GEOMETRIC LIMITS OF KNOT COMPLEMENTS

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Abstract. We prove that any complete hyperbolic 3–manifold with finitely generated fundamental group, with a single topological end, and which embeds into $\mathbb{S}^3$ is the geometric limit of a sequence of hyperbolic knot complements in $\mathbb{S}^3$. In particular, we derive the existence of hyperbolic knot complements which contain balls of arbitrarily large radius. We also show that a complete hyperbolic 3–manifold with two convex cocompact ends cannot be a geometric limit of knot complements in $\mathbb{S}^3$.

1. Introduction

In this paper we study geometric properties of hyperbolic knot complements. Unless explicitly stated, a *knot complement* is understood to be the complement of a non-trivial knot in the 3–sphere $\mathbb{S}^3$. A non-trivial knot $K \subset \mathbb{S}^3$ is *hyperbolic* if its complement M_K admits a complete hyperbolic metric. Many knots are hyperbolic; in fact, among the 1.701.936 prime knots with 16 or fewer crossings, all but 32 are hyperbolic [23]. In general, Thurston proved that every knot $K \subset \mathbb{S}^3$ which is neither a torus nor a satellite knot is hyperbolic.

By the Mostow–Prasad theorem, the hyperbolic metric on a hyperbolic knot complement is unique. In particular it follows from work of Gordon and Luecke [20] that the hyperbolic metric on the complement of a hyperbolic knot in $\mathbb{S}^3$ is a complete invariant of the knot. However, it remains a difficult problem to use geometric arguments in the classification of knots. One central difficulty is in distinguishing hyperbolic knot complements from general finite volume hyperbolic manifolds.

Question 1. *Which properties of the hyperbolic metric distinguish knot complements from other hyperbolic 3–manifolds?*

There are some results suggesting that within the world of finite volume hyperbolic 3–manifolds, hyperbolic knot complements form a very special class. For instance, Reid [40] proved that the complement of the figure-8 knot is the only hyperbolic arithmetic knot complement. Similarly, every hyperbolic knot complement admits infinitely many non-isometric finite degree covers, but at most three of them are knot complements [19].

In this paper, we investigate the question above by studying geometric limits of hyperbolic knot complements. We find that many manifolds arise as these geometric limits, which implies that knot complements can admit metrics with unusual and perhaps unexpected geometric properties. We also investigate manifolds that cannot arise in this manner.

Recall that a hyperbolic manifold M is a geometric limit of a sequence of hyperbolic manifolds M_i, if there are basepoints $p \in M$ and $p_i \in M_i$ such that larger and larger balls about p in M have, as i tends to ∞, better and better almost isometric embeddings into balls about p_i in M_i. Thus if a manifold is a geometric limit of knot complements, then there exist hyperbolic knots whose geometric properties are very close to those of the limiting manifold.

Our first result asserts that surprisingly many hyperbolic 3–manifolds arise as geometric limits of knot complements. Specifically, we prove:

Theorem 1.1. *Let N be a complete hyperbolic 3–manifold with finitely generated fundamental group and a single topological end. If N is homeomorphic to a submanifold of $\mathbb{S}^3$, then N is a geometric limit of a sequence of hyperbolic knot complements in $\mathbb{S}^3$.*

Before moving on, observe that applying Theorem 1.1 to $N = \mathbb{H}^3$ we obtain:

Corollary 1.2. *For every $R > 0$ there exists a hyperbolic knot complement M_K and $x \in M_K$ with injectivity radius $\mathrm{inj}(x, M_K) > R$.*

The statement of Theorem 1.1 is not optimal. Namely, we can construct many other geometric limits of knot complements by observing that every hyperbolic 3–manifold which is a geometric limit of manifolds satisfying the conditions in Theorem 1.1 is also a geometric limit of hyperbolic knot complements. For example, this idea can be used to prove that every hyperbolic manifold N homeomorphic to the trivial interval bundle over a closed surface which has at least one degenerate end is also a limit of knot complements. Again the same reasoning shows that there are hyperbolic 3–manifolds with finitely generated fundamental group and arbitrarily many ends which are geometric limits of knot complements. We discuss these in section 7.

We then turn to the converse problem, and ask which hyperbolic 3–manifolds cannot be the geometric limit of hyperbolic knot complements. Besides obvious topological obstructions, we find there are some less obvious geometric obstructions. In particular, we find:

Theorem 1.3. *Let M be a hyperbolic 3–manifold. If the manifold M has at least two convex cocompact ends, then M is not the geometric limit of any sequence of hyperbolic knot complements in $\mathbb{S}^3$.*

Recall that if M has two convex cocompact ends, then every regular neigbhorhood of the convex core $CC(M)$ in M has at least two compact boundary components, where $CC(M)$ is the smallest closed totally convex subset of M.

Organization and overview. The following gives a road map for this paper. In section 2, we remind the reader of basic and not so basic facts on hyperbolic 3–manifolds and their deformation theory.

The main result of section 4, namely Proposition 4.1, is that any manifold satisfying the hypotheses of Theorem 1.1 is the geometric limit of hyperbolic manifolds which, besides satisfying the same hypotheses, have the property that their single topological end is degenerate. We also prove Proposition 4.4: that if M is a hyperbolic 3–manifold with a degenerate end, and if $\{M_{\gamma_i}\}$ is any sequence of hyperbolic 3–manifolds homeomorphic to the interior of M with parabolics γ_i converging to the ending lamination of M, then M is a geometric limit of $\{M_{\gamma_i}\}$.

Thus at this step, it remains to show that there are geometric limits of knot complements homeomorphic to M and where certain curves are parabolic. This is done in Proposition 5.5, in section 5. We start constructing the desired manifold as a limit of hyperbolic link complements. In order to go from link complements to knot complements, we perform hyperbolic Dehn filling on certain components. The links are chosen in such a way that the slopes of Dehn filling get long. We derive from a version of Hodgson and Kerckhoff's quantified Dehn filling theorem that the filled manifolds have the same geometric limit.

Theorem 1.1 follows easily from Proposition 4.1, Proposition 5.5 and Proposition 4.4. For the convenience of the reader, we invert the logical order, deriving Theorem 1.1 from these results in section 3.

In section 6 we prove Theorem 1.3. The proof is based on the fact that the closure of the complement of the convex–hull of a subset of $\partial_\infty \mathbb{H}^3$ is a locally CAT(-1)–manifold. This fact allows us to adapt an argument due to Marc Lackenby [28].

Finally, in section 7 we discuss briefly Corollary 1.2 and some other consequences of Theorem 1.1 and Theorem 1.3. We also propose some questions.

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2. Hyperbolic 3–manifolds and geometric limits

Throughout this note we will only consider knot complements in $\mathbb{S}^3$.

Under a *hyperbolic 3–manifold* we understand a connected and orientable, complete Riemannian manifold M of constant curvature -1. Equivalently, M is isometric to $\mathbb{H}^3/\Gamma$, where Γ is a discrete, torsion free subgroup of the group of isometries of hyperbolic 3–space $\mathbb{H}^3$.

By Mostow–Prasad rigidity, any two homotopy equivalent, finite volume hyperbolic 3–manifolds are isometric. On the other hand, if the volume of M is infinite, then there is a rich and well–developed deformation theory.

In this section, we review a few aspects of this deformation theory. We refer to [33] for details of facts which are, by now, classical, and to [31] for more recent results.

2.1. **Tameness.**

Given a hyperbolic 3–manifold M with finitely generated fundamental group, there is an associated compact 3–manifold with boundary $\overline{M}$, the *manifold compactification of* M, such that M is homeomorphic to the interior of $\overline{M}$. This is the result of the *tameness theorem*, proved by Agol [1], and Calegari and Gabai [13]. It is well known that $\overline{M}$ is unique up to homeomorphism, that every component of $\partial\overline{M}$ has at least genus 1 and that $\chi(\partial\overline{M}) = 0$ if and only if M has finite volume.

Recall that a non-trivial element $\gamma \in \pi_1(M)$ is *parabolic* if it is freely homotopic to curves in M of arbitrarily short length. A subsurface $P \subset \partial\overline{M}$ is the *parabolic locus of* M provided it satisfies:

(1) P consists of all toroidal components of $\partial\overline{M}$ and a collection of homotopically essential, disjoint, non-parallel annuli.
(2) If γ is a homotopically non-trivial curve in P, then any element in $\pi_1(M)$ represented by γ is parabolic.
(3) If $\gamma \subset M$ represents a parabolic element and γ_i is a sequence of curves freely homotopic to γ and with $\ell_M(\gamma_i) \to 0$, then for every regular neighborhood U of P in $\overline{M}$, there is i_U with $\gamma_i \subset U$ for all $i \geq i_U$.

If $\overline{M}$ is the manifold compactification of the hyperbolic 3–manifold M, then any two incompressible subsurfaces $P_1, P_2 \subset \partial\overline{M}$ satisfying conditions (1)–(3) are isotopic to each other within $\partial\overline{M}$.

In other words, given a hyperbolic 3–manifold M with finitely generated fundamental group, $(\overline{M}, P)$ is uniquely determined up to homeomorphism of pairs. In particular, given a hyperbolic 3–manifold M, we will abuse terminology only slightly when we use the definite article in the sentence: *$\overline{M}$ is the manifold compactification of M and $P \subset \partial\overline{M}$ is the parabolic locus.*

Definition. *Let M be a hyperbolic 3–manifold with finitely generated fundamental group; let $\overline{M}$ be its manifold compactification and $P \subset \partial\overline{M}$ its parabolic locus. The components of $\partial\overline{M} \smallsetminus P$ are the* geometric ends *of M.*

Observe that whenever the parabolic locus of M is empty, then geometric and topological ends coincide.

2.2. **Convex core.** Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3–manifold and recall that we can identify the boundary at infinity of $\mathbb{H}^3$ with the complex projective line $\partial_\infty\mathbb{H}^3 = \mathbb{C}P^1$. Denote by Λ_Γ the *limit set* of Γ and let $CH(\Lambda_\Gamma) \subset \mathbb{H}^3$ be its convex hull. The quotient $CC(M) = CH(\Lambda_\Gamma)/\Gamma$ is the *convex core of M*. Equivalently, $CC(M)$ is the smallest closed, totally convex subset of M, i.e. the smallest closed subset which contains all closed geodesics. The manifold M is *convex cocompact* if its convex core is compact. Observe that if M is convex cocompact then its parabolic locus is empty.

More generally, a hyperbolic 3–manifold M with finitely generated fundamental group is *geometrically finite* if its convex core has finite volume $\mathrm{vol}(CC(M)) < \infty$. One important class of geometrically finite manifolds are the maximal cusps. A hyperbolic 3–manifold M is said to be a *maximal cusp* if $(\overline{M}, P)$ is such that every component of $\partial\overline{M} \smallsetminus P$ is homeomorphic to a 3–punctured sphere. Note that if M is a maximal cusp, then its convex core $CC(M)$ must have totally geodesic boundary, with each boundary component isometric to the unique hyperbolic structure on a 3–punctured sphere.

2.3. **Degenerate ends.** Throughout the paper, we will be mostly interested in hyperbolic 3–manifolds which have a single geometric end. To avoid unnecessary notation and terminology, the following definition is tailored to the particular situation of this paper:

Definition. *Let M be a hyperbolic 3–manifold with finitely generated fundamental group and a single geometric end. The geometric end of M is said to be* degenerate *if M does not contain any proper totally convex subset, or equivalently, if $M = CC(M)$.*

Intuitively, manifolds with degenerate ends are more complicated than convex cocompact manifolds. However, the former are much more rigid than the latter; this is going to be crucial in this paper. We find the first manifestation of the rigidity of degenerate ends in the following result, due to Thurston [43] and Canary [16], which we state in the situation we are interested in:

Covering theorem. *Assume that M is a hyperbolic 3–manifold with finitely generated fundamental group and with a unique geometric end. Assume that the geometric end of M is degenerate and that $\pi\colon M \to N$ is a Riemannian cover where N has infinite volume. Then the cover π is finite–to–one.*

Suppose now that M has a single geometric end which is degenerate. Then we have an associated *ending lamination* λ, whose definition we recall.

We may identify $\overline{M}$ with a submanifold of M such that the complement $M \setminus \overline{M}$ has a product structure. Observe that any curve $\gamma \subset M \setminus \overline{M}$ corresponds to a unique homotopy class of curves in $\partial\overline{M}$. Now if the end of M is degenerate, then there is a sequence of closed geodesics γ_i with bounded length, exiting the end, represented by simple closed curves in $\partial\overline{M} \setminus P$. A subsequence of the curves γ_i converges in $\mathcal{PML}(\partial\overline{M} \setminus P)$ to a filling measured lamination whose support, λ, does not depend on the sequence γ_i. The lamination λ is the *ending lamination* of M. See Thurston [43], Bonahon [6], or Canary [15] for basic properties of the ending lamination.

The ending lamination theorem, due to Minsky [35] and Brock, Canary, and Minsky [8], asserts that every hyperbolic 3–manifold is determined up to isometry by its topology and end invariants. We state this theorem again in the particular case we are interested in.

Ending lamination theorem. *Let $\overline{M}$ be a compact 3–manifold with boundary, and $P \subset \partial\overline{M}$ a possibly empty subsurface with $\partial\overline{M} \setminus P$ connected. Assume that M and M' are hyperbolic 3–manifolds homeomorphic to the interior of $\overline{M}$ with parabolic locus P. Assume also that the geometric ends of M and M' are degenerate and have the same ending lamination. Then M and M' are isometric.*

2.4. **Doubly incompressible laminations.** Before going any further recall the following definition:

Definition. *Let $\overline{M}$ be a compact 3–manifold whose interior admits a hyperbolic structure. Let $P \subset \partial\overline{M}$ be a possibly empty subsurface consisting of all toroidal components of $\partial\overline{M}$ and a collection of disjoint,* *non-parallel, homotopically essential annuli. We say that* $(\overline{M}, P)$ *is* acylindrical *if*

(1) *every component of* $\partial\overline{M} \setminus P$ *is incompressible in* $\overline{M}$*,*
(2) *every properly embedded annulus* $(A, \partial A) \subset (\overline{M}, \partial\overline{M} \setminus P)$ *is isotopic relative to the boundary to an annulus contained in the boundary* $\partial\overline{M}$*, and*
(3) *there is no properly embedded Möbius band with boundary in* $\partial\overline{M} \setminus P$*.*

A generalization of this notion is due to Kim, Lecuire, and Ohshika [25], who defined a measured lamination $\alpha \in \mathcal{PML}(\partial\overline{M} \setminus P)$ to be *doubly incompressible*, if there is $\eta > 0$ such that $i(\alpha, \partial E) > \eta$ for E any essential annulus, Möbius band, or disc. Observe that if $\gamma \subset \partial\overline{M} \setminus P$ is doubly incompressible when considered as an element in $\mathcal{PML}(\partial\overline{M} \setminus P)$, then $(\overline{M}, \mathcal{N}(\gamma) \cup P)$ is acylindrical; here $\mathcal{N}(\gamma)$ is a regular neighborhood of γ in $\partial\overline{M}$.

The following result, which we state only in the setting we are interested in, asserts essentially that ending laminations are doubly incompressible.

Theorem 2.1 (Canary)**.** *Assume that a hyperbolic 3–manifold M with finitely generated fundamental group has a single geometric end and that this end is degenerate with ending lamination λ. If α is any measured lamination with support λ, then α is doubly incompressible.*

Moreover, if γ_i is any sequence of simple closed curves in $\partial\overline{M} \setminus P$ converging in $\mathcal{PML}(\partial\overline{M} \setminus P)$ to α, then $(\overline{M}, \mathcal{N}(\gamma_i) \cup P)$ is acylindrical for all sufficiently large i. Here, $\mathcal{N}(\gamma_i)$ is a regular neighborhood of γ_i in $\partial\overline{M}$.

Remark. If $\partial\overline{M} \setminus P$ is incompressible, then Theorem 2.1 follows from the work of Thurston [43]. Canary proved the first claim of Theorem 2.1 if $\partial\overline{M}$ is compressible and the parabolic locus is empty. In fact, in this case he showed that λ belongs to the so–called Masur domain, a subset of the set of doubly incompressible laminations [15]. Since the Masur domain is open [32, 37], the second claim follows. Canary's argument goes through without any problems in the presence of parabolics.

2.5. **Pleated surfaces and non-realized laminations.** Assume that M is a hyperbolic 3–manifold with manifold compactification $\overline{M}$ and parabolic locus P, and let $\overline{S}$ be a compact surface with interior $S = \overline{S} \setminus \partial\overline{S}$. A *pleated surface* is a map

$$\phi\colon\, S \to M$$

such that there is a finite volume hyperbolic metric σ on S such that the following holds:

(1) The image under ϕ of a boundary parallel curve in S represents a parabolic element in $\pi_1(M)$.
(2) The map $\phi\colon (S,\sigma) \to M$ preserves the lengths of paths.
(3) Every point in x is contained in an arc κ, geodesic with respect to σ, such that the restriction of ϕ to κ is an isometric embedding.

See for instance [14] for basic properties of pleated surfaces.

If we fix a homotopy class of maps $[S \to M]$ then we say that a lamination $\lambda \subset S$ is *realized* in M by a pleated surface $\phi\colon S \to M$ in the correct homotopy class if:

(4) The restriction of ϕ to each leaf of λ is an isometric immersion.

The following is a very technical result which essentially asserts that if a filling lamination is not realized, then it is the ending lamination.

Proposition 2.2 (Non-realized implies ending lamination)**.** *Assume that $\overline{M}$ is a compact 3–manifold and $P \subset \partial\overline{M}$ is a subsurface such that there is some hyperbolic manifold with manifold compactification $\overline{M}$ and parabolic locus P. Assume moreover that $S = \partial\overline{M} \smallsetminus P$ is connected.*

Let N be a hyperbolic 3–manifold and $f\colon \overline{M} \to N$ a homotopy equivalence mapping each curve in P to a parabolic element in $\pi_1(N)$. Assume that that there is a filling doubly incompressible lamination $\lambda \subset S = \partial\overline{M} \smallsetminus P$ which is not realized by any pleated surface homotopic to the restriction of f to S. Then the following holds:

- *$\overline{M}$ is the manifold compactification of N and P its parabolic locus,*
- *the only geometric end of N is degenerate, and has ending lamination λ.*

Proposition 2.2 is due to Thurston [43] if $\pi_1(M)$ does not split as a free product. If $\pi_1(M)$ splits as a free product but is not free, then Proposition 2.2 is due to Kleineidam and Souto [27]. The case that $\pi_1(M)$ is free has been treated by Namazi and Souto [36] and Ohshika.

Remark. Recall that the manifold compactification and the parabolic locus of a hyperbolic 3–manifold are only determined up to homeomorphism. Therefore, to be slightly more precise, the statement of Proposition 2.2 should be that there is a homeomorphism $F\colon N \to \overline{M} \smallsetminus \partial\overline{M}$ that satisfies properties of the Proposition.

2.6. **Geometric limits.** A sequence (M_i, p_i) of pointed hyperbolic 3–manifolds *converges geometrically*, or equivalently, converges in the *pointed Gromov–Hausdorff topology*, to a pointed manifold (M, p) if for every $\epsilon > 0$ and every compact set $K \subset M$ with $p \in K$, there exists $i_{\epsilon,K}$ such that for all $i \geq i_{\epsilon,K}$ there is a $(1+\epsilon)$–bilipschitz embedding

$$f_i\colon\ (K, p) \hookrightarrow (M_i, p_i).$$

Observe that if (M_i, p_i) converges geometrically to (M, p) and $q \in M$ is a second base point, then there are points $q_i \in M_i$ such that (M_i, q_i) converges geometrically to (M, q). On the other hand, it is not difficult to construct sequences of manifolds (M_i) and two sequences of base points $p_i, q_i \in M_i$ such that the (M_i, p_i) and (M_i, q_i) converge geometrically to non-homeomorphic limits. This last remark explains the undetermined article in the following definition:

Definition. *A (connected) hyperbolic 3–manifold M is a* geometric limit *of a sequence of hyperbolic manifolds (M_i) if there are base points $p \in M$ and $p_i \in M_i$ such that (M_i, p_i) converges geometrically to (M, p).*

The following is an obvious but extremely useful lemma.

Lemma 2.3. *Assume that a manifold M is a geometric limit of a sequence of hyperbolic 3–manifolds M_i and that each M_i is a geometric limit of a sequence of hyperbolic knot complements. Then M is also a geometric limit of a sequence of hyperbolic knot complements.* □

2.7. **Algebraic limits.** We recall now a second concept of convergence.

Let M be a hyperbolic 3–manifold. Let $AH(M)$ denote the space of conjugacy classes of discrete faithful representations of $\pi_1(M)$ into $\mathrm{PSL}_2\,\mathbb{C}$. Give $AH(M)$ the quotient topology induced by the compact–open topology on the space of discrete faithful representations. Convergence of representations in $AH(M)$ is called *algebraic convergence*. If $\{\rho_n\}$ converges algebraically to ρ, then the manifold $\mathbb{H}^3/\rho(\pi_1(M))$ is the *algebraic limit* of the 3–manifolds $\mathbb{H}^3/\rho_n(\pi_1(M))$. See [34] or [5] for a description of algebraic convergence from the point of view of the involved hyperbolic manifolds.

Density theorem. *Assume that $\mathbb{Z}^2 \not\leq \pi_1(M)$. Then the set of those $\rho \in AH(M)$ such that the associated manifold $\mathbb{H}^3/\rho(\pi_1(M))$ is convex cocompact is dense in $AH(M)$.*

The density theorem follows from the proof of the tameness conjecture by Agol, Calegari and Gabai, the proof of the ending lamination theorem by Brock, Canary, and Minsky, and the work of Kim, Lecuire and Ohshika. The final step needs the more general form of Proposition 2.2 above due to Namazi, Ohshika and Souto. See [31] for the relation between all these results.

Before moving on we state a very weak form of the continuity of Thurston's length function (see Brock [10] for an extensive discussion of the length function).

Proposition 2.4. *Assume that $\overline{M}$ is a compact 3–manifold and $P \subset \partial\overline{M}$ is a subsurface such that there is some hyperbolic manifold with manifold compactification $\overline{M}$ and parabolic locus P. Assume moreover that $S = \partial\overline{M} \setminus P$ is connected and let $\lambda \subset S$ be a filling doubly incompressible lamination.*

Let $\{M_i\}$ be a sequence of hyperbolic 3–manifolds in $AH(\overline{M})$ such that every curve in P is parabolic in M_i for all i and assume that the sequence $\{M_i\}$ converges algebraically to some manifold M in which λ is realized. Then we have

$$\lim_i l_{M_i}(\gamma_i) = \infty$$

for every sequence of simple closed curves $\{\gamma_i\}$ in S converging to λ in $\mathcal{PML}(S)$. Here $l_{M_i}(\gamma_i)$ is the infimum of the lengths in M_i of all curves freely homotopic to γ_i.

2.8. **Strong limits.** Algebraic limits are not necessarily the same as geometric limits, see for example [43, Chapter 9].

Definition. *Suppose $\{M_n\}$ is a sequence of hyperbolic 3–manifolds that converges algebraically and geometrically to the manifold M. Then we say $\{M_n\}$* converges strongly *to M.*

In the course of the proof of Theorem 1.1 we will need to be able to deduce that some algebraically convergent sequences also converge strongly. Our main tool is the following result which follows easily from the Canary–Thurston covering theorem.

Proposition 2.5. *Assume that a sequence $\{\rho_i\}$ in $AH(M)$ converges algebraically to some $\rho \in AH(M)$ and suppose that $CC(\mathbb{H}^3/\rho(\pi_1(M))) = \mathbb{H}^3/\rho(\pi_1(M))$. Then the sequence $\{\rho_i\}$ converges strongly to ρ.*

See [2, 3, 26] for related, much more powerful results.

Proposition 2.5 does not apply if $\mathbb{H}^3/\rho(\pi_1(M))$ is a maximal cusp. However, in this case, we have the following weaker result which follows directly from [4, Prop. 3.2]:

Proposition 2.6. *Assume that a sequence $\{\rho_i\}$ in $AH(M)$ converges algebraically to some $\rho \in AH(M)$ and suppose that $\mathbb{H}^3/\rho(\pi_1(M))$ is a maximal cusp. Then, up to passing to a subsequence, the hyperbolic manifolds $\mathbb{H}^3/\rho_i(\pi_1(M))$ converge geometrically to some N such that there is an isometric embedding $CC(\mathbb{H}^3/\rho(\pi_1(M))) \hookrightarrow N$.*

Before going on to more interesting topics, we recall the following strong density theorem:

Strong density theorem. *Assume that* $\mathbb{Z}^2 \not\leq \pi_1(M)$. *Then any* $\rho \in AH(M)$ *is the strong limit of a sequence* $\{\rho_i\}$ *in* $AH(M)$ *such that for each* i *the associated manifold* $\mathbb{H}^3/\rho_i(\pi_1(M))$ *is convex cocompact.*

The strong density theorem follows directly from the Density theorem and work of Brock and Souto [9, Theorem 1.4].

3. Proof of the main theorem

In this section we reduce the proof of Theorem 1.1 to various results obtained in the two subsequent sections.

Theorem 1.1. *Let* N *be a complete hyperbolic 3–manifold with finitely generated fundamental group and a single topological end. If* N *is homeomorphic to a submanifold of* $\mathbb{S}^3$, *then* N *is a geometric limit of a sequence of hyperbolic knot complements.*

Proof. Assume that N is as in the statement of Theorem 1.1 and let $\overline{N}$ be its manifold compactification.

First, we note that we need not consider trivial cases. If N has abelian fundamental group, then using for instance Klein combination, one obtains a sequence $\{N_i\}$ of say genus 2 handlebodies converging geometrically to N. By Lemma 2.3, it suffices to prove that each one of the N_i is a geometric limit of hyperbolic knot complements. From now on we assume that $\pi_1(N)$ is not abelian. Similarly, if $\partial\overline{N}$ is a torus, then the fact that N embeds into $\mathbb{S}^3$ implies that N is itself a knot complement. So we may assume that $\partial\overline{N}$ is a connected surface of genus at least two. In other words, we may assume than N has infinite volume.

In section 4, we will prove:

Proposition 4.1. *Let* N *be a complete, infinite volume, hyperbolic 3–manifold with non-abelian, finitely generated fundamental group and a single topological end. Assume that* N *is homeomorphic to a submanifold of* $\mathbb{S}^3$. *Then* N *is a geometric limit of a sequence* $\{N_i\}$ *of hyperbolic 3–manifolds such that for all* i *the following holds:*

- N_i *has finitely generated fundamental group, no parabolics and a single end,*
- *the end of* N_i *is degenerate, and*
- N_i *admits an embedding into* $\mathbb{S}^3$.

Combining Proposition 4.1 and Lemma 2.3, we see that it suffices to prove Theorem 1.1 for those manifolds N which, besides satisfying the assumptions in the theorem, have empty parabolic locus and degenerate end. Assume that we have such a manifold N and let $\lambda \subset \partial\overline{N}$ be the ending lamination of its unique end. Denote also by λ some measured lamination supported by λ and choose once and forever a sequence $\{\gamma_i\}$ of simple closed curves in $\partial\overline{N}$ converging to λ in $\mathcal{PML}(\partial\overline{N})$. By Theorem 2.1, for all sufficiently large i, the pair $(\overline{N}, \mathcal{N}(\gamma_i))$ is acylindrical. Here, $\mathcal{N}(\gamma_i)$ is a regular neighborhood of γ_i in $\partial\overline{N}$. In section 5 we will show:

Proposition 5.5. *Let $\overline{N}$ be a compact irreducible and atoroidal submanifold of $\mathbb{S}^3$ with connected boundary of genus at least two, and let $\eta \subset \partial\overline{N}$ be a simple closed curve with $(\partial\overline{N}, \mathcal{N}(\eta))$ acylindrical and $\partial\overline{N} \setminus \eta$ connected. Then there is a sequence of hyperbolic knot complements $\{M_{K_i}\}$ converging geometrically to a hyperbolic manifold N_η homeomorphic to the interior of $\overline{N}$, such that η represents a parabolic element in N_η.*

With γ_i as above and i sufficiently large, let N_{γ_i} be the hyperbolic 3–manifold provided by Proposition 5.5. The content of Proposition 4.4, proved in section 4, is that the manifolds N_{γ_i} converge geometrically to N.

Proposition 4.4. *Suppose M is a hyperbolic 3–manifold with empty parabolic locus and a single end. Assume that the end of M is degenerate with ending lamination λ and suppose $\{\gamma_n\}$ is a sequence of simple closed curves in $\partial\overline{M}$ converging in $\mathcal{PML}(\partial\overline{M})$ to a projective measured lamination supported by λ. If $\{M_n\}$ is any sequence of hyperbolic 3–manifolds homeomorphic to M, such that γ_n is parabolic in M_n for all n, then M is a geometric limit of the sequence $\{M_n\}$.*

By Proposition 5.5, each of the N_{γ_i} is a limit of hyperbolic knot complements and by Proposition 4.4 the N_{γ_i} converge geometrically to N. Lemma 2.3 concludes the proof of Theorem 1.1. □

4. Three facts on geometric limits

The main goals of this section are Proposition 4.1 and Proposition 4.4. We also prove a technical result needed in the proof of Proposition 5.5 later on.

4.1. **Approximating by manifolds with degenerate ends.**

The statement of Proposition 4.1 below is summarized by saying that every hyperbolic manifold satisfying the assumptions of Theorem 1.1 can be approximated by a sequence of manifolds which, besides satisfying the same assumptions, have the property that their unique end is degenerate.

Proposition 4.1. *Let N be a complete, infinite volume, hyperbolic 3–manifold with non–abelian, finitely generated fundamental group and a single topological end. Assume that N is homeomorphic to a submanifold of $\mathbb{S}^3$. Then N is a geometric limit of a sequence $\{N_i\}$ of hyperbolic 3–manifolds such that for all i the following holds:*

- *N_i has finitely generated fundamental group, no parabolics and a single end,*
- *the end of N_i is degenerate, and*
- *N_i admits an embedding into $\mathbb{S}^3$.*

Proof. Observe that it follows from the Strong Density Theorem that the manifold N can be approximated by homeomorphic convex cocompact manifolds. In other words, we may assume that N is convex cocompact.

Below we will use an argument of Brooks [12] to prove:

Lemma 4.2. *Let N be a complete, convex cocompact hyperbolic 3–manifold with finitely generated fundamental group and a single topological end. Assume that N is homeomorphic to a submanifold of $\mathbb{S}^3$. Then N is a geometric limit of the sequence of convex cores $\{CC(N_i)\}$ associated to a sequence of hyperbolic 3–manifolds N_i such that the following holds:*

- *N_i has finitely generated fundamental group and a single end,*
- *N_i admits an embedding into $\mathbb{S}^3$, and*
- *N_i is a maximal cusp.*

Assuming Lemma 4.2 we conclude the proof of Proposition 4.1. Let N_i be one of the maximal cusps provided by Lemma 4.2. By work of Canary, Culler, Hersonsky, and Shalen [17, Lemma 15.2], N_i is the algebraic limit of a sequence $\{N^i_n\}_n$ of hyperbolic 3–manifolds homeomorphic to M, without parabolics, and whose only end is degenerate. By Proposition 2.6, there is a subsequence of $\{N^i_n\}_n$, say the whole sequence, converging geometrically to some manifold N^i_∞ into which the convex core $CC(N_i)$ of the maximal cusp N_i can be embedded. Since the convex cores $CC(N_i)$ converge geometrically to N, and each one of them is contained in a geometric limit of the sequence $\{N^i_n\}$, we deduce that there is a diagonal sequence $\{N^i_{n_i}\}$ converging geometrically to N. This diagonal sequence satisfies the requirements of the proposition. □

It remains to prove Lemma 4.2. Before launching the proof we need to remind the reader of a couple of definitions. Assume for the sake of concreteness that $N = \mathbb{H}^3/\Gamma$ is convex cocompact. Recall that the *discontinuity domain* $\Omega_\Gamma = \partial_\infty \mathbb{H}^3 \setminus \Lambda_\Gamma$ is the complement of the limit set Λ_Γ of Γ in the boundary at infinity $\partial_\infty \mathbb{H}^3$. Identifying $\partial_\infty \mathbb{H}^3 = \mathbb{C}P^1$ we obtain that the surface $\partial_\infty N = \Omega_\Gamma/\Gamma$ is not only a Riemann surface but has also a canonical projective structure. A *circle in* $\partial_\infty N$ is a topological circle which is actually round with respect to the canonical projective structure. A *circle packing* is a collection of circles in $\partial_\infty N$ bounding disjoint disks, such that every component of the complement of the union of all these disks are curvilinear triangles. We will derive Lemma 4.2 from the following fact due to Brooks [12]:

Theorem 4.3 (Brooks)**.** *For all $\epsilon > 0$ there is a convex cocompact hyperbolic 3–manifold N_ϵ homeomorphic to N such that the following holds: N_ϵ is $(1+\epsilon)$–bilipschitz to N, and $\partial_\infty N_\epsilon$ admits a circle packing such that each disk has at most diameter ϵ with respect to the canonical hyperbolic metric of $\partial_\infty N$.*

This theorem is not stated exactly in this way in [12] but it follows easily from the arguments used to prove Theorem 3 therein.

Proof of Lemma 4.2. Let N be convex cocompact, and let $\partial_\infty N$ be its conformal boundary with induced projective structure. Choose ϵ_i positive and tending to 0, and for all i let N_i^1 be the manifold provided by Theorem 4.3, and let $\mathcal{C}_i$ be the corresponding circle packing of $\partial_\infty N_i^1$. Observe that N is a geometric limit of the sequence N_i^1.

Each of the circles in the circle packing $\mathcal{C}_i$ bounds a properly embedded totally geodesic plane in N_i^1. Paint such a plane black. Any two of these black planes either coincide or are disjoint. Moreover observe that if x, y, and z are the vertices of one of the (triangular) interstices of the packing $\mathcal{C}_i$, say bounded by the circles C_1, C_2, and C_3, then the ideal triangle with vertices x, y, z is perpendicular to the corresponding black planes P_1, P_2, P_3. Paint carefully the triangles red.

Since each of the circles has at most diameter $\epsilon_i \to 0$, for all $d > 0$ there is some i_d such that for $i \geq i_d$, none of the disks or triangles enters the radius d neighborhood of the convex core $CC(N_i^1)$ of N_1^1. Cut N_i open along all these planes and triangles and let N_i^2 be the closure of the component containing $CC(N_i^1)$.

The boundary of N_i^2 consists of totally geodesic pieces, some black and some red. Let N_i^3 be the double of N_i^2 along the black boundary, which consists of subsets of the disks. Since the red ideal triangles are perpendicular to the black planes of the boundary of N_i^2, the manifold N_i^3 has totally geodesic (red) boundary. Moreover, since N_i^3 is obtained by doubling N_i^2, each of the boundary components of N_i^3 is the double of one of the red triangles in the boundary of N_i^2. Thus every boundary component of N_i^3 is a 3–punctured sphere. Hence N_i^3 is the convex core of a maximal cusp for all i.

Additionally, observe that the manifold compactification of N_i^3 is homeomorphic to two copies of $\overline{N}$ to which we have attached 1–handles, one for each circle in the circle packing. In particular, N_i^3 embeds in $\mathbb{S}^3$ and has a single end.

By construction, N is a geometric limit of the sequence N_i^1. The condition that for all d, black disks and red triangles are eventually of distance greater than d from $CC(N_i^1)$ implies that N is also a geometric limit of the sequence N_i^2. Each N_i^2 embeds into N_i^3. Thus N is a geometric limit of the manifolds N_i^3 as well. □

Remark. Observe that in the proof of Lemma 4.2 we used the fact that the manifold N had only a single end to conclude that the manifolds N_i^3 embed into $\mathbb{S}^3$. For manifolds with more ends this argument fails. However, in some cases it should be possible to by-pass this problem by choosing the circle packings $\mathcal{C}_i$ with more care than we did. For instance, if N admits a fixed–point free involution τ which preserves each boundary component, then we can consider only τ-equivariant circle packings and glue each black tile with its image under τ. This shows for instance that whenever $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ is a torsion free, cocompact Fuchsian group such that the quotient surface $\mathbb{H}^2/\Gamma$ admits an orientation preserving fixed–point free involution, then the hyperbolic 3–manifold $\mathbb{H}^3/\Gamma$ is the geometric limit of a sequence of hyperbolic 3–manifolds N_i, homeomorphic to submanifolds of $\mathbb{S}^3$, with two topological ends which are degenerate.

4.2. **Approximating by manifolds with prescribed cusps.** We show now that given a manifold M with a single degenerate end, we may obtain M as a limit of hyperbolic manifolds with few restrictions on their geometry.

Proposition 4.4. *Suppose M is a hyperbolic 3–manifold with empty parabolic locus and a single end. Assume that the end of M is degenerate with ending lamination λ and suppose $\{\gamma_n\}$ is a sequence of simple closed curves in $\partial\overline{M}$ converging in $\mathcal{PML}(\partial\overline{M})$ to a projective measured lamination supported by λ. If $\{M_n\}$ is any sequence of hyperbolic 3–manifolds homeomorphic to M, such that γ_n is parabolic in M_n for all n, then M is a geometric limit of the sequence $\{M_n\}$.*

Proof. Observe that it suffices to show that every subsequence of the sequence $\{M_n\}$ contains a further subsequence which converges geometrically to M. In particular, we may pass to subsequences as often as we wish.

Abusing notation, let λ also denote the measured lamination with support λ which is the limit of the γ_n. By Theorem 2.1, the measured lamination λ is doubly incompressible. In particular, it follows from the work of Kim, Lecuire, and Ohshika [25, Theorem 2] that every subsequence of $\{M_n\}$ has a subsequence, say the whole sequence, which converges algebraically to a hyperbolic 3–manifold M_A homotopy equivalent to $\overline{M}$.

If λ were realized in M_A, then by Proposition 2.4

$$\lim_n l_{M_n}(\gamma_n) = \infty.$$

On the other hand, since γ_n is parabolic in M_n we have that

$$l_{M_n}(\gamma_n) = 0$$

for all n. Hence λ cannot be realized. It now follows from Proposition 2.2 that M_A is homeomorphic to M, has no parabolics, and its end is degenerate with ending lamination λ. In particular, Proposition 2.5 implies that the sequence M_n converges strongly to M_A. Moreover, since M and M_A have the same ending lamination λ, the ending lamination theorem implies that M and M_A are isometric. □

4.3. **Geometric limits of gluings.** We now study some of the geometric limits of sequences of hyperbolic manifolds obtained by gluing two manifolds by higher and higher powers of a pseudo-Anosov; compare with [41]. We prove:

Proposition 4.5. *Assume that $(\overline{N}_1, P_1)$ and $(\overline{N}_2, P_2)$ are acylindrical, that $S = \partial\overline{N}_1 \setminus P_1$ is connected, that $\partial\overline{N}_2 \setminus P_2$ has a component S' homeomorphic to S, and fix a homeomorphism $\phi\colon S \to S'$. Fix also a pseudo-Anosov mapping class $\psi\colon S \to S$ and consider the 3–manifold*

$$N^n = \overline{N}_1 \cup_{\phi\circ\psi^n} \overline{N}_2$$

obtained by gluing $\overline{N}_1$ and $\overline{N}_2$. Assume finally that for each n there is a hyperbolic 3–manifold M_n and a π_1–injective embedding $N^n \hookrightarrow M_n$, with all curves in $P_1 \cup P_2$ mapped to parabolics in M_n.

Then some geometric limit M_G of the sequence $\{M_n\}$ is homeomorphic to the interior of $\overline{N}_1$, has parabolic locus P_1, and its only geometric end is degenerate.

The proof of Proposition 4.5 is similar to the proof of Proposition 4.4. This result will play a key role in the proof of Proposition 5.5.

Proof. We are going to prove that every subsequence of the sequence $\{M_n\}$ has some further subsequence converging geometrically to a hyperbolic 3–manifold M_G homeomorphic to the interior of $\overline{N}_1$, with parabolic locus P_1, and whose unique geometric end is degenerate with ending lamination equal to the repelling lamination of ψ. The ending lamination theorem implies then that any two of these geometric limits are isometric; this shows that M_G is a geometric limit of the whole sequence $\{M_n\}$. In particular, as in the proof of Proposition 4.4, we may pass to subsequences as often as we wish.

Before going any further, observe that the assumption that $\partial\overline{N}_1 \setminus P_1$ and $\partial\overline{N}_2 \setminus P_2$ are incompressible implies that $\pi_1(\overline{N}_1)$ injects into $\pi_1(N^n)$ and hence into $\pi_1(M_n)$. The assumption that $(\overline{N}_1, P_1)$ and $(\overline{N}_2, P_2)$ are actually acylindrical implies furthermore:

Lemma 4.6. *Let $\overline{N}_1$ and M_n be as in the statement of Proposition 4.5 and identify $\pi_1(\overline{N}_1)$ with a subgroup of $\pi_1(M_n)$. If two elements $\gamma, \gamma' \in \pi_1(\overline{N}_1)$ are conjugate in $\pi_1(M_n)$ for some n, then they are conjugate in $\pi_1(\overline{N}_1)$. Equivalently, if two essential curves $\gamma, \gamma' \subset \overline{N}_1$ are freely homotopic within M_n they are freely homotopic within $\overline{N}_1$.* □

Consider now the coverings M_n^1 and M_n^2 of M_n corresponding to the subgroups $\pi_1(\overline{N}_1)$ and $\pi_1(\overline{N}_2)$ of $\pi_1(M_n)$ respectively. Since $(\overline{N}_1, P_1)$ and $(\overline{N}_2, P_2)$ are acylindrical, Thurston's compactness theorem (see [24]) applies. Hence, passing to a subsequence we may assume that the two sequences $\{M_n^1\}$ and $\{M_n^2\}$ converge algebraically. Let M_G be the algebraic limit of the sequence $\{M_n^1\}$. Every curve in P_1 represents a parabolic element in M_G because this is the case in M_n^1 for all n. On the other hand, fix a simple closed curve $\alpha \subset S' \subset \partial\overline{N}_2 \setminus P_2$. The compactness of the sequence $\{M_2^n\}$ implies that there is some L with $l_{M_2^n}(\alpha) \leq L$ for all n. After the identification $\phi \circ \psi^n$, the curve α becomes the curve $\psi^{-n}(\phi^{-1}(\alpha)) \subset S \subset \partial\overline{N}_1 \setminus P_1$. In particular, we have

$$l_{M_1^n}(\psi^{-n}(\phi^{-1}(\alpha))) \leq L \quad \text{for all n.}$$

Let λ be the repelling lamination of ψ and observe that

$$\lim_{n\to\infty} \psi^{-n}(\phi^{-1}(\alpha)) = \lambda \quad \text{in } \mathcal{PML}(S).$$

A small variation of the argument used in the proof of Proposition 4.4 shows that λ is not realized in M_G. Again as in the proof of Proposition 4.4, we derive from Proposition 2.2 that P_1 is the parabolic locus of M_G and that its unique geometric end is degenerate with ending lamination λ. Also as in the proof of Proposition 4.4, Proposition 2.5

shows that the sequence $\{M_n\}$ does not only converge algebraically but also geometrically to M_G.

We have proved that the sequence of covers M_n^1 of M_n has the desired geometric limit. We claim that this is also the case for the original sequence $\{M_n\}$. Passing perhaps to a further subsequence, we may assume that the manifolds $\{M_n\}$ converge geometrically to some manifold M_G' covered by M_G. Since the unique geometric end of M_G is degenerate, we deduce from the Thurston–Canary covering theorem that the cover $M_G \to M_G'$ is finite–to–one. We claim that this cover is trivial. Otherwise, there are two elements $\gamma, \gamma' \in \pi_1(M_G) = \pi_1(\overline{N}_1)$ which are conjugated in $\pi_1(M_G')$ but not in $\pi_1(M_G)$. In particular, we deduce from the geometric convergence of the sequence $\{M_n\}$ to M_G that for all sufficiently large n the elements $\gamma, \gamma' \in \pi_1(\overline{N}_1) \subset \pi_1(M_n)$ are also conjugated in $\pi_1(M_n)$. This contradicts Lemma 4.6. Thus the covering $M_G \to M_G'$ is trivial and hence $M_G' = M_G$. In other words, M_G is a geometric limit of the sequence $\{M_n\}$. □

5. Constructing some limits of knot complements

The goal of this section is to prove Proposition 5.5. In order to do so, we start studying certain multicurves in the boundary of a compression body. These multicurves are used to prove first a version of Proposition 5.5 for links. The desired knots are then obtained from these links by Dehn-filling.

5.1. **Curves on a compression body.** We prove now a topological fact used in the proof of Proposition 5.3. The setting is the following: Let Σ be a closed surface and C the compression body obtained by gluing $\Sigma \times [0,1]$ and $\mathbb{S}^1 \times \mathbb{D}^2$ along disks $D_1 \subset \Sigma \times \{1\}$ and $D_2 \subset \partial(\mathbb{S}^1 \times \mathbb{D}^2)$. We denote by $\partial_e C$ the compressible boundary component of C, by $\partial_i C$ the incompressible one, and by D the disk $D_1 = D_2 \subset C$.

Fix a simple curve $\alpha \subset \partial_e C$ disjoint from D, contained in the component of $C \setminus D$ corresponding to $\mathbb{S}^1 \times \mathbb{D}^2$, representing a generator of $\mathbb{Z} = \pi_1(\mathbb{S}^1 \times \mathbb{D}^2)$. We will use repeatedly the following fact, which is easily verified by considering the two components of $C \setminus D$.

Lemma 5.1. *The only embedded essential disks in $C \setminus D$ are isotopic to D or intersect α.* □

Let γ be any simple closed curve in $\partial_e C$ which intersects α exactly once, with $i(\gamma, \partial D) > 0$, and such that, after isotoping γ so that it intersects ∂D minimally, $\gamma \cap (\Sigma \times \{1\} \setminus D_1)$ contains at least two nonisotopic properly embedded arcs. Here $i(\cdot,\cdot)$ is the geometric intersection number. Finally, denote by β the boundary of a regular neighborhood of $\alpha \cup \gamma$. We prove:

Proposition 5.2. *Let α and β be as above. Then there exists a non-separating curve $\eta \subset \partial_i C$ such that $(C, \mathcal{N}(\alpha \cup \beta \cup \eta))$ is acylindrical.*

Proof. We first prove prove that the multicurve $\alpha \cup \beta \subset \partial_e C$ intersects every properly embedded essential disk, Möbius band, and annulus $(A, \partial A) \subset (C, \partial_e C)$.

Claim 1. The multicurve $\alpha \cup \beta$ intersects every properly embedded essential disk in C.

Proof of Claim 1. Let Δ be a properly embedded essential disk in C such that $\partial\Delta$ does not meet $\alpha \cup \beta$.

If $\Delta \cap D = \emptyset$, then Δ is an embedded essential disk in $C \smallsetminus D$, hence by Lemma 5.1, Δ is isotopic to D or meets α. By assumption it does not meet α. But $i(\beta, \partial D) = 2i(\gamma, \partial D) > 0$, by choice of β (and γ), hence Δ cannot be isotopic to D. Thus $\Delta \cap D \neq \emptyset$.

Consider the intersections $\Delta \cap D$. These consist of closed curves and arcs. Using the irreducibility of C, along with an innermost disk argument, we may assume no components of $\Delta \cap D$ are closed curves (else they can be isotoped off). So consider arcs of intersection. There is an innermost arc of intersection on Δ, which bounds a disk $E_1 \subset \Delta$ disjoint from D, and some disk $E_2 \subset D$. Together, these two disks form a disk E in C. By pushing E_2 off D slightly, we may assume E is embedded in $C \smallsetminus D$.

If E is not essential, then together E_1 and E_2 and their boundary curve bound a ball in C. We may isotope E_1 through this ball to decrease the number of intersections of Δ and D.

So assume E is essential. Then by Lemma 5.1, E is isotopic to D or meets α. First, E cannot meet α, for by assumption E_1 doesn't meet α, and E_2 is a subset of D, which does not meet α. So E is isotopic to D. But then again we may isotope E_1 through D, reducing the number of intersections of Δ and D.

Repeating this argument a finite number of times, we find $\Delta \cap D$ must be empty, which is a contradiction. □

Claim 2. The multicurve $\alpha \cup \beta$ intersects every properly embedded Möbius band in $(C, \partial_e C)$.

Proof of Claim 2. Let $(M, \partial M) \subset (C, \partial_e C)$ be a properly embedded Möbius band. Since C is orientable, M has to be one-sided. In particular, the homology class $[M] \in H_2(C, \partial_e C; \mathbb{Z}/2\mathbb{Z})$ is non-trivial; by duality, there is some class $[c]$ in $H_1(C;\mathbb{Z}/2\mathbb{Z})$ with $[c]\cap[M]=1$ mod 2. The first homology of C is generated by $H_1(\partial_i C)$ and $[\alpha]$. Since $\partial_i C\cap M=\emptyset$, we deduce that α has to intersect M. $\square$

Claim 3. The multicurve $\alpha\cup\beta$ intersects every properly embedded essential annulus $(A,\partial A)\subset(C,\partial_e C)$ which is disjoint of D.

Proof of Claim 3. Let $(A,\partial A)$ be an essential annulus in $(C,\partial_e C)$, and assume $\partial A\cap(\alpha\cup\beta)=\emptyset$ and $A\cap D=\emptyset$.

First, if A is contained in the component corresponding to $\mathbb{S}^1\times\mathbb{D}^2$, then because $\partial A\cap\alpha=\emptyset$, ∂A must be parallel to α or ∂D. In either case, A will be inessential. Thus A will lie in the component of $C\smallsetminus D$ corresponding to $\Sigma\times[0,1]$. In particular, the annulus A is isotopic, relative to its boundary, to an annulus A' contained in $\Sigma\times\{1\}$. The assumption that A is essential in C implies that $D_1\subset A'$ (recall $D_1\subset\Sigma\times\{1\}$ is the disk $D_1=D$ in C).

Cut open the surface $\partial_e C$ along $\partial A'=\partial A$ and let X be the component containing ∂D. Observe that $Y=X\cap(\Sigma\times\{1\}\smallsetminus D_1)$ is a pair of pants. By assumption, the curve β is disjoint of ∂A and by construction intersects ∂D. In particular, $\beta\subset X$. This implies that the curve γ is also contained in X. By assumption, $\gamma\cap(\Sigma\times\{1\}\smallsetminus D_1)$ contains at least two nonisotopic properly embedded essential arcs. Hence $\gamma\cap Y$ contains at least two nonisotopic arcs. This is impossible, since Y is a pair of pants and all the arcs in $\gamma\cap Y$ end in the same boundary component. This contradiction concludes the proof of Claim 3. $\square$

Claim 4. The multicurve $\alpha\cup\beta$ intersects every properly embedded essential annulus $(A,\partial A)\subset(C,\partial_e C)$.

Proof of Claim 4. Let $(A,\partial A)$ be an essential annulus in $(C,\partial_e C)$, and assume $\partial A\cap(\alpha\cup\beta)=\emptyset$. By Claim 3, we may assume that $A\cap D\neq\emptyset$. As in the proof of Claim 1, consider the arcs and curves of intersection.

Because A is essential, and any curve of intersection bounds a disk in D, it must bound a disk in A, and thus we may isotope A so that there are no closed curves of intersection with A.

Suppose there is an arc τ of $A\cap D$ such that both endpoints of τ lie on the same component of ∂A. Then τ, along with a portion of ∂A, bounds a disk in A. The only components of $A\cap D$ that lie in that disk will also have endpoints on the same component of ∂A. Thus we may assume τ is an innermost arc of intersection, which, together with a portion of ∂A, bounds a disk in A disjoint from D. The arc τ also bounds a disk on D. The union of these two disks can be pushed off of D slightly to give an embedded disk E in $C\smallsetminus D$.

Now, if E is inessential, we can isotope A through D to reduce the number of intersections. So assume E is essential. Then by Lemma 5.1, E is isotopic to D or meets α. E cannot meet α, for neither A nor D meet α. Thus E is isotopic to D. But then again we may isotope A through D and reduce the number of intersections $A \cap D$.

So we may assume no arcs of $A \cap D$ have both endpoints on the same component of ∂A.

Thus assume each arc of intersection has endpoints on distinct components of ∂A. Let τ_1 and τ_2 be consecutive arcs of intersection, such that they bound a disk on A disjoint from D. Note τ_1 and τ_2 will also bound disjoint disks on D. Putting these disks together and pushing off D slightly, again we have a disk E embedded in $C \smallsetminus D$.

If E is essential in $C \smallsetminus D$, then again Lemma 5.1 gives a contradiction: E cannot meet α because A and D do not, and if E is isotopic to D then we can isotope A through D, reducing the number of intersections.

So E is inessential in $C \smallsetminus D$. But then we may isotope the portion of A between τ_1 and τ_2 to lie on the boundary $\partial_e C$. This is true of any consecutive arcs τ_1 and τ_2 on A. Since A can be cut into pieces, each of which is bounded by consecutive arcs, A is not an essential annulus. This contradiction finishes the proof of the claim. □

We can now conclude the proof of Proposition 5.2. Let $\mathcal{N}(\alpha \cup \beta)$ be a regular neighborhood of the multicurve $\alpha \cup \beta$. By Claim 1, $\partial C \smallsetminus \mathcal{N}(\alpha \cup \beta)$ is incompressible. Assume that $(C, \mathcal{N}(\alpha \cup \beta))$ is not acylindrical. Then we can consider the JSJ–splitting of C relative to a regular neighborhood of $\mathcal{N}(\alpha \cup \beta)$; let S be the union of the Seifert pieces. Since C is not itself Seifert fibered, S is a proper subset of C. The boundary of S consists of a collection of properly embedded essential annuli; it is easy to see that this implies that the interior boundary $\partial_i C = \Sigma \times \{0\}$ of C is not contained in S. On the other hand, all the boundary annuli of S have at least one of their boundary components in $\partial_i C$. Let X be the boundary in $\partial_i C$ of $\partial_i C \cap S$. Any curve η which, together with X, fills the surface $\partial_i C$ has the property of intersecting every properly embedded essential annulus in $(C, \mathcal{N}(\alpha \cup \beta))$. In particular, $(C, \mathcal{N}(\alpha \cup \beta \cup \eta))$ is acylindrical; this concludes the proof of Proposition 5.2. □

5.2. **Limits of link complements.** In this section we prove a version of Proposition 5.5 for links instead of knots. As mentioned above, we will obtain the desired knots needed to prove Proposition 5.5 using Dehn filling. In order to be able to do so, we need to construct links satisfying certain conditions. One of those conditions is that a certain slope has a long *normalized length*, as in [22].

Definition. *Let M be a hyperbolic 3–manifold with a rank 2 cusp T, and let H be an embedded horoball neighborhood of the cusp with boundary ∂H. Let s be a* slope *on the cusp , that is, an isotopy class of simple closed curves on T. The* normalized length *of s is defined to be the length of a geodesic representative of s on ∂H, divided by the square root of the area of the torus ∂H. Note that this definition is independent of choice of horoball neighborhood H.*

We prove now:

Proposition 5.3. *Let $\overline{N}$ be a compact irreducible and atoroidal submanifold of $\mathbb{S}^3$ with connected boundary of genus at least 2, and let $\eta \subset \partial\overline{N}$ be a simple closed curve with $(\partial\overline{N}, \mathcal{N}(\eta))$ acylindrical and $\partial\overline{N} \smallsetminus \eta$ connected. Then there is a sequence of hyperbolic link complements M_{L_i} converging geometrically to a hyperbolic manifold N_η homeomorphic to the interior of $\overline{N}$ and such that η represents a parabolic element in N_η.*

Moreover, the links L_i have exactly four components, α_i, β_i, κ_i, and η_i. The normalized length of the standard meridian on η_i is increasing in an unbounded manner, and α_i and β_i bound disjoint disks in the complement of η_i, α_i and β_i in $\mathbb{S}^3$.

The construction given in the proof of Proposition 5.3 is quite involved. There are simpler ways to build N_η as the geometric limit of link complements. However, our choice of link complements and of N_η needs the extra care so that we may turn these links into knots to prove Proposition 5.5 in the following section.

Before launching the proof of Proposition 5.3, let Σ be a closed surface homeomorphic to $\partial\overline{N}$, C the compression body considered in the previous section and $\alpha, \beta \subset \partial_e C$ and $\eta' \subset \partial_i C$ the curves provided by Proposition 5.2. Let also $\phi\colon \partial\overline{N} \to \partial_i C$ be any homeomorphism with $\phi(\eta) = \eta'$. Denote by X_ϕ the manifold obtained by gluing $\overline{N}$ and C via ϕ and removing a regular neighborhood of the the curve $\eta = \eta'$. For the sake of book–keeping we denote by α_ϕ and β_ϕ the curves in ∂X_ϕ corresponding to α and β. Observe that the pair $(X_\phi, \mathcal{N}(\alpha_\phi \cup \beta_\phi))$ is acylindrical.

Lemma 5.4. *The embedding $\iota\colon \overline{N} \hookrightarrow \mathbb{S}^3$ extends to an embedding $\iota'\colon X_\phi \hookrightarrow \mathbb{S}^3$ with the property that the curves $\iota'(\alpha_\phi)$ and $\iota'(\beta_\phi)$ bound properly embedded disjoint disks in the closure of $\mathbb{S}^3 \smallsetminus \iota'(X_\phi)$.*

Proof. Using the embedding $C \hookrightarrow X_\phi$, identify the disk D with a disk in X_ϕ, and let Y_ϕ be the component of $X_\phi \smallsetminus D$ containing $\overline{N}$.

The manifold Y_ϕ is homeomorphic to the complement in $\overline{N}$ of the regular neighborhood of some curve contained in the interior of $\overline{N}$ and isotopic to η. In particular, the embedding $\iota\colon \overline{N} \hookrightarrow \mathbb{S}^3$ extends to an embedding $\hat{\iota}\colon Y_\phi \hookrightarrow \mathbb{S}^3$.

The other component, say U, of $X_\phi \smallsetminus D$ containing α, is homeomorphic to $\mathbb{S}^1 \times \mathbb{D}^2$. In particular, we can embed U in $\mathbb{S}^3 \smallsetminus \hat{\iota}(Y_\phi)$ in such a way that the image of the curve α bounds a properly embedded disk Δ in the complement of U and $\hat{\iota}(Y_\phi)$.

In order to extend these two embeddings to an embedding $\iota'\colon X_\phi \hookrightarrow \mathbb{S}^3$ we map the 1–handle joining Y_ϕ and U to a 1–handle in $\mathbb{S}^3 \smallsetminus (\hat{\iota}(Y_\phi) \cup U)$ whose core is disjoint from Δ.

So far, we have an embedding $\iota'\colon X_\phi \hookrightarrow \mathbb{S}^3$ with the properly that $\iota'(\alpha_\phi)$ bounds an embedded disk Δ in the complement of the image of ι'. Recall that the curve β is obtained as the boundary of the regular neighborhood of $\alpha \cup \gamma$ where γ intersects α once. The boundary of a regular neighborhood in $\mathbb{S}^3 \smallsetminus \iota'(X_\phi)$ of $\iota'(\gamma) \cup \Delta$ is a properly embedded disk with boundary β, which is disjoint of Δ. □

Remark. Neither the constructed embedding ι', nor the image of the curve α_ϕ depend on ϕ. However, the curve $\iota'(\beta_\phi)$ is very sensitive to ϕ.

We are now ready to prove Proposition 5.3:

Proof of Proposition 5.3. Choose a homeomorphism ϕ as above once and for ever. Choose a mapping class $\psi\colon \partial_i C \to \partial_i C$ with $\psi(\eta') = \eta'$ and whose restriction to $\partial_i C \smallsetminus \eta'$ is pseudo-Anosov. Consider the sequence of manifolds $X_{\psi^n \circ \phi}$.

For all n, choose once and for ever a knot $\kappa_n \subset \mathbb{S}^3 \smallsetminus \iota'(X_{\psi^n \circ \phi})$ which intersects every properly embedded essential disk, annulus and Möbius band therein. Consider the link

$$L_n = \eta \cup \iota'(\alpha_{\psi^n \circ \phi}) \cup \iota'(\beta_{\psi^n \circ \phi}) \cup \kappa_n$$

By choice of κ_n, the complement $M_{L_n} = \mathbb{S}^3 \smallsetminus L_n$ of this link is irreducible and atoroidal. In particular, by Thurston's hyperbolization theorem (see [24, 38, 39]), M_{L_n} admits a complete hyperbolic metric.

Observe that by construction, $\overline{N}$ (minus a small regular neighborhood of the boundary) is a π_1-injective submanifold of M_{L_n}. Moreover, it follows from the construction that the sequences of manifolds M_{L_n} and submanifolds $\overline{N} \subset M_{L_n}$ satisfy the conditions of Proposition 4.5.

It follows that the sequence $\mathbb{S}^3 \smallsetminus L_n$ has a geometric limit N_η with the following property:

(*) N_η is homeomorphic to the interior of $\overline{N}$, η is parabolic in N_η and the only geometric end of N_η is degenerate.

It remains only to show that the normalized length of the standard meridian of the link component corresponding to η tends to infinity. This is due to the fact that these manifolds approach N_η, whose end is degenerate, and hence in the geometric limit, the link components tending to η tend to a rank 1 cusp, which is an infinite annulus. This rank 1 cusp has fixed translation length, but unbounded length in another direction.

We claim that the unbounded direction corresponds to the limit of standard meridians of η in M_{L_n}. That is, note a standard meridian of η in M_{L_n} intersects $\partial\overline{N}$ twice in $\iota'(X_{\psi^n \circ \phi})$. The geometric limit N_η is homeomorphic to the interior of $\overline{N}$, hence a curve meeting $D\overline{N}$ twice must have unbounded length.

Because the translation length of the rank 1 cusp is fixed, lengths of curves meeting the meridian once have bounded length for large n in M_{L_n}. Hence the normalized length of the meridian must become arbitrarily long. □

5.3. **Proof of Proposition 5.5.** The goal of this section should be clear from the title.

Proposition 5.5. *Let $\overline{N}$ be a compact irreducible and atoroidal submanifold of $\mathbb{S}^3$ with connected boundary of genus at least 2, and let $\eta \subset \partial\overline{N}$ be a simple closed curve with $(\partial\overline{N}, \mathcal{N}(\eta))$ acylindrical and $\partial\overline{N} \setminus \eta$ connected. Then there is a sequence of hyperbolic knot complements $\{M_{K_i}\}$ converging geometrically to a hyperbolic manifold N_η homeomorphic to the interior of $\overline{N}$ and such that η represents a parabolic element in N_η.*

Proof. By Proposition 5.3, we may assume there is a sequence of hyperbolic link complements $\{M_{L_i}\}$ converging geometrically to N_η homeomorphic to the interior of $\overline{N}$ and such that η represents a parabolic element in N_η. Moreover, we know that the link L_i has four components α_i, β_i, κ_i, and η_i, that the normalized length of the standard meridian of η_i grows unbounded, and that α_i and β_i bound disjoint disks in the complement of α_i, β_i, and η_i in $\mathbb{S}^3$. To prove the proposition, we will perform hyperbolic Dehn filling on the link components corresponding to α_i, β_i, and η_i. We will use the following version, whose proof is written carefully in Aaron Magid's thesis [30, Theorem 4.3], of Hodgson and Kerckhoff's quantified Dehn-filling theorem [22]. See also Bromberg [11, Theorem 2.5].

Theorem 5.6. *Let $J > 1$ and ϵ positive and smaller than the Margulis constant. Then there is some $L > 16\pi^2 + 2\pi/\epsilon$ such that if M is a finite* *volume hyperbolic 3–manifold, and* $s \subset \partial \overline{M}$ *is a slope with normalized length at least* L*, then:*

(1) *The interior* M_s *of the Dehn filled manifold* $\overline{M}(s)$ *obtained from* $\overline{M}$ *by surgery along* s *is hyperbolic,*
(2) *the geodesic* $\gamma_s \subset M_s$ *isotopic to the core of the attached solid torus has at most length* $(2\pi)/(L^2 - 16\pi^2) < \epsilon$*, and*
(3) *there is a* J*-bi-Lipschitz embedding* $\phi\colon M \smallsetminus \mathbb{T}_{s,\epsilon} \hookrightarrow M_s$ *of the complement in* M *of the component in* $M^{<\epsilon}$ *corresponding to the filled cusp into the Dehn filled manifold* M_s*.*

Let $M_{\alpha_i\cup\beta_i\cup\kappa_i}$ be the manifold obtained from M_{L_i} by performing surgery on M_{L_i} along the standard meridian of η_i. Since the normalized length of this meridian grows unbounded, we deduce from Theorem 5.6 that for all sufficiently large i the manifold $M_{\alpha_i\cup\beta_i\cup\kappa_i}$ is hyperbolic and that N_η is also a geometric limit of the sequence $\{M_{\alpha_i\cup\beta_i\cup\kappa_i}\}$.

Fixing now i, consider the manifold $M_{\alpha_i\cup\beta_i}$ obtained by filing the standard meridian of the knot κ_i. In other words, $M_{\alpha_i\cup\beta_i} = \mathbb{S}^3 \smallsetminus (\alpha_i \cup \beta_i)$. Since the two link components α_i and β_i bound disjoint embedded disks, we see that the manifold $M_{\alpha_i\cup\beta_i}$ is homeomorphic to the interior connected sum $(\mathbb{D}^2 \times \mathbb{S}^1)\#(\mathbb{D}^2 \times \mathbb{S}^1)$ of two solid tori. In particular, there are infinitely many ways in which one can Dehn fill each of the two ends of $M_{\alpha_i\cup\beta_i}$ so that we obtain the 3–sphere $\mathbb{S}^3$. Each one of these fillings yields a new embedding of $M_{\alpha_i\cup\beta_i}$ into $\mathbb{S}^3$; observe that the knot $\kappa_i \subset M_{\alpha_i\cup\beta_i}$ is mapped under these new embeddings to a knot which may not be isotopic to the original κ_i.

Hence for each i the hyperbolic manifold $M_{\alpha_i\cup\beta_i\cup\kappa_i}$ admits infinitely many Dehn fillings of the cusps associated to α_i and β_i so that the obtained manifold is homeomorphic to the complement of a knot in $\mathbb{S}^3$. Choosing a sequence of more and more complicated Dehn fillings in each of the cusps α_i and β_i, we obtain a sequence $\{M_{K_{i,j}}\}_j$ of knot complements. We deduce from Theorem 5.6, or even from the classical version of Thurston's Dehn filling theorem [43], that for each fixed i the following holds:

- For all sufficiently large j, say for all j, the knot complement $M_{K_{i,j}}$ is hyperbolic, and
- the sequence of knot complements $\{M_{K_{i,j}}\}$ converges geometrically to $M_{\alpha_i\cup\beta_i\cup\kappa_i}$.

Since by construction the sequence $\{M_{\alpha_i\cup\beta_i\cup\kappa_i}\}$ converges to N_η, the claim of Proposition 5.5 follows now from Lemma 2.3. □

6. Convex submanifolds and the proof of Theorem 1.3

In this section we prove Theorem 1.3. However, before doing so we have to establish a (perhaps well–known or possibly of independent interest) property of certain convex subsets in 3–dimensional hyperbolic space. Essentially we prove that the completion of the complement of the convex hull of any subset of $\partial_\infty \mathbb{H}^3$ is a locally CAT(-1) space. This fact allows us to adapt an argument due to Lackenby to prove Theorem 1.3. We end this section with a few unrelated observation on the complements of convex submanifolds of hyperbolic 3–manifolds.

6.1. **Convex hulls.** Let $X \subset \partial_\infty \mathbb{H}^3$ be a closed subset in the boundary at infinity of $\mathbb{H}^3$ and assume that X contains at least three points. Let U be a connected component of the complement $\mathbb{H}^3 \setminus CH(X)$ of the convex hull $CH(X)$ of X in $\mathbb{H}^3$. We consider U with the induced interior metric and let $\overline{U}$ be its metric completion.

Remark. Recall that the completion of a subspace of a metric space may be different from its closure. Think of the open interval $(0,1)$ in the circle $\mathbb{R}/\mathbb{Z}$.

We prove:

Proposition 6.1. *Let $X \subset \partial_\infty \mathbb{H}^3$ be a closed set in the boundary at infinity of $\mathbb{H}^3$ containing at least three points, and U a connected component of the complement $\mathbb{H}^3 \setminus CH(X)$ of the convex hull $CH(X)$ of X in $\mathbb{H}^3$. The metric completion $\overline{U}$ of U is a locally CAT(-1) manifold with totally geodesic boundary $\partial\overline{U} = \overline{U} \setminus U$, under the path metric inherited from $\mathbb{H}^3$.*

Before launching the proof of Proposition 6.1 we would like to remark that this is an almost exclusively 3–dimensional phenomenon. For instance, it was pointed out to us by Larry Guth that if X is any finite set in $\partial_\infty \mathbb{H}^4$ then the complement of $CH(X)$ is not aspherical and hence cannot be a locally CAT(-1) manifold. It would be interesting to determine for which subsets of $\partial_\infty \mathbb{H}^n$, $n \geq 4$, Proposition 6.1 remains valid.

Proof. The assumption that X has at least three points implies that $CH(X)$ is either a totally geodesic surface (if X is contained in a round circle in $\partial_\infty \mathbb{H}^3$) or a convex set with non-empty interior.

Assume that we are in the first case, or equivalently that $CH(X) \subset \mathbb{H}^2$. If $CH(X) = \mathbb{H}^2$ then each component of $\mathbb{H}^3 \setminus CH(X)$ is an open halfspace, its closure is a closed halfspace, and we have nothing to prove. If $CH(X)$ is a proper subset of $\mathbb{H}^2$ then $U = \mathbb{H}^3 \setminus CH(X)$ is connected and its metric completion $\overline{U}$ is homeomorphic to the complement of a regular neighborhood of $CH(X)$ in $\mathbb{H}^3$. In particular, $\overline{U} \smallsetminus U$ is the double of $CH(X)$. In fact, the map $\overline{U} \to \mathbb{H}^3$ induces the "folding the double" map $\overline{U} \smallsetminus U \to CH(X)$. The double $D\overline{U}$ of $\overline{U}$ is a hyperbolic cone–manifold with empty boundary and all cone angles equal to 4π. This implies local uniqueness of geodesics, and hence $D\overline{U}$ is CAT(-1) (compare with [21]). The local uniqueness of geodesics in $D\overline{U}$ implies that $\overline{U}$ is totally convex and $\partial\overline{U}$ totally geodesic under the path metric on $\overline{U}$ inherited from the hyperbolic metric. Since totally convex subsets of locally CAT(-1) spaces are locally CAT(-1), the claim follows in this case.

In some sense, the case that $CH(X)$ has nonempty interior is less confusing. In this case the metric completion $\overline{U}$ of a component U of $\mathbb{H}^3 \smallsetminus CH(X)$ is equal to its closure in $\mathbb{H}^3$. Moreover, a theorem of Thurston (see for example [18]) asserts that $\partial\overline{U} = \overline{U} \smallsetminus U = \partial CH(X)$ is, with respect to its intrinsic distance, a complete hyperbolic surface.

Now choose a nested collection of finite subsets X_i of X with dense union. In other words,

$$X_1 \subset X_2 \subset X_3 \subset \dots \quad \text{and} \quad X = \overline{\cup_{i=1}^{\infty} X_i} \tag{6.1}$$

We may assume without loss of generality that none of the sets X_i is contained in a round circle, since X is not. For each i, the set $U_i = \mathbb{H}^3 \smallsetminus CH(X_i)$ is connected and contains $\mathbb{H}^3 \smallsetminus CH(X)$. When i tends to ∞, the closures $\overline{U}_i$ of U_i converge in the pointed Hausdorff topology to the closure $\overline{\mathbb{H}^3 \smallsetminus CH(X)}$ of $\mathbb{H}^3 \smallsetminus CH(X)$. Moreover, for every $p \in \mathbb{H}^3 \smallsetminus CH(X)$ there is $\epsilon > 0$ such that for all i sufficiently large, $B_p(\epsilon, \mathbb{H}^3) \cap \overline{U}_i$ and $B_p(\epsilon, \mathbb{H}^3) \cap (\mathbb{H}^3 \smallsetminus CH(X))$ are simply connected. In other words, the sequence of universal covers of $\overline{U}_i$ converge in the pointed Hausdorff topology to the universal cover of $\overline{\mathbb{H}^3 \smallsetminus CH(X)}$. It follows now from [7, II, Theorem 3.9] that it suffices to show that each of the $\overline{U}_i$ is locally CAT(-1).

In other words, in order to conclude the proof of Proposition 6.1 it remains to prove it for finite sets $X \subset \partial_\infty \mathbb{H}^3$ which are not contained in a round circle. Under this assumption, $CH(X)$ is a convex ideal polyhedron with non-empty interior. In particular $CH(X)$ has finitely many totally geodesic faces, finitely many geodesic edges and no vertices (other than the ideal vertices at infinity). Convexity of $CH(X)$ implies that every interior dihedral angle of the polyheldron is less than π. The closure $\overline{U}$ of the complement U of $CH(X)$ in $\mathbb{H}^3$ is just the complement in $\mathbb{H}^3$ of the interior of the polyhedron $CH(X)$. Doubling $\overline{U}$ we obtain a hyperbolic cone–manifold $D\overline{U}$ with empty boundary and cone angles greater than 2π. As above, the doubled $D\overline{U}$ is CAT(-1) under the induced path metric and, again as above, we deduce that $\overline{U}$ itself is CAT(-1) with totally geodesic boundary under the induced path metric. This concludes the proof of Proposition 6.1. □

We conclude with the following slightly more general version of Proposition 6.1:

Corollary 6.2. *Let $\{X_i\}$ be a sequence of closed subsets of $\partial_\infty \mathbb{H}^3$ and assume that for every $x \in \mathbb{H}^3$ there is some $\epsilon_x > 0$ such that the ball $B_{\mathbb{H}^3}(x, \epsilon_x)$ intersects at most one of the convex hulls $CH(X_i)$. Let U be a connected component of $\mathbb{H}^3 \setminus \cup_i CH(X_i)$ and $\widetilde{U}$ its universal cover. Then $\overline{\widetilde{U}}$, the completion of $\widetilde{U}$ with respect to the lifted path metric, is a CAT(-1) space.*

Proof. The assumption that each point in $x \in \mathbb{H}^3$ is the center of some ball which only intersects one of the convex hulls $CH(X_i)$ implies that each point in $x \in \overline{\widetilde{U}}$ has a small neighborhood isometric to the completion of the universal cover of the complement of $CH(X_i)$. If X_i consists of only two points, then it follows from Soma [42] that x has a small CAT(-1) neighborhood in $\overline{\widetilde{U}}$. If X_i has at least three points then we obtain the same consequence from Proposition 6.1. Using this local description of the completion of the universal cover, it is easy to see that every curve in $\overline{\widetilde{U}}$ can be homotoped to the a curve in $\widetilde{U}$ and hence that $\overline{\widetilde{U}}$ is simply connected. In other words, $\overline{\widetilde{U}}$ is a simply connected locally CAT(-1) space and hence is CAT(-1) by the Hadamard–Cartan theorem. □

6.2. **Proof of Theorem 1.3.** We prove now Theorem 1.3.

Theorem 1.3. *Let M be a hyperbolic 3–manifold. If the manifold M has at least two convex cocompact ends, then M is not the geometric limit of any sequence of hyperbolic knot complements in $\mathbb{S}^3$.*

Proof. Let M be as in the statement of the theorem and assume that there is a sequence of hyperbolic knot complements $\{M_{K_i}\}$ converging geometrically to M. Let $\mathcal{N}(CC(M))$ be a regular neighborhood of $CC(M)$ with smooth strictly convex boundary and let Σ_1 and Σ_2 be two compact components of $\partial\mathcal{N}(CC(M))$. Fix now a compact, connected 3–dimensional submanifold W of M whose boundary contains both Σ_1 and Σ_2. For some d to be determined below, let $W_d = \{x \in M | d_M(x, W) \leq d\}$. By geometric convergence, for large i we have better and better almost isometric embeddings

$$f_i\colon W_d \to M_{K_i}.$$

We may assume, passing to a subsequence, that for all i the surfaces $f_i(\Sigma_1)$ and $f_i(\Sigma_2)$ are locally convex.

The two surfaces $f_i(\Sigma_1)$ and $f_i(\Sigma_2)$ are closed and disjoint. Since the complement of a knot in $\mathbb{S}^3$ does not contain non-separating closed surfaces, we obtain that $f_i(\Sigma_1)$ and $f_i(\Sigma_2)$ separate the knot complement M_{K_i} into three pieces. Let V_i^0 be the component containing $f_i(W)$. If $M_{K_i} \setminus V_i^0$ has an unbounded component, let V_i^1 be this component; otherwise choose V_i^1 to be either remaining component. Set $V_i = V_i^0 \cup V_i^1$ and observe that it is convex, and that its boundary is one of the two surfaces $f_i(\Sigma_1), f_i(\Sigma_2)$. Up to relabeling and passing to a subsequence, we may assume $\partial V_i = f_i(\Sigma_1)$ for all i.

It follows from the convexity of V_i that its fundamental group $\pi_1(V_i)$ injects into $\pi_1(M_{K_i})$. Let M_{V_i} be the associated cover of M_{K_i} and lift the inclusion $V_i \hookrightarrow M_{K_i}$ to the inclusion $V_i \hookrightarrow M_{V_i}$. The convexity of V_i implies that the convex core $CC(M_{V_i})$ of M_{V_i} is contained in V_i. In particular, the restriction of the covering $M_{V_i} \to M_{K_i}$ to $CC(M_{V_i})$ is injective. Before moving on we observe:

(1) The submanifolds $CC(M_{V_i})$ and V_i of M_{K_i} are isotopic.
(2) The boundary of $CC(M_{V_i})$ is a closed connected surface S_i. The surfaces S_i are uniformly bi-Lipschitz equivalent to Σ_1. In particular, there is some uniform δ with $\mathrm{inj}(S_i) \geq \delta$ for all i.
(3) The surface $\partial CC(M_{V_i})$ has, for large i, a collar of at least width d in $M_{K_i} \setminus CC(M_{V_i})$, since recall f_i is an almost isometric embedding of W_d, containing W. In particular, any essential arc $(\kappa, \partial\kappa) \subset (M_{K_i} \setminus CC(M_{V_i}), \partial CC(M_{V_i}))$ has, for all sufficiently large i length at least $2d$.

Let now $U_i' = M_{K_i} \setminus V_i$ and $U_i = M_{K_i} \setminus CC(M_{V_i})$ be the complements of V_i and $CC(M_{V_i})$ in M_{K_i} respectively. It follows from (1) that U_i and U_i' are isotopic and from the construction that $U_i' \subset U_i$. Let $\overline{U}_i = U_i \cup S_i$ be the closure of U_i in M_{K_i}. Now, $\overline{U}_i$ is the completion of the complement of a convex set $CC(M_{V_i})$ in the hyperbolic manifold M_{K_i}. Lift to the universal $\mathbb{H}^3$ of M_{K_i}. Lifts of $CC(M_{V_i})$ are convex hulls of sets X_j in $\partial\mathbb{H}^3$. By Corollary 6.2, the completion of the universal cover of a component of $\mathbb{H}^3 \setminus \cup_j CH(X_j)$ is a CAT(-1) space with geodesic boundary. This is the universal cover of $\overline{U}_i$. Hence with respect to the path metric induced from the interior metric, $\overline{U}_i$ is a locally CAT(-1) manifold with totally geodesic boundary.

It is a well–known fact that a compact 3–manifold which admits a hyperbolic metric with respect to which the boundary is totally geodesic is irreducible, atoroidal and has incompressible and acylindrical boundary. The argument applies verbatim to locally CAT(-1) metrics with again totally geodesic boundary. We deduce hence that $\overline{U}_i$ is irreducible, atoroidal and has incompressible and acylindrical boundary.

Via the embedding $M_{K_i} \hookrightarrow \mathbb{S}^3$, we consider $\overline{U}_i$ as a submanifold of the 3–sphere and let $H_i = \mathbb{S}^3 \setminus \overline{U}_i$ be its complement. Since a component of ∂H_i has at least genus 2 we deduce that H_i cannot be simply connected. In particular, the homomorphism

$$\pi_1(H_i) \to \pi_1(\overline{U}_i \cup H_i) = \pi_1(\mathbb{S}^3) = 1$$

cannot be injective. We deduce as in [28, Section 2] that the acylindrical manifold $\overline{U}_i$ contains an immersed incompressible, boundary incompressible planar surface

$$(X_i, \partial X_i) \subset (\overline{U}_i, \partial \overline{U}_i)$$

with negative Euler characteristic $\chi(X_i) = 2 - k_i$; here k_i is the number of boundary components of X_i.

We proceed now as for example in [42] to obtain a CAT(-1) metric with geodesic boundary on X_i. Start with a triangulation of X_i with a single vertex at each boundary component and no vertices in the interior. Homotope the map $X_i \to \overline{U}_i$ so that the boundary curves go to geodesics in $\partial \overline{U}_i$. Then, keeping the boundary fixed, homotope the remaining edges of the 1–skeleton of X_i to geodesic arcs. Finally homotope each of faces of the triangulation to a ruled surface.

Pulling back the CAT(-1) metric of $\overline{U}_i$ we obtain a CAT(-1) metric on X_i with geodesic boundary. The Gauß–Bonnet theorem implies then that

$$\mathrm{vol}(X_i) \leq 2\pi(k_i - 2) < 2\pi k_i \tag{6.2}$$

Assume that $(\kappa, \partial\kappa) \subset (X_i, \partial X_i)$ is an essential arc. Since the surface X_i is boundary incompressible we deduce from (3) that the image of κ has at least length $2d$. On the other hand, the map $X_i \to \overline{U}_i$ is 1–Lipschitz. Hence each of the boundary components of X_i has an embedded collar of width d and all these collars are disjoint. At the same time, each one of the boundary components is an essential curve in $S_i = \partial \overline{U}_i'$ and hence has at least length δ by (2). We deduce that each one of the k collars has at least volume $d\delta$ and that all these collars are disjoint. Thus

$$\mathrm{vol}(X_i) \geq k_i d\delta.$$

In particular, if d is sufficiently large we obtain a contradiction to the area bound (6.2). This concludes the proof of Theorem 1.3. □

At this point we would like to add a few observations on the geometry of complements of convex submanifolds in hyperbolic 3–manifolds. For example, remark that during the proof of Theorem 1.3 we have essentially obtained the following result:

Proposition 6.3. *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3–manifold and $V \subset M$ a 3–dimensional submanifold with locally convex, compact boundary and non-abelian fundamental group $\pi_1(V)$. Consider $U = M \smallsetminus V$ and let $\overline{U} = U \cup \partial V$ be its metric completion. The embedding $U \hookrightarrow M$ is isotopic (but perhaps not ambient isotopic) to a second embedding $\phi\colon U \hookrightarrow M$ with the following properties:*

1. *$U \subset \phi(U)$ and moreover, if ∂V is smooth and none of its components are totally geodesic, then $\phi(U) \smallsetminus U$ is homeomorphic to $\partial\overline{U} \times \mathbb{R}$.*
2. *ϕ extends continuously to a locally injective $\bar{\phi}\colon \overline{U} \to M$. Moreover, unless some component of V is a regular neighborhood of a non-separating totally geodesic surface, the map $\bar{\phi}$ is injective when restricted each connected component of $\overline{U}$.*
3. *If we endow $\overline{U}$ with the unique interior distance such that the map $\bar{\phi}$ preserves the lengths of curves, then $\overline{U}$ is a CAT(-1) space with totally geodesic boundary.* □

As a consequence of Proposition 6.3 and Soma [42] we have:

Corollary 6.4. *Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3–manifold and $V \subset M$ a 3–dimensional submanifold with locally convex, compact boundary. Assume that no component of V is simply connected and let $\overline{U}$ be the metric completion of $U = M \smallsetminus V$. Then $\overline{U}$ is irreducible, atoroidal and has incompressible and acylindrical boundary.* □

7. Various

In this section we discuss some of the examples mentioned in the introduction, we make some more or less interesting remarks and we ask a few questions.

We start constructing hyperbolic knots whose complements have very big injectivity radius at some point. In order to do so, it suffices to remark that $\mathbb{H}^3$ is a complete hyperbolic 3–manifold with trivial, and hence finitely generated, fundamental group which is homeomorphic to a ball and hence has a single end and embeds into $\mathbb{S}^3$. In other words, it satisfies the conditions of Theorem 1.1 and hence $\mathbb{H}^3$ is a geometric limit of some sequence of hyperbolic knot complements $\{M_{K_i}\}$. Recall that this means that there are $p \in \mathbb{H}^3$ and $p_i \in M_{K_i}$ such that for all r and ϵ and all sufficiently large i the ball $B_p(r) \subset \mathbb{H}^3$ with center p and radius r can be $(1+\epsilon)$ isometrically embedded into M_{K_i} by a map which maps p to p_i. This implies that for all sufficiently large i, the set of points in M_{K_i} which is at distance at most say $R/2$ of p_i is simply connected. In other words, for all large i we have $\text{inj}(p_i, M_{K_i}) \geq R/2$. Since R was arbitrary we obtain:

Corollary 1.2. *For every $R > 0$ there is a hyperbolic knot complement M_K and $x \in M_K$ with injectivity radius $\text{inj}(x, M_K) > R$.* □

The proof of Corollary 1.2 is somehow disappointing because the involved knots are produced in a very very indirect way. We ask:

Question 2. *Give an explicit construction of knots as in Corollary 1.2.*

The same strategy used to prove Corollary 1.2 can be used to show that there are hyperbolic knots whose complement has arbitrarily large Heegaard genus, arbitrarily large volume, arbitrarily many arbitrarily small eigenvalues of the Laplacian, arbitrarily many arbitrarily short geodesics, contains surfaces with arbitrarily small principal curvatures... It should be said that knots with most of these properties were either known to exist (see for instance [29]) or no one bothered to try to construct them before.

We show now how to prove that some other hyperbolic 3–manifolds not covered by Theorem 1.1 are also geometric limits of knot complements.

It is a by now almost folklore fact that every hyperbolic 3–manifold M homeomorphic to the trivial interval bundle $\Sigma_g \times \mathbb{R}$ over a closed surface of genus g and which has at least one degenerate end is the geometric limit of a sequence of hyperbolic 3–manifolds $\{M_i\}$ where each of M_i is homeomorphic to the interior of a handlebody of genus g (compare for instance with [5, Section 3]). Each one of the manifolds M_i satisfies the assumption of Theorem 1.1 and hence is a geometric limit of knot complements. It follows now from Lemma 2.3:

Corollary 7.1. *Every hyperbolic 3–manifold M homeomorphic to $\Sigma_g \times \mathbb{R}$ and which has at least one degenerate end is the geometric limit of a sequence of hyperbolic knot complements.* □

More generally, if M is homeomorphic to the interior of a compression body with exterior boundary of genus g and such that each of the interior ends is degenerate, then M the geometric limit of a sequence $\{M_i\}$ where each of the M_i is homeomorphic to a genus g handlebody (compare again [5, Section 3]). Paying the price of having a large exterior boundary, we have compression bodies with as many interior boundary components as we wish. In particular we deduce:

Corollary 7.2. *For every n and g there is a hyperbolic 3–manifold M with at least n ends which have neighborhoods homeomorphic to $\Sigma_g \times \mathbb{R}_+$ and which arise as geometric limits of knot complements.* □

The statement of Corollary 7.1 and Corollary 7.2 become more interesting when compared with Theorem 1.3 since the latter shows that for instance not every hyperbolic manifold homeomorphic to $\Sigma_g \times \mathbb{R}$ is a geometric limit of knot complements.

Question 3. *Assume that M has finitely generated fundamental group, embeds into the sphere and has more than one end. What are the possible hyperbolic metrics on M which make M a geometric limit of knot complements? Are there restrictions on the possible ending laminations of the degenerate ends?*

Question 4. *Is there an irreducible and atoroidal 3–manifold with finitely generated fundamental group which embeds into $\mathbb{S}^3$ and is not homeomorphic to any geometric limit of knot complements?*

In relation with these two last questions, we would like to observe that in the course of the proof of Theorem 1.1 we proved something slightly stronger than what we state. In fact, if M is as in the statement of the theorem and we fix an embedding $\overline{M} \hookrightarrow \mathbb{S}^3$ of the manifold compactification of M into the sphere, then we can find knots K_i in the complement of $\overline{M}$ in $\mathbb{S}^3$ such that M is a geometric limit of the M_{K_i}. In other words, if we identify $\overline{M}$ with a standard compact core of M, then the initial embedding of $\overline{M}$ into $\mathbb{S}^3$ and the embedding obtained by composing the homeomorphism $\overline{M} \hookrightarrow M$ with the almost isometric embedding $M \hookrightarrow M_{K_i}$ provided by geometric convergence, followed by the standard embedding $M_{K_i} \hookrightarrow \mathbb{S}^3$, are isotopic.

It seems that whenever M has at least two ends the situation dramatically changes. In particular, question 3 and question 4 may actually turn to be questions on the possible re-embeddings of submanifolds of the sphere.

Also related to question 3 and question 4 but in a little different spirit we ask:

Question 5. *Is there a geometric limit of knot complements M with finitely generated $\pi_1(M)$ which has two geometrically finite geometric ends contained in two different topological ends?*

For the sake of playfulness, or what is almost the same, to answer a question of Yair Minsky, we discuss which hyperbolic 3–manifolds of the form $\mathbb{H}^3/\Gamma$, with $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ a torsion–free Fuchsian group, arise as geometric limits of knot complements. If the surface $\mathbb{H}^2/\Gamma$ is compact, then $\mathbb{H}^3/\Gamma$ has two convex cocompact ends and hence is not a limit of knot complements by Theorem 1.3. On the other hand, if the surface $\mathbb{H}^2/\Gamma$ is not compact then let $S_1 \subset S_2 \subset \dots$ be a nested sequence of compact connected π_1–injective subsurfaces of $\mathbb{H}^2/\Gamma$ with $\mathbb{H}^2/\Gamma = \cup S_i$. Let $\Gamma_1 \subset \Gamma_2 \subset \dots$ be the associated sequence of subgroups $\Gamma_i = \pi_1(S_i)$ of Γ. Then the hyperbolic manifold $\mathbb{H}^3/\Gamma$ is the geometric limit of the sequence $\{\mathbb{H}^3/\Gamma_i\}$. On the other hand, each of the manifolds $\mathbb{H}^3/\Gamma_i$ is homeomorphic to a handlebody and hence is a geometric limit of knot complements by Theorem 1.1. Combining these two observations we obtain:

Corollary 7.3. *Let $\Gamma \subset \mathrm{PSL}_2\,\mathbb{R}$ be a torsion free, but possibly infinitely generated, Fuchsian group. Then the hyperbolic manifold $\mathbb{H}^3/\Gamma$ is a geometric limit of knot complements if and only if the surface $\mathbb{H}^2/\Gamma$ is open.* □

At this point we would like to mention that the argument used to prove Theorem 1.1 together with the remark after the proof of Proposition 4.1 imply that if $\mathbb{H}^2/\Gamma$ is a closed hyperbolic surface which admits an orientation preserving involution without fixed points, then $\mathbb{H}^3/\Gamma$ is the geometric limit of a sequence of link complements where each link has two components.

During the discussion of Corollary 7.3 we observed that Theorem 1.1 can also be used to prove that some hyperbolic 3–manifolds with infinitely generated fundamental group are geometric limits of knot complements. It may be that every one–ended hyperbolic 3–manifold which embeds into the sphere is a geometric limit of still one–ended hyperbolic 3–manifolds which embed into the sphere and have finitely generated fundamental group. Hence we ask:

Question 6. *Is it true that every one–ended hyperbolic 3–manifold which embedds into $\mathbb{S}^3$ is a limit of knot complements? In other words, does Theorem 1.1 holds for manifolds with infinitely generated π_1?*

On the other hand, it could be that some hyperbolic 3–manifold M which does not embed into the sphere is a geometric limit of hyperbolic 3–manifolds which embed into the sphere. This prompts the following question:

Question 7. *Is it true that every geometric limit of hyperbolic knot complements embeds into $\mathbb{S}^3$?*

The answer to question 7 is most likely negative.

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