A REMARK ABOUT THE CURVE COMPLEX

JUAN SOUTO

Let from now on S_g be the closed genus $g \geq 1$ surface and denote by $\mathcal{S}(S_g)$ the set of isotopy classes of simple closed essential curves. Recall that the *curve complex* $\mathcal{C}(S_g)$ is the simplicial complex with set of vertices $\mathcal{S}(S_g)$ where a subset $\{\gamma_1, \dots, \gamma_r\} \subset \mathcal{S}(S_g)$ spans a simplex if the isotopy classes γ_i and γ_j have disjoint representatives; in other words,

$$\iota(\gamma_i, \gamma_j) = 0$$

for all i, j where $\iota(\cdot, \cdot)$ is the geometric intersection number.

Harer [1] proved that the curve complex $\mathcal{C}(S_g)$ is homotopy equivalent to a wedge of spheres of dimension $2g-2$. In particular, for $g = 1$ the curve complex is not connected. In order to by-pass this problem, it is customary to change the definition saying that $\{\gamma_1, \dots, \gamma_r\} \subset \mathcal{S}(S_1)$ span a simplex if $\iota(\gamma_i, \gamma_j) = 1$ for all i, j. After this modification, the curve complex of the torus becomes not only connected but also contractible.

Motivated by the last observation we define for $d \geq 0$ the *d-curve complex* $\mathcal{C}_d(S_g)$ as the simplicial complex with again vertex set $\mathcal{S}(S_g)$ and where a subset $\{\gamma_1, \dots, \gamma_r\} \subset \mathcal{S}(S)$ spans a simplex in $\mathcal{C}_d(S_g)$ if $\iota(\gamma_i, \gamma_j) \leq d$ for all i, j. The mapping class group $\mathrm{Map}(S)$ of S acts on $\mathcal{C}_d(S)$ for all d. Moreover, if $d < d'$ then every subset of $\mathcal{S}(S)$ spanning a simplex in $\mathcal{C}_d(S)$ also spans a simplex in $\mathcal{C}_{d'}(S)$ and therefore there is a $\mathrm{Map}(S)$-equivariant simplicial inclusion $\mathcal{C}_d(S) \hookrightarrow \mathcal{C}_{d'}(S)$. We prove:

Theorem 1. *Let S be a closed surface of genus $g \geq 1$. The natural inclusion $\mathcal{C}_0(S) \hookrightarrow \mathcal{C}_1(S)$ is homotopic to a constant map. In other words, $\mathcal{C}_0(S)$ is contractible within $\mathcal{C}_1(S)$.*

Before launching the proof of Theorem 1 we recall the definition of the *curve and arc complex* of a punctured surface. Under a punctured surface we understand a closed surface S together with a finite set V of marked points in S. A simple closed curve in $S \setminus V$ is peripheral if its bounds, in S, a disk containing at most 1 point of V; a non-peripheral curve is *essential.* A properly embedded simple arc in (S, V) is an arc whose interior is embedded and contained in $S \setminus V$, and with possibly equal endpoints in V. Such an arc is essential if either the end-points are different or if it does not bound a disk in S whose interior is disjoint of V. Two essential arcs or curves are disjoint if they have disjoint interiors and they are isotopic if they are properly isotopic in $S \setminus V$. Let $\mathcal{S}(S, V)$ be the set of isotopy classes of simple essential curves and arcs in (S, V) and let $\mathcal{A}(S, V)$ be the simplicial complex with vertices $\mathcal{S}(S, V)$ and where $\{\alpha_1, \dots, \alpha_r\} \subset \mathcal{S}(S, V)$ spans a simplex if the proper isotopy classes α_i and α_j have disjoint representatives for all i, j. The complex $\mathcal{A}(S, V)$ is the *curve-and-arc complex* of (S, V). In [2], Harcher proved:

Theorem 2. *Let S be a closed surface of genus $g \geq 1$ and $V \subset S$ a non-empty finite set. Then $\mathcal{A}(S, V)$ is contractible.*

After this discussion we prove Theorem 1.

Proof of Theorem 1. Since the case of the torus is well-known we assume from now on that $g \geq 2$ and fix a hyperbolic metric on S. In particular, we can identify $\mathcal{S}(S)$ as the set of simple closed geodesics in S. There are countably many such geodesics and hence most points in S are not contained in any element in $\mathcal{S}(S)$; let $*$ be such a point.

The choice of the point $*$ yields an obviously injective map $\mathcal{S}(S) \to \mathcal{S}(S, *)$. Moreover, if a subcollection of $\mathcal{S}(S)$ spans a simplex in $\mathcal{C}_0(S)$ then its image also spans a simplex in $\mathcal{A}(S, *)$. In other words, the map $\mathcal{S}(S) \to \mathcal{S}(S, *)$ extends to a simplicial map $\mathcal{C}_0(S) \to \mathcal{A}(S, *)$.

On the other hand, there is a clearly non-injective map $\mathcal{S}(S, *) \to \mathcal{S}(S)$ as follows. By definition, if $\alpha \in \mathcal{S}(S, *)$ is a closed curve then, when considered in S, α is homotopically essential and hence it represents an element in $\mathcal{S}(S)$. If $\alpha \in \mathcal{S}(S, *)$ is an arc then its end-points coincide and hence we obtain a simple closed curve in S which is again homotopically essential. Observe that the composition of the maps $\mathcal{S}(S) \to \mathcal{S}(S, *) \to \mathcal{S}(S)$ is the identity.

Since $\mathcal{A}(S, *)$ is contractible, we can extend the map $\mathcal{S}(S, *) \to \mathcal{S}(S)$ to a continuous map $\mathcal{A}(S, *) \to \mathcal{C}_0(S)$ but this map is not simplicial. The problem is that two essential arcs α, β in $(S, *)$ are, by definition, disjoint if they have disjoint interiors and hence the two corresponding simple closed curves in S may intersect. But they only intersect once. In other words, if a collection in $\mathcal{S}(S, *)$ spans a simplex in $\mathcal{A}(S, *)$ then it also spans a simplex in the 1-curve complex $\mathcal{C}_1(S)$. We have thus the following commutative diagram

$$\begin{tikzcd}
\mathcal{C}_0(S) \arrow[r] \arrow[dr] & \mathcal{A}(S, *) \arrow[d] \\
 & \mathcal{C}_1(S)
\end{tikzcd}$$

where the horizontal and the vertical arrows are the maps we just constructed and the slanted one is the canonical inclusion of $\mathcal{C}_0(S)$ to $\mathcal{C}_1(S)$. The claim follows from the contractibility of $\mathcal{A}(S,*)$. □

We should remark that the proof above can be adapted to show that for all $d \geq 0$ the map $\mathcal{C}_d(S) \to \mathcal{C}_{d+1}(S)$ is homotopically trivial.

We end with some questions:

Question 1: What is the dimension of $\mathcal{C}_1(S)$*?* In contrast to the standard curve complex $\mathcal{C}_0(S)$, in $\mathcal{C}_1(S)$ there are maximal simplexes of different dimension; we are interested in maximal possible dimension of a simplex in $\mathcal{C}_1(S)$. Equivalently, the questions is how many simple curves intersecting pairwise at most once can one find in a surface of genus g. This problem was considered by Benson Farb and Chris Leininger. They obtained a lower bound which is quadratic in the genus and a surprisingly exponential upper bound.

Question 2: Is $\mathcal{C}_1(S)$*, or if not* $\mathcal{C}_d(S)$ *for some* d*, contractible?* It is easy to see that $\mathcal{C}_1(S)$ is simply connected. I don't know anything else but I suspect that the answer is yes.

REFERENCES

[1] J. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. 84 (1986), no. 1, 157–176.

[2] A. Hatcher, *On triangulations of surfaces*, Topology Appl. 40 (1991), no. 2, 189–194.