\gamma))$. The geodesic in $M_\sigma = \mathbb{H}^3/\sigma(\pi_1(\Sigma_g))$ corresponding to the free homotopy class of $\sigma(\gamma)$ is the projection to M_σ of $\mathrm{Axis}(\sigma(\gamma))$ and hence it has self-intersections if and only if there is $\eta \in \pi_1(\Sigma_g)$ such that the axis $\mathrm{Axis}(\sigma(\gamma))$ and $\mathrm{Axis}(\sigma(\eta\gamma\eta^{-1}))$ are not equal but intersect each other. Observe that $\mathrm{Axis}(\sigma(\gamma)) = \mathrm{Axis}(\sigma(\eta\gamma\eta^{-1}))$ if and only if γ and η commute, i.e. $[\gamma, \eta] = 1$.

Given $\gamma, \eta \in \pi_1(\Sigma)$ with γ represented by an essential simple closed curve and such that $[\gamma, \eta] \neq 1$ set

$$\mathcal{X}(\gamma, \eta) = \{\sigma \in QH_g | \, \mathrm{Axis}(\sigma(\gamma)) \cap \mathrm{Axis}(\sigma(\eta\gamma\eta^{-1})) \neq \emptyset\}$$

By the preceding discussion, $\mathcal{X} \cap QH_g$ is the union of the sets $\mathcal{X}(\gamma, \eta)$ over all possible choices for γ and η. If $\sigma \in QH_g$ is a fuchsian representation, i.e. $\sigma(\pi_1(\Sigma_g))$ stabilizes a totally geodesic copy of $\mathbb{H}^2$ in $\mathbb{H}^3$, then the geodesic corresponding to $\sigma(\gamma)$ in M_σ is simple. In particular $\sigma \notin \mathcal{X}(\gamma, \eta)$ and hence the later is a proper subset of QH_g. We claim that $\mathcal{X}(\gamma, \eta)$ is analytic.

The axis $\mathrm{Axis}(\sigma(\gamma))$ is determined by the pair $(\theta^+_{\sigma(\gamma)}, \theta^-_{\sigma(\gamma)})$ of attractive and repelling fixed points of $\sigma(\gamma)$ on $\mathbb{C}P^1 = \partial_\infty \mathbb{H}^3$. A simple computation shows that the map

$$QH_g \to \mathbb{C}P^1 \times \mathbb{C}^1, \; \sigma \mapsto (\theta^+_{\sigma(\gamma)}, \theta^-_{\sigma(\gamma)})$$

is analytic. If $\mathrm{Axis}(\sigma(\gamma)) \cap \mathrm{Axis}(\sigma(\eta\gamma\eta^{-1})) \neq \emptyset$ then both geodesics lie in a totally geodesic copy of $\mathbb{H}^2$ in $\mathbb{H}^3$. In particular, the cross-ratio of the points $(\theta^+_{\sigma(\gamma)}, \theta^-_{\sigma(\gamma)}, \theta^+_{\sigma(\eta\gamma\eta^{-1})}, \theta^-_{\sigma(\eta\gamma\eta^{-1})})$

$$\frac{\left(\theta^+_{\sigma(\gamma)} - \sigma(\eta)\theta^+_{\sigma(\gamma)}\right)\left(\theta^-_{\sigma(\gamma)} - \sigma(\eta)\theta^-_{\sigma(\gamma)}\right)}{\left(\theta^+_{\sigma(\gamma)} - \sigma(\eta)\theta^-_{\sigma(\gamma)}\right)\left(\theta^-_{\sigma(\gamma)} - \sigma(\eta)\theta^+_{\sigma(\gamma)}\right)}$$

is real. A computation in the upper half plane shows now that in fact the two lines $\mathrm{Axis}(\sigma(\gamma))$ and $\mathrm{Axis}(\sigma(\eta\gamma\eta^{-1}))$ intersect if and only if the cross-ratio is not only real but also < 1. This shows that $\mathcal{X}(\gamma, \eta)$ is the pre-image of $(-\infty, 1)$ under the analytic map

$$QH_g \to \mathbb{C}, \quad \sigma \mapsto \frac{\left(\theta^+_{\sigma(\gamma)} - \theta^+_{\sigma(\eta\gamma\eta^{-1})}\right)\left(\theta^-_{\sigma(\gamma)} - \theta^-_{\sigma(\eta\gamma\eta^{-1})}\right)}{\left(\theta^+_{\sigma(\gamma)} - \theta^-_{\sigma(\eta\gamma\eta^{-1})}\right)\left(\theta^-_{\sigma(\gamma)} - \theta^+_{\sigma(\eta\gamma\eta^{-1})}\right)}$$

This proves that $\mathcal{X}(\gamma, \eta)$ is analytic. We have proved that $\mathcal{X}$ is the countable union of proper analytic subsets in the the open connected subset QH_g of the algebraic variety $\mathrm{Hom}(\pi_1(\Sigma_g), \mathrm{PSL}_2\,\mathbb{C})$. In particular, $\mathcal{X}$ has empty interior. □

Before moving on to more interesting topics we would like to ask:

Question 1: Is $\mathcal{X}$ dense?

Question 2: Is the interior of $\mathcal{X} \cap (AH_g \setminus QH_g)$ in $AH_g \setminus QH_g$ empty?

3. A knottedness criterium

In this section we prove the main technical ingredient in the proof of Theorem 1.1 but before doing so we need to recall some facts about the thin-thick decomposition of hyperbolic 3-manifolds. Recall that for ϵ positive and small, the ϵ-thin part $N^{<\epsilon}$ of a hyperbolic 3-manifold is the set of point with injectivity radius $\mathrm{inj}_N(x) < \epsilon$. It is due to Margulis that there is a universal constant μ such that for every $\epsilon \leq \mu$ positive and every (orientable) hyperbolic 3-manifold N we have: the fundamental group of every component U of the ϵ-thin part $N^{<\epsilon}$ is abelian. The bounded components of $N^{<\mu}$ are called *Margulis tubes* and the unbounded components *cusps*. Every Margulis tube U is a standard neighborhood of a simple closed geodesic, the *soul* of U.

Below we will say that a hyperbolic metric ρ on M_g is *quasi-fuchsian* if the manifold (M_g, ρ) is isometric to M_σ for some $\sigma \in QH_g$. Observe that the μ-thin part of (M_g, ρ) consists only of bounded components whenever ρ is quasi-fuchsian.

Proposition 3.1. *For all l there is a positive $\epsilon < \epsilon_{Otal}$ such that if $\gamma \subset M_g$ is a simple closed primitive geodesic with respect to some quasi-fuchsian metric ρ with $\epsilon_{Otal} < l_\rho(\gamma) < l$, then γ knotted unless γ is contained in an embedded level surface $S \subset (M, \rho)^{>\epsilon}$.*

Recall that $\epsilon_{Otal} < \mu$ is the constant in the theorem of Otal [Ota95] mentioned in the introduction. Under a level surface we understand a surface $S \subset M_g$ such that the inclusion $S \hookrightarrow M_g$ is a homotopy equivalence. We only assume that ρ is quasi-fuchsian in order to have a less cumbersome statement.

Proof. Given $l > 0$ we choose $\epsilon < \epsilon' < \epsilon_{Otal}$ such that for every hyperbolic 3-manifold N the following holds:

- If a curve $\eta \subset N$ with at most length l intersects a bounded component U of the ϵ'-thin part $N^{\leq \epsilon'}$, then η is homotopic to a power of the soul of U.
- If U is a bounded component of $N^{<\epsilon'}$ whose soul σ_U is shorter than ϵ, then $\cosh(d_N(\sigma_U, \partial U)) - 1 \geq 2(g-1)$.

The existence of the constant ϵ and ϵ' follows from the Margulis lemma.

Let now ρ be a quasi-fuchsian metric on M_g and assume that γ is a simple unknotted primitive geodesic in (M_g, ρ) with length $\epsilon_{Otal} < l_\rho(\gamma) < l$. Recall that $(M_g, \rho)^{<\epsilon'}$ consists only of bounded components. By the choice of ϵ' the geodesic γ is contained in $(M_g, \rho)^{>\epsilon'}$. Choose δ very small and positive such that

$$\mathcal{N}_\delta(\gamma) = \{x \in (M_g, \rho) | d_\rho(x, \gamma) < \delta\}$$

is a regular neighborhood of the simple geodesic γ disjoint of $(M_g, \rho)^{<\epsilon'}$. For $\delta' < \delta$ we can find as in [CG06, 1.4] as metric $\hat{\rho}$ on $\hat{M}_g = M_g \setminus \mathcal{N}_{\delta'}(\gamma)$ with the following properties:

- $\hat{\rho}$ is non-positively curved,
- $(\hat{M}_g, \hat{\rho})$ has totally geodesic flat boundary, and
- ρ and $\hat{\rho}$ coincide on $M_g \setminus \mathcal{N}_\delta(\gamma)$.

The assumption that γ is unknotted, implies that there is an embedded level surface $\Sigma_1 \subset M_g$ with $\gamma \subset \Sigma_1$. Up to isotopy we may assume that $\Sigma_1 \cap \mathcal{N}_{\delta'}(\gamma)$ is an annulus A such that ∂A is geodesic in the flat torus $(\partial \hat{M}_g, \hat{\rho})$.

Then $\hat{\Sigma}_1 = \Sigma_1 \setminus A \subset \hat{M}_g$ is an incompressible and ∂-incompressible properly embedded surface with geodesic boundary. It follows then from [CG06, Lemma 1.20] that the surface $\hat{\Sigma}_1$ is properly isotopic to an embedded surface $\hat{\Sigma}_2$ with the following properties:

- $\hat{\Sigma}_2 \cap \partial \hat{M} = \partial \hat{\Sigma}_2 = \partial \hat{\Sigma}_1 = \partial A$, and
- $\hat{\Sigma}_2$ is a minimal surface with respect to the metric $\hat{\rho}$.

Let U be the closure of a component of $(M_g, \rho)^{<\epsilon'}$ whose soul σ_U has at most length ϵ; observe that by construction $U \subset \hat{M}_g$ and that $\rho = \hat{\rho}$ on U; in particular $\hat{\rho}$ is hyperbolic on U. We claim that $\hat{\Sigma}_2$ is properly isotopic by an isotopy supported by a regular neighborhood of U to a surface which does intersects U. Up to reducing U slightly we may assume that $\hat{\Sigma}_2$ intersects ∂U tranversally. Observe that it follows from the incompressibility of $\hat{\Sigma}_2$ and the convexity of U that $\hat{\Sigma}_2 \cap U$ consist of a collection of disks and ∂-parallel annuli. In particular, $\hat{\Sigma}_2$ can be isotoped out of U once we prove that no component of $\hat{\Sigma}_2 \cap U$ is a meridional disk of U. Seeking a contradiction assume that such a disk $D \subset \hat{\Sigma}_2 \cap U$ exists and let $x \in D \cap \sigma_U$ be a point where D intersects x. By construction, the boundary of ∂D is disjoint of the interior of the ball $B_x(d_\rho(\sigma_U, \partial U))$ centered at x and with radius $d_\rho(\sigma_U, \partial U)$. In particular, the monotonicity formula shows that the disk D has at least the same area as a disk in $\mathbb{H}^2$ of radius $d_\rho(\sigma_U, \partial U)$. In other words,

$$\operatorname{area}(D) \geq 2\pi(\cosh(d_\rho(\sigma_U, \partial U)) - 1) > 4\pi(g-1)$$

by the choice of ϵ.

The Gauß-equation implies that minimal surfaces have extrinsic negative curvature. In particular we obtain the following estimates for the sectional curvature $\kappa_{\hat{\Sigma}_2}$ of $\hat{\Sigma}_2$:

$$\kappa_{\hat{\Sigma}_2}(p) \leq 0 \qquad \text{for all } p \in \hat{\Sigma}_2 \text{ and}$$

$$\kappa_{\hat{\Sigma}_2}(p) \leq -1 \qquad \text{for all } p \in D \subset \hat{\Sigma}_2$$

Integrating we obtain

$$\int_{\hat{\Sigma}_2} \kappa_{\hat{\Sigma}_2}(p) d\operatorname{vol}_{\hat{\Sigma}_2} \leq \int_D \kappa_{\hat{\Sigma}_2}(p) d\operatorname{vol}_{\hat{\Sigma}_2} < 4\pi(1-g)$$

On the other hand, since $\partial\hat{\Sigma}_2$ is geodesic in $(\hat{M}_g, \hat{\rho})$ and hence in $\hat{\Sigma}_2$, the Gauß-Bonnet theorem implies that

$$\int_{\hat{\Sigma}_2} \kappa_{\hat{\Sigma}_2}(p) d\operatorname{vol}_{\hat{\Sigma}_2} = 2\pi\chi(\hat{\Sigma}_2) = 2\pi\chi(\hat{\Sigma}_1) = 2\pi\chi(\Sigma_1) = 4\pi(1-g)$$

This yields a contradiction and proves that $\hat{\Sigma}_2 \cap U$ does not contain meridional disks. In particular, $\hat{\Sigma}_2 \cap U$ consists of ∂-parallel annuli and inessential disks. This proves that $\hat{\Sigma}_2$ can be isotoped, by an isotopy supported by a regular neighborhood of U in $\hat{\Sigma}_2$ to a surface disjoint of U.

We can proceed in this way with all the components of $M^{<\epsilon'}$ whose soul has at most length ϵ. At the end of the day we obtain that $\hat{\Sigma}_2$ is properly isotopic to a surface $\hat{\Sigma}_3$ with $\partial\hat{\Sigma}_3 = \partial\hat{\Sigma}_2 = \partial A$ and $\hat{\Sigma}_3 \cap M^{<\epsilon} = \emptyset$. We consider the surface S obtained by gluing the annulus A to $\hat{\Sigma}_3$. By construction the surface S contains γ, is contained in $M^{\geq\epsilon}$ and isotopic to S in M; hence S is the desired embedded level surface. This concludes the proof of Proposition 3.1. □

4. Finding knotted geodesics

In this section we use the facts proved above to find knotted geodesics in hyperbolic manifolds homeomorphic to M_g. In order to do so we repeat essentially word-by-word a well-known construction due to Kerchhoff and Thurston [KT90, Section 3].

Let $\eta \subset M_g$ be an homotopically essential simple closed curve which is isotopic to a non-separating curve in ∂M_g. Consider the manifold $\bar{N}_g = \bar{M}_g \setminus \mathcal{N}(\eta)$ obtained by removing a regular neighborhood of η from $\bar{M}_g$. Observe that since η is unknotted, it is contained in an embedded level surface $\Sigma \subset M_g$ and in a properly embedded essential annulus $(A, \partial A) \subset (\bar{M}_g, \partial \bar{M}_g)$. Up to isotopy we may assume that $\mathcal{N}(\eta)$ intersects both Σ and A in annuli C_Σ, C_A with $C_\Sigma \cap C_A = \eta$. In particular, $A \setminus C_A$ consists of two properly embedded annuli A^+ and A^- in $\bar{N}_\eta$ going from $\partial\mathcal{N}(\eta)$ to $\partial \bar{N}_\eta \setminus \partial\mathcal{N}(\eta)$; one to the top and the other to the bottom boundary. The two curves $\Sigma \cap C_\Sigma$ are homotopic in $\bar{N}_\eta$ in many ways. Let W be a properly immersed annulus in $\bar{N}_\eta$ with boundary ∂C_Σ and such that every vertical segment, i.e. from boundary to boundary, has algebraic intersection number 1 with A^- and 2 with A^+. Let now Σ' be the surface obtained by gluing the annulus W to $\Sigma \setminus C_\Sigma$. We have maps of both pieces of Σ' into $\bar{N}_\eta$ which coincide on the gluing curves. In particular we obtain a continuous map of Σ' into $\bar{N}_\eta$ which can be homotoped to an immersion $\iota : \Sigma' \to \bar{N}_\eta$ with $\iota(\Sigma') \cap \partial \bar{N}_\eta = \emptyset$. Choose also once and for ever an homeomorphism

$$\phi : \Sigma \to \Sigma'$$

mapping $\Sigma \setminus C_\Sigma$ to itself.

If $\gamma \subset \Sigma$ is a closed curve which intersects η exactly once, then $\iota(\phi(\gamma))$ has algebraic intersection number 1 with A^- and 2 with A^+. In particular, $\iota(\phi(\gamma))$ cannot be homotoped within $\bar{N}_\eta$ into $\partial \bar{N}_\eta$.

We choose a basis for the homology of $\partial\mathcal{N}(\eta)$ as follows: μ is the meridian and λ is one of the components of ∂C_Σ. It is well-known that for every $n \in \mathbb{Z}$ the manifold $\bar{N}_g^{\mu+n\lambda}$ obtained from $\bar{N}_g$ by Dehn-filling the curve $\mu + n\lambda$ is homeomorphic to $\bar{M}_g$; fix a homeomorphism

$$\psi_n : \bar{N}_g^{\mu+n\lambda} \to \bar{M}_g$$

It is not difficult to see that the map

$$\Sigma \to \Sigma' \to \bar{N}_\eta \to \bar{N}_g^{\mu+n\lambda} \simeq \bar{M}_g$$

obtained by composing ϕ, ι and the inclusion of $\bar{N}_\eta$ into $\bar{N}_g^{\mu+n\lambda}$ is a homotopy equivalence. In particular, the so obtained immersion of Σ in $\bar{M}_g$ is homotopic to an embedding into $\partial \bar{M}_g$.

This concludes the topological part of the construction. We construct now the desired hyperbolic metrics. It follows from Thurston's hyperbolization theorem that the interior N_g of $\bar{N}_g$ admits a geometrically finite hyperbolic metric ρ_∞. We identify $\bar{N}_g$ with a standard compact core, i.e. a submanifold of N_g whose complement has a product structure. The Dehn-filling theorem implies then that there is a sequence of quasi-fuchsian metrics ρ_n on M_g such that for every δ there is n_δ such that for all $n \geq n_\delta$ there is a $(1+\delta)$-bi-Lipschitz embedding

$$\psi'_n : (\bar{N}_g, \rho_\infty) \to (M_g, \rho_n)$$

in the isotopy class of the restriction of the homeomorphism ψ_n above to $\bar{N}_n$. This property is stable under small perturbations of the metrics ρ_n. In other words, we may assume by Proposition 2.1 that the metrics ρ_n have the property that every geodesic in (M_g, ρ_n) which is homotopic to a simple closed curve in $\bar{M}_g$ is simple itself.

Finally remark that the component of $\phi'_n(\bar{N}_g)$ in M_g homeomorphic to a solid torus contains the ρ_n-geodesic η_n homotopic to η. Observe that the length $l_{\rho_n}(\eta_n)$ tends to 0.

To conclude the construction choose once and for ever a simple closed curve γ in Σ which intersects η exactly once. Then consider the curve $\iota(\psi(\gamma)) \subset \bar{N}_\eta$ and the corresponding geodesic γ_∞ in (N_η, ρ_∞). Up to enlarging the compact core $\bar{N}_g$ of N_g we may assume that γ_∞ is contained in the domain the maps ϕ'_n. Let γ_n be the geodesic in (M_g, ρ_n) homotopic to $\phi'_n(\gamma_\infty)$. Since the maps ϕ'_n are closer and closer to isometric embeddings, we deduce from the stability of geodesics that for large n:

a) $\phi'_n(\gamma)$ and γ_n are homotopic within $\phi'_n(\bar{N}_\eta)$.

By a) and the topological part of the construction we have then:

b) The geodesic γ_n is homotopic in $\bar{M}_g$ to a simple curve in $\partial \bar{M}_g$.
c) The geodesic γ_n cannot be homotoped into an embedded level surface within $M_g \setminus \eta_n$.

The choice of the metrics ρ_n together with b) implies finally that:

d) The geodesic γ_n is simple.

Having b), c) and d), it follows directly from Proposition 3.1 that for sufficiently large n the geodesic γ_n is knotted in (M_g, ρ_n). This concludes the proof of Theorem 1.1. □

Remark. It should be remarked that the fact that the fact that the geodesic γ_n is knotted for large n can be deduced also from Thurston's Dehn-filling theorem or, as pointed to us by Cameron Gordon, using purely topological arguments.

We conclude this section and the paper with some questions:

Question 3. Assume that ρ is a hyperbolic metric on M_g with the property that every geodesic which is homotopic to a simple curve in $\partial\bar{M}_g$ is itself simple and unknotted. Is ρ fuchsian?

Question 4. Fix $L > 1$ and a hyperbolic metric ρ on M_g. We say that a curve $\gamma \subset M_g$ is L-truly knotted if every simple curve γ' homotopic to γ in M_g by a homotopy whose tracks have at most length L is knotted. Is it true that for every L, l and g there is ϵ such that if γ is primitive geodesic with length $\epsilon < l_\rho(\gamma) < l$ which is linked with the set of geodesics shorter than ϵ, then γ is L-truly knotted?

The four questions above have been formulated in such a way that the authors think that the answer should be yes.

References

[Bon86] F. Bonahon, *Bouts des variétés hyperboliques de dimension* 3, Ann. of Math. (2) 124 (1986), no. 1, 71–158

[BB04] J. Brock and K. Bromberg, *On the density of geometrically finite Kleinian groups*, Acta Math. 192, (2004), 33-93.

[CG06] D. Calegari and D. Gabai, *Shrinkwrapping and the taming of hyperbolic 3-manifolds*, J. Amer. Math. Soc. 19 (2006), no. 2, 385–446

[Can93] R. D. Canary, *Ends of hyperbolic* 3*-manifolds*, J. Amer. Math. Soc. 6 (1993), no. 1, 1–35.

[Can96] R. D. Canary, *A covering theorem for hyperbolic 3-manifolds and its applications*, *Topology* **35** (1996).

[KT90] S. Kerckhoff and W. Thurston, *Noncontinuity of the action of the modular group at Bers' boundary of Teichmfller space*, Invent. Math. 100 (1990), no. 1, 25–47.

[Mar74] A. Marden, *The geometry of finitely generated kleinian groups*, Ann. of Math. (2) 99 (1974), 383–462.

[Ota95] J.-P. Otal, *Sur le nouage des géodésiques dans les variétés hyperboliques*, C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), no. 7, 847–852.

Ken Bromberg, Department of Mathematics, University of Utah

Juan Souto, Department of Mathematics, University of Michigan