# A note on the tameness of hyperbolic 3-manifolds

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**Abstract**

Among other related results we prove that a hyperbolic 3-manifold which admits an exhaustion by nested cores is tame.

*Key words:* hyperbolic manifolds, almost compact manifolds, tameness conjecture

## 1 Introduction

An irreducible 3–manifold $M$ with finitely generated fundamental group is *tame* if it is homeomorphic to the interior of a compact manifold with boundary. Sometimes tame manifolds are said to be *almost compact*. The first example of a 3–manifold which is not tame was given by Whitehead; McMillen proved that there are uncountably many 3–manifolds which are not tame (see (ST89) and the references therein). However Marden (Mar74) proved that every geometrically finite hyperbolic 3–manifold is tame, and, in fact, he conjectured:

TAMENESS CONJECTURE *Every complete hyperbolic 3–manifold with finitely generated fundamental group is tame.*

Thurston (Thu) and Canary (Can93) proved that the tameness of a hyperbolic 3–manifold has strong analytic and geometric consequences.

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Bonahon (Bon86) showed that a complete hyperbolic 3–manifold $M$ with finitely generated fundamental group is tame if $\pi_1(M)$ does not split as a free product or as an HNN–extension. In the cases which are not covered by Bonahon's theorem not much is known; there are some partial results due to Canary, Minsky, Ohshika and others (see (CM96; Ohs; KS; BBES)).

Given a complete hyperbolic 3–manifold $M$ with finitely generated fundamental group, there is a compact 3–dimensional submanifold $C \subset M$ such that the inclusion of $C$ in $M$ is a homotopy equivalence (Sco73); the submanifold $C$ is said to be a *core* of $M$. It is well–known (Bon86) that the ends of $M$ are in one–to–one correspondence with the components of $M - C$. Further, every component of $M - C$ is bounded by a unique component of $\partial C$. If $[E]$ is an end of $M$ then we will denote by $E$ and $\partial E$ the corresponding components of $M - C$ and of $\partial C$.

The manifold $M$ is tame if and only if every end $[E]$ has a neighborhood homeomorphic to the trivial interval bundle over a closed surface; an end having such a neighborhood is said to be *tame*. In this paper we show

**Theorem 1** *Let $[E]$ be an end of a complete, orientable hyperbolic 3–manifold $M$ with finitely generated fundamental group and let $C$ be a core of $M$. If there is a sequence $(S_i)_i$ of embedded surfaces that are incompressible in $M - C$, which have bounded Euler–characteristic and converge to $[E]$ when $i$ goes to $\infty$, then the end $[E]$ is tame.*

A sequence $(S_i)_i$ of surfaces in $M$ is said to *converge to the end* $[E]$ if for every neighborhood $U$ of $[E]$ there is $i_U$ with $S_i \subset U$ for all $i \geq i_U$.

Theorem 1 leads to

**Corollary 2** *If a complete orientable hyperbolic 3–manifold $M$ with finitely generated fundamental group is a nested union of cores, then $M$ is tame.*

We say that $M$ is a nested union of cores if there is a sequence $(C_i)_i$ of cores such that $C_i \subset C_{i+1}$ for all $i$ and such that $M = \cup C_i$.

Surprisingly, there are examples which show that Corollary 2 fails if $M$ is not hyperbolic (Tuc73; CM96) (see example 4 below).

The conditions in Theorem 1 on the surfaces $S_i$ are difficult to check in practice. We prove the following theorem which is more likely to be used (see (KS)):

**Theorem 3** *Let $[E]$ be an end of a complete oriented hyperbolic 3–manifold $M$ such that $\pi_1(M)$ is finitely generated but not free. If there is a sequence $(X_i)_i$ of surfaces homotopic in $M$ to $\partial E$ which converge to $[E]$ when $i$ goes to*

*$\infty$, then the end $[E]$ is tame.*

If the fundamental group of $M$ is a free group, then Theorem 3 is still true under the additional assumption that the surfaces $X_i$ are not homologically trivial in $M - C$, i.e. $0 \neq [X_i] \in H_2(M - C; \mathbb{Z})$.

We now outline the proof of Theorem 1. Let $M$, $C$, $[E]$ and $(S_i)_i$ be as in Theorem 1; without loss of generality we may assume that $S_i$ and $S_j$ are disjoint for all $i \neq j$. The surfaces $S_i$ are incompressible in $M - C$ but they may be compressible in $M$. We construct a branched cover $\pi : M^3 \to M$ to which $S_i$ lifts homeomorphically to an incompressible surface $S_i^3$ for sufficiently large $i$, say for all $i$. We prove that the surfaces $S_i^3$ represent only finitely many homotopy classes; this is the bulk of the proof of Theorem 1. Thus we may assume, choosing a convenient subsequence, that the surfaces $S_i^3$ and $S_j^3$ are homotopic in $M^3$ for all $i, j$. It is due to Waldhausen (Wal68) that the embedded, homotopic and incompressible surfaces $S_i^3$ and $S_j^3$ bound a submanifold of $M^3$ homeomorphic to $S_i^3 \times [0, 1]$. The covering $\pi : M^3 \to M$ is one–to–one on this interval bundle; thus, the surfaces $S_i$ and $S_j$ bound an interval bundle in $M$ for all $i \neq j$. This shows that the end $[E]$ is tame.

In order to illustrate the difficulties that we face we present the following example (see (CM96)).

**Example 4** *Let $\mathbb{D}^2 \times \mathbb{S}^1$ be a full torus and let $K \subset \mathbb{D}^2 \times \mathbb{S}^1$ be a knot which is homotopic but not isotopic to the soul $\{0\} \times \mathbb{S}^1$. For instance, let $K$ be the knot drawn in figure 1. Let $U$ be a regular neighborhood of $K$ in $\mathbb{D}^2 \times \mathbb{S}^1$ and let $Y$ be the complement of $U$. The manifold $Y$ has two boundary components which we call the interior boundary $\partial_{\mathrm{int}} Y$ and the exterior boundary $\partial_{\mathrm{ext}} Y$.*

*Fig. 1. The manifold $Y$. The dotted line represents the interior boundary $\partial_{\mathrm{i}} Y$.*

*We can glue a full torus $\mathbb{D}^2 \times \mathbb{S}^1$ to the interior boundary of $Y$ in such a way that we obtain again a full torus; we will denote the full torus obtained by this process by $Y \cup_{\partial_{\mathrm{i}} Y} (\mathbb{D}^2 \times \mathbb{S}^1)$. Let now $C_0 = \mathbb{D}^2 \times \mathbb{S}^1$ be a full torus and define inductively $C_i = Y \cup_{\partial_{\mathrm{i}} Y} C_{i-1}$; the manifold $C_i$ is a full torus and the embedding $C_{i-1} \hookrightarrow C_i$ is a homotopy equivalence. Consider the sequence*

$$C_0 \subset C_1 \subset C_2 \subset C_3 \subset \ldots$$

*and the union $M = \cup_i C_i$. The embedding $C_i \hookrightarrow M$ is a homotopy equivalence for all $i$, i.e. $C_i$ is a core of $M$ for all $i$; thus $M$ is a nested union of cores.*

*It is easy to see that $\pi_1(M - C_0)$ is not finitely generated; this shows that $M$ is not tame. In particular, this example shows that Corollary 2 fails if $M$ is* *not hyperbolic. Remark that the surfaces $\partial C_i$ and $\partial C_j$ are not homotopic in $M - C_0$ for $i \neq j$.*

We conclude the introduction with a brief plan of the paper.

Section 2 contains some facts and definitions which we asume to be well–known.

In section 3 we discuss simplicial ruled surfaces and obtain a finiteness result (Proposition 8) for those surfaces. This result is probably also well–known but as far as we know it can not be found in the literature.

In section 4 we prove Theorem 1; Corollary 2 and Theorem 3 are proved in section 5.

## 2 Preliminaries

### *2.1 Manifolds with pinched negative curvature*

A complete Riemannian 3–dimensional manifold $M$ has pinched negative curvature if there are constants $a, b$ with $-b^2 \leq \kappa_M \leq -a^2 < 0$, where $\kappa_M$ is the sectional curvature of $M$. If $M$ has constant curvature $-1$, then it is locally isometrically covered by hyperbolic space $\mathbb{H}^3$ and $M$ is said to be hyperbolic. Every simply connected complete 3–manifold with pinched negative curvature is diffeomorphic to $\mathbb{R}^3$ and, thus, is tame. More generally, it is known that every complete 3–manifold with pinched negative curvature and virtually abelian fundamental group is tame. From now on we will only consider open 3–manifolds $M$ with non virtually abelian fundamental group; equivalently $\pi_1(M)$ contains a free group.

The $\epsilon$–thin part $M^{<\epsilon}$ of a Riemannian manifold $M$ is the set of points $x$ in $M$ with injectivity radius $\text{inj}_M(x)$ less than $\epsilon$. The complement of the $\epsilon$–thin part is the $\epsilon$–thick part $M^{\geq\epsilon}$.

Let $M$ be a complete, orientable, 3–dimensional Riemannian manifold with pinched negative curvature. It is due to Margulis (BGS85) that there is a constant $\mu > 0$ –called the *Margulis constant*–, depending only on the curvature bounds such that for all $\epsilon < \mu$ the fundamental group of every component $U$ of $M^{<\epsilon}$ is abelian. This implies that every component $U$ of $M^{<\epsilon}$ belongs to one of the following types (BP92):

*Tube*: $\pi_1(U) = \mathbb{Z}$ and $U$ is bounded; then $U$ is homeomorphic to a relative compact full torus in $M$.

*Rank–one cusp*: $\pi_1(U) = \mathbb{Z}$ and $U$ is unbounded; then $U$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{R} \times (0, \infty)$ and its boundary in $M$ is an embedded annulus.

*Rank–two cusp*: $\pi_1(U) = \mathbb{Z}^2$; then $U$ is unbounded and homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1 \times (0, \infty)$.

Even in the case that $M$ is hyperbolic and $\pi_1(M)$ is finitely generated, there can be infinitely many components in the $\epsilon$–thin part of $M$ for every positive $\epsilon$, smaller than the Margulis constant (BO88). However Sulivan proved

**Sullivan's finiteness theorem**(Sul81) *A complete hyperbolic 3–manifold $M$ with finitely generated fundamental group has only finitely many cusps.*

### *2.2 Surfaces in 3–manifolds*

An orientable 3–dimensional manifold $M$ is said to be *irreducible* if every embedded sphere bounds a ball. A 3–manifold is irreducible if and only if its universal cover is irreducible. In particular, every 3–manifold which admits a complete Riemannian metric of pinched negative curvature is irreducible.

Throughout the paper we will only consider irreducible 3–manifolds.

A map $S \to M$ from a connected surface $S \neq \mathbb{D}^2, \mathbb{S}^2, \mathbb{R}P^2$ into an irreducible 3–manifold $M$ is *$\pi_1$–injective* if the induced map $\pi_1(S) \to \pi_1(M)$ is injective; for simplicity we say that $S$ is $\pi_1$–injective in $M$. A $\pi_1$–injective properly embedded surface $S$ in $M$ is said to be *incompressible*. A properly embedded surface $S \subset M$ is incompressible if and only if for every embedded disk $D$ in $M$ with $D \cap S = \partial D$ there is a disk $\bar{D} \subset S$ with $\partial \bar{D} = \partial D$. In this case the sphere $\bar{D} \cup D$ bounds a ball in $M$ since $M$ is assumed to be irreducible.

Suppose that $M$ carries a complete metric with pinched negative curvature. Let $\bar{S}$ be a compact surface with interior $S$. A map $f : S \to M$ is said to be *weakly type preserving* if the image of every boundary–homotopic curve is essential and can be homotoped into a cusp of $M$. An essential simple closed curve $\gamma$ in $S$ is an *accidental parabolic* if it is not boundary–homotopic in $\bar{S}$ but $f(\gamma)$ can be homotoped into a cusp of $M$. We say that the map $f$ is *type preserving* if it is weakly type preserving and has no accidental parabolics.

### *2.3 Cores*

Scott (Sco73) proved that every irreducible 3–manifold $M$ with finitely generated fundamental group contains a compact submanifold $C$ such that the embedding of $C$ into $M$ is a homotopy equivalence; $C$ is said to be a *core* of $M$. Furthermore, it is due to McCullough, Miller and Swarup (MMS85) that for any two cores $C_1, C_2$ there is a homeomorphism $h : C_1 \to C_2$ such that $i_2 \circ h$ is homotopic to $i_1$ where $i_j$ is the inclusion of $C_j$ into $M$.

Now suppose that $M$ is irreducible with finitely generated fundamental group and with non–empty boundary $\partial M$, and that $\Sigma \subset \partial M$ is a compact surface. Under these assumptions, McCullough (McC86) proved that there is a core $C$ of $M$ with $C \cap \partial M = \Sigma$; the pair $(C, \Sigma)$ is called a *relative core* of $(M, \partial M)$.

Let $M$ be a hyperbolic manifold with finitely generated fundamental group and let $M_0$ be the complement in $M$ of the disjoint union of the rank–one and rank–two cusps. Sullivan's finiteness theorem implies that the boundary $\partial M_0$ of $M_0$ is a disjoint union of finitely many tori and annuli. Let $P \subset \partial M_0$ be a compact subsurface such that the inclusion $P \hookrightarrow \partial M_0$ is a homotopy equivalence; $P$ is the union of the toroidal components of $\partial M_0$ and of compact annuli corresponding to the annular components of $\partial M_0$. If $C$ is a core of $M_0$ with $C \cap \partial M_0 = P$, then the inclusion of pairs $(C, P) \hookrightarrow (M_0, \partial M_0)$ is a homotopy equivalence.

### *2.4 Ends*

Let $M$ be an irreducible 3–manifold and let $C$ be a core. Bonahon (Bon86) proved that there is a one–to–one correspondence between the ends of $M$, the components of $M - C$ and the components of $\partial C$. See (Bon86) for a precise definition of *end*. An end $[E]$ of $M$ is said to be *tame* if it has a neighborhood which is homeomorphic to the trivial interval bundle over a closed surface. Moreover, if $[E]$ is a tame end of $M$, then it has a neighborhood homeomorphic to $\partial E \times \mathbb{R}$ where $\partial E$ is the corresponding component of $\partial C$. A 3–manifold is tame if and only if every end is tame (Tuc73).

### *2.5 Compression bodies*

A *compression body* $B$ is a compact, irreducible 3–manifold which has a boundary component $\partial_{\text{ext}} B$ whose fundamental group surjects onto $\pi_1(B)$; $\partial_{\text{ext}} B$ is said to be the *exterior boundary* of $B$. If the exterior boundary is incompressible, then $B$ is homeomorphic to the trivial interval bundle over $\partial_{\text{ext}} B$; such a compression body is said to be *trivial*. In the case that $B$ is non–trivial, the exterior boundary is the unique compressible component of $\partial B$.

Every non–trivial compression body $B$ contains a non–empty collection $\mathcal{D}$ of disjoint, non–parallel, properly embedded essential disks $(D_i, \partial D_i) \subset (B, \partial_{\text{ext}} B)$

such that $B-\mathcal{D}$ is either homeomorphic to a closed ball or to the disjoint union of trivial compression bodies. The compression body $B$ is a *handlebody* if $B-\mathcal{D}$ is a closed ball; equivalently $\pi_1(B)$ is a free group. If $B$ is not a handlebody, then the *interior boundary* $\partial_{\text{int}}B = \partial B - \partial_{\text{ext}}B$ is not empty. In particular we deduce that a compression body $B$ is a handlebody if and only if the exterior boundary $\partial_{\text{ext}}B$ is homologically trivial in $B$, i.e. $0 = [\partial_{\text{ext}}B] \in H_2(B;\mathbb{Z})$.

Ohshika (Ohs, p.39) proved that the exterior boundary has the following useful property:

**Lemma 5 (Ohshika)** *Let $B$ be a compression body and let $S$ be a connected closed surface such that there is a surjective homomorphism $\pi_1(S) \to \pi_1(B)$, then $\chi(S) \leq \chi(\partial_{\text{ext}}B)$.*

For a more detailled discussion about the topology of compression bodies see (MM86; CM).

Bonahon (Bon83, Appendix B) proved:

**Theorem (Bonahon)** *Let $C$ be a compact, orientable, irreducible 3–manifold and let $S$ be a component of $\partial C$. There is a compression body $B_S \subset C$ whose exterior boundary is $S$ and such that $\pi_1(B_S)$ injects into $\pi_1(C)$; furthermore $B_S$ is unique up to isotopy.*

The compression body $B_S$ provided by Bonahon's theorem is called the *relative compression body* associated to $S$. If the relative compression body $B_S$ is a handlebody for some component $S$ of $\partial C$, then $B_S = C$ and hence $\pi_1(C)$ is free.

Masur, Otal and others (Mas86; Ota88; KS02; KS) have extensively studied simple closed curves on the exterior boundary of compression bodies. From their results and from Bonahon's theorem follows the next Proposition (Can89; Can93) which is going to be crucial in the present paper:

**Proposition 6** *Let $C$ be a compact, orientable and irreducible 3–manifold. There is a collection $\Gamma$ of disjoint simple closed curves on $\partial C$ with the following properties:*

*(1)* $\Gamma$ *intersects at least three times every essential simple closed compressible curve on $\partial C$,*
*(2)* $\Gamma$ *intersects the boundary of every essential and properly embedded annulus $(A, \partial A) \subset (C, \partial C)$ and*
*(3)* $0 = [\Gamma] \in H_1(C;\mathbb{Z})$.

*Furthermore, if $C$ is the core of a complete hyperbolic manifold $M$ then it is possible to choose $\Gamma$ such that it is freely homotopic in $M$ to a union $\Gamma_*$ of* *primitive geodesics* □

## 3 Simplicial ruled surfaces

In this section $N$ will be a complete, oriented, 3–dimensional Riemannian manifold with pinched negative curvature $-b^2 \leq \kappa_N \leq -a^2 < 0$.

Let $\Delta$ be a triangle which is foliated by segments with an endpoint at a vertex $v$ of $\Delta$ and the other endpoint at the the edge of $\Delta$ opposite to $v$. An immersion $f : \Delta \to N$ is said to be a *ruled triangle* if every edge of $\Delta$ and every leaf of the foliation is mapped to a geodesic segment. The image of a ruled triangle is called a ruled triangle as well. Sometimes we will also allow that the map $f$ is only defined on $\Delta - v$. In this case we have the additional condition on $f$ to be proper and that $f$ maps the edges adjacent to $v$ to asymptotic geodesic rays in $N$.

Fig. 2. Ruled triangles

If $f : \Delta \to N$ is a ruled triangle, then the Riemannian metric of $N$ induces a smooth metric on $\Delta$ with curvature bounded from above by $-a^2$.

Let now $\bar{S}$ be a closed surface, $\mathcal{V} \subset \bar{S}$ a finite collection of points and $S = \bar{S} - \mathcal{V}$. A continuous map $\phi : S \to N$ is a *pre–simplicial ruled surface* if the following conditions hold:

- The boundary of every small disk centered at a point in $\mathcal{V}$ is mapped by $\phi$ to an essential curve in $N$.
- There is a triangulation $\mathcal{T}$ of $\bar{S}$ which contains $\mathcal{V}$ in the set of vertices such that $\phi|_\Delta$ is a simplicial ruled triangle for every face $\Delta$ of $\mathcal{T}$.

See Hatcher (Hat91) for a precise definition of triangulation. Remark that it follows from the definition that every pre–simplicial ruled surface is proper and weakly type preserving.

If $\phi : S \to N$ is a pre–simplicial ruled surface we will often say that $S$ itself is a pre–simplicial ruled surface in $N$.

As remarked above, the Riemannian metric of $N$ induces a metric on every pre–simplicial ruled surface $S$ in $N$ and this metric is smooth with curvature bounded from above by $-a^2$ on every face of $\mathcal{T}$. In particular, this metric has well–defined cone–angles at every point. A pre–simplicial hyperbolic surface $S$ in $N$ is a *simplicial ruled surface* if the cone angles are at least $2\pi$ at every point. The distance induced on the universal cover of a simplicial ruled surface in $N$ is complete and has curvature $\leq -a^2$ in the sense of Alexandroff. In particular, it follows from the Gauß–Bonnet theorem (Bon86; Can89) that for every simplicial ruled surface $S$ in $N$ holds:

$$\mathrm{vol}(S) \leq \frac{2\pi}{a^2}|\chi(S)|$$

Existence of simplicial ruled surfaces is guaranteed by the following result:

**Lemma 7** *(Bon86; Can89) Given a $\pi_1$–injective weakly type preserving surface $S$ in $N$ with $-\infty < \chi(S) < 0$, then there is a simplicial ruled surface $S'$ in $N$ homotopic to $S$.* $\square$

The main goal of this section is the following finiteness result for simplicial ruled surfaces, which is probably well–known.

**Proposition 8** *For all compact subsets $K \subset N$ and all $A > 0$ there are only finitely many proper homotopy classes of type preserving $\pi_1$–injective simplicial ruled surfaces $S$ in $N$ with $|\chi(S)| \leq A$ and $S \cap K \neq \emptyset$.*

Recall that a simple closed curve $\gamma$ on a $\pi_1$–injective surface $S$ in $N$ is an *accidental parabolic* if it is not boundary–homotopic in $S$ but can be homotoped into a cusp of $N$. A simplicial ruled surface $S$ is *type preserving* if it has no accidental parabolics.

We emphasize that, in Proposition 8, we do not assume that the fundamental group $\pi_1(N)$ is finitely generated. In the case that $\pi_1(N)$ is finitely generated we obtain from Proposition 8 the following:

**Corollary 9** *Assume that $N$ is hyperbolic and has finitely generated fundamental group. For all $A > 0$ there are only finitely many proper homotopy classes of type preserving $\pi_1$–injective simplicial ruled surfaces $S$ in $N$ with $|\chi(S)| \leq A$.*

**Proof** Let $N_0$ denote the complement in $N$ of a union of disjoint horospherical neighborhoods of the cusps of $N$; the boundary $\partial N_0$ of $N_0$ is then a finite collection of tori and annuli. Let $P \subset \partial N_0$ be a compact subsurface such that the inclusion $P \hookrightarrow \partial N_0$ is a homotopy equivalence, and let $C$ be a core of $N_0$ with $C \cap \partial N_0 = P$; the map of pairs $(C, P) \to (N_0, \partial N_0)$ is a homotopy equivalence.

Thurston's hyperbolization theorem (Ota96) implies that the interior of $C$ admits a complete geometrically finite hyperbolic metric $g_C$ such that the homotopy equivalence $(C, g_C) \to N$ is type–preserving. This homotopy equivalence induces a bijection between the sets of proper homotopy classes of type preserving $\pi_1$–injective simplicial ruled surfaces $S$ in $N$ and in $(C, g_C)$.

If follows from Proposition 8 that the set of proper homotopy classes of type preserving $\pi_1$–injective simplicial ruled surfaces $S$ in $N$ and in $(C, g_C)$ with $|\chi(S)| \leq A$ is finite because $(C, g_C)$ is geometrically finite. $\square$

Proposition 8 is an analogue of a finiteness result for pleated surfaces due to Thurston (Thu). When we try to mimic Thurston's arguments we meet the difficulty that simplicial ruled surfaces do not have curvature bounded from below. This implies that, at least a priori, it is not reasonable to expect that a geometric limit of simplicial ruled surfaces has nice properties. We obtain a priori estimates on the geometry of simplicial ruled surfaces and this makes possible to by–pass this difficulty.

Recall that the $\epsilon$–thin part $S^{<\epsilon}$ (resp. $N^{<\epsilon}$) is the set of all points in $S$ (resp. $N$) with injectivity radius $\mathrm{inj}_S(x) < \epsilon$ (resp. $\mathrm{inj}_N(x) < \epsilon$). The $\epsilon$–thick part $S^{\geq\epsilon}$ (resp. $N^{\geq\epsilon}$) is the complement of the $\epsilon$–thin part.

**Proof of Proposition 8** Given $A > 0$ and $K \subset N$ compact let $\mathfrak{S}$ be the set of all $\pi_1$–injective, type–preserving simplicial ruled surfaces $S$ in $N$ with $|\chi(S)| \leq A$ and $S \cap K \neq \emptyset$. Further, let $\mu > 0$ be a Margulis constant for $N$ and assume that

$$\mu < \mathrm{inj}_N(x) \ \text{ for all } x \in K \tag{1}$$

As the proof of Proposition 8 is quite long we outline first the strategy of the proof. We are going to show that there is a constant $L > 0$ such that for all $S \in \mathfrak{S}$ and all $x \in S \cap K$ there are generators of $\pi_1(S, x)$ with length less than $L$. This implies, that, up to conjugacy, there are only finitely many subgroups of $\pi_1(N)$ which can be represented as the image of $\pi_1(S) \to \pi_1(N)$ for $S \in \mathfrak{S}$. Proposition 8 follows because two surfaces $S, S' \in \mathfrak{S}$ are proper homotopic in $N$ if and only if the images of $\pi_1(S)$ and $\pi_1(S')$ in $\pi_1(N)$ are conjugated in $\pi_1(N)$.

**Lemma 10** *For all $S \in \mathfrak{S}$ holds $S^{<\mu} \subset N^{<\mu}$ and every component of $S^{<\mu}$ is homeomorphic to an annulus. Moreover, $S^{<\mu}$ has at most $3A$ components.*

**Proof** Given a point $x$ in $S^{<\mu}$, let $\gamma \subset S$ be an essential loop based at $x$ with length $l_S(\gamma) \leq 2\mu$. The loop $\gamma$ is not homotopically trivial in $S$ and $\pi_1(S)$ injects into $\pi_1(N)$, thus $\gamma$ is not homotopically trivial in $N$; therefore we obtain

$$2\,\mathrm{inj}_N(x) \leq l_N(\gamma) = l_S(\gamma) = 2\mu.$$

This proves that $S^{<\mu} \subset N^{<\mu}$.

Let now $U$ be a component of $S^{<\mu}$; by the above there is a component $V$ of $N^{<\mu}$ with $U \subset V$. In particular, the image of $\pi_1(U) \to \pi_1(N)$ is contained in an abelian subgroup of $\pi_1(N)$. Since $\pi_1(S)$ injects into $\pi_1(N)$ it follows that the image of $\pi_1(U) \to \pi_1(S)$ is a cyclic group.

The convexity of the distance function on the universal cover of $S$ implies that $U$ is an annulus. This argument also shows that any two components of $S^{<\mu}$ are non–parallel; the estimate on the number of components of $S^{<\mu}$ follows. □

We continue with the proof of Proposition 8. Given $S \in \mathfrak{S}$ let $\mathcal{U}(S)$ be the union of all unbounded components of $S^{<\mu}$. By Lemma 10, the surface $S - \mathcal{U}(S)$ is homotopy–equivalent to $S$.

Remark that (1) implies that $S - \mathcal{U}(S)$ intersects $K$ for every $S \in \mathfrak{S}$.

**Lemma 11** *There is $D > 0$ with* $\operatorname{diam}(S - \mathcal{U}(S)) \leq D$ *for all $S \in \mathfrak{S}$.*

This lemma is a generalization of Thurston's bounded diameter lemma (Bon86; Thu).

**Proof** Suppose that there is a sequence $(S_i)$ in $\mathfrak{S}$ with $\operatorname{diam}(S_i - \mathcal{U}(S_i)) \to \infty$ when $i \to \infty$. Choose $x_i \in K \cap (S_i - \mathcal{U}(S_i))$ for all $i$ and let $y_i \in S_i - \mathcal{U}(S_i)$ be such that $d_{S_i}(x_i, y_i) \to \infty$ when $i \to \infty$. For all $i$, let

$$\gamma_i : [0, d_{S_i}(x_i, y_i)] \to S_i$$

be a minimizing geodesic joining $x_i$ and $y_i$. Before going further, remark that length of $\gamma_i[0, d_{S_i}(x_i, y_i)] \cap \mathcal{U}(S_i)$ is bounded from above by a constant depending on $\mu$ and on the number of components of $\mathcal{U}(S_i)$ because $\gamma_i$ minimizes. Since the number of components of $\mathcal{U}(S_i)$ is at most $3A$ we deduce that there is $C > 0$ with

$$\text{length of } (\gamma_i[0, d_{S_i}(x_i, y_i)] \cap \mathcal{U}(S_i)) \leq C \quad \text{for all } i. \tag{2}$$

Now we claim that there is a subsequence of $(S_i)$, say the whole sequence, with

$$\lim_{t \to \infty} \limsup_{i \to \infty} \operatorname{inj}_{S_i}(\gamma_i(t)) = 0 \tag{3}$$

If (3) fails to be true, then there is $\epsilon > 0$, a sequence $t_j \to \infty$ and a subsequence of $(S_i)$, say the whole sequence, with

$$\liminf_{i \to \infty} \operatorname{inj}_{S_i}(\gamma_i(t_j)) > \epsilon$$

for all $j$; moreover we may assume $t_{j+1} - t_j > 2\epsilon$. In particular, there is for all $n$ some $i_n$ with $\mathrm{inj}_{S_{i_n}}(\gamma_{i_n}(t_j)) > \epsilon$ for $j = 1, \dots, n$. Since $\gamma_i$ minimizes, the balls $B_{S_{i_n}}(\gamma_{i_n}(t_j), \epsilon)$ in $S_{i_n}$ with center $\gamma_{i_n}(t_j)$ and radius $\epsilon$ are pairwise disjoint for $j = 1, \dots, n$. Canary (Can89) and Bonahon (Bon86) proved

**Lemma 12** *For every simplicial ruled surface $S$ in $N$, and every $x \in S^{\geq \epsilon}$ we have* $\mathrm{vol}_S(B_S(x, \epsilon)) \geq \pi\epsilon^2$. □

Thus we obtain for all $n$ a surface $S_{i_n} \in \mathfrak{S}$ with

$$\mathrm{vol}(S_{i_n}) \geq \sum_{j=1}^{n} \mathrm{vol}_{S_{i_n}}(B_{S_{i_n}}(\gamma_{i_n}(t_j), \epsilon)) \geq n\pi\epsilon^2$$

contradicting

$$\mathrm{vol}(S) \leq \frac{2\pi}{a^2} A \quad \text{for all } S \in \mathfrak{S}.$$

We have proved (3).

Now, by compactness of $K$, we may assume that the points $x_i = \gamma_i(0) \in K$ converge and that the curves

$$\gamma_i : [0, d_{S_i}(x_i, y_i)] \to S_i \subset N$$

converge to a 1–Lipschitz map $\gamma_\infty : [0, \infty) \to N$. We obtain from (3)

$$\lim_{t\to\infty} \mathrm{inj}_N(\gamma_\infty(t)) = \lim_{t\to\infty} \limsup_{i\to\infty} \mathrm{inj}_N(\gamma_i(t)) \leq \lim_{t\to\infty} \limsup_{i\to\infty} \mathrm{inj}_{S_i}(\gamma_i(t)) = 0$$

This implies that there are $t_\infty > 0$ and an unbounded component $V$ of $N^{<\mu}$ with $\gamma_\infty[t_\infty, \infty) \subset V$. Moreover, by (2) and (3), there is $t_0 > t_\infty$ with $\gamma_i(t_0) \notin \mathcal{U}(S_i)$ for all $i$ and with

$$\liminf_{i\to\infty} \mathrm{inj}_{S_i}(\gamma_i(t_0)) < \frac{\mu}{2}.$$

In particular, there $i$ with $\mathrm{inj}_{S_i}(\gamma_i(t_0)) < \mu$, with $\gamma_i(t_0) \in V$ and with $\gamma_i(t_0) \notin \mathcal{U}(S_i)$.

Let now $\eta$ be a simple essential loop in $S_i$ based at $\gamma_i(t_0)$ with length less than $2\mu$. The Margulis lemma, implies that $\eta$ represents an element in $\pi_1(V)$. Since $V$ is unbounded we deduce that $\eta$ can be homotoped into a cusp of $N$. As $\gamma_i(t_0) \notin \mathcal{U}(S_i)$ we deduce that $\eta$ is not boundary homotopic in $S_i$. In other words, $\eta$ is an accidental parabolic in $S_i$, contradicting the assumption that $S_i \in \mathfrak{S}$.

This concludes the proof of Lemma 11. □

We continue with the proof of Proposition 8. Consider the set

$$K' = \{x \in N | d_N(x, K) \leq D\}$$

where $D$ is the constant provided by Lemma 11. We have proved that $S - \mathcal{U}(S) \subset K'$ for all $S \in \mathfrak{S}$. In particular, we deduce from the compactness of $K'$ and from Lemma 10 that there is a positive constant $\delta > 0$ with $\mathrm{inj}_S(x) \geq \delta$ for all $x \in S - \mathcal{U}(S)$ and all $S \in \mathfrak{S}$.

Lemma 12 implies that there is a constant $N > 0$ such that $S - \mathcal{U}(S)$ can be covered by at most $N$ balls of radius $\delta$ for all $S \in \mathfrak{S}$. In particular, we obtain that for all $S \in \mathfrak{S}$ and all $x \in K \cap (S - \mathcal{U}(S))$ there are generators of $\pi_1(S - \mathcal{U}(S), x)$ with length less than $L = L(N, \delta)$; recall that $S - \mathcal{U}(S)$ is homotopy equivalent to $S$.

This implies that there are only finitely many conjugacy classes of subgroups of $\pi_1(N)$ which can be represented as $\pi_1(S)$ with $S \in \mathfrak{S}$. Proposition 8 follows because two surfaces $S, S' \in \mathfrak{S}$ are homotopic if and only if the images of $\pi_1(S)$ and $\pi_1(S')$ are conjugated in $\pi_1(N)$. $\square$

Now we assume that the manifold $N$ is hyperbolic. A simplicial ruled surface in a hyperbolic 3–manifold is said to be a *simplicial hyperbolic surface*. If $S$ is a simplicial hyperbolic surface, then the faces of the associated triangulation are mapped to totally geodesic triangles. Simplicial hyperbolic surfaces are useful because simple moves performed on their associated triangulations can be translated to homotopies through simplicial hyperbolic surfaces. A homotopy $(S_t)$ with $S_t$ simplicial hyperbolic for all $t$ is said to be an *interpolation*. The local model of an interpolation is sketched in figure 3.

Fig. 3. A move on a triangulation and the associated interpolation

The following propostion is due to Canary (Can96, Section 5) and Canary–Minsky (CM96, Proposition 4.5).

**Proposition 13** *Let $N$ be an oriented complete hyperbolic 3–manifold and let $S$ be a simplicial hyperbolic surface in $N$:*

*(1) If $S$ is $\pi_1$–injective and type–preserving and $S'$ is a second simplicial hyperbolic surface in $N$ which is homotopic to $S$, then there is an interpolation $(S_t)$ with $S_0 = S$ and $S_1 = S'$.*
*(2) Suppose that $S$ is $\pi_1$–injective in $N$ but not type–preserving. If there is a compact submanifold $K \subset N$ with $S \cap K = \emptyset$ such that every curve $\gamma$ in $S$ which is homotopic in $N - K$ into a cusp of $N$ is boundary parallel in*

*$S$, then, there is an interpolation $(S_t)$ with $S_0 = S$ and $S_1 \cap \mathcal{N}_1(K) \neq \emptyset$, where $\mathcal{N}_1(K) = \{x \in N | d(x, K) \leq 1\}$.*

*(3) Suppose that $S$ is not $\pi_1$–injective in $N$ but there is a compact submanifold $K \subset N$ with $S \cap K = \emptyset$ such that $\pi_1(S)$ injects into $\pi_1(N - K)$ and every curve $\gamma$ in $S$ which is homotopic in $N - K$ into a cusp of $N$ is boundary parallel in $S$. Then, there is an interpolation $(S_t)$ with $S_0 = S$ and $S_1 \cap \mathcal{N}_1(K) \neq \emptyset$.*

## 4 Proof of the Main Theorem

In this section we prove

**Theorem 1** *Let $[E]$ be an end of a complete, orientable hyperbolic 3–manifold $M$ with finitely generated fundamental group and let $C$ be a core of $M$. If there is a sequence $(S_i)_i$ of embedded surfaces that are incompressible in $M - C$, which have bounded Euler–characteristic and converge to $[E]$ when $i$ goes to $\infty$, then the end $[E]$ is tame.*

In the sequel let $M$, $C$, $[E]$ and $(S_i)$ be as in the statement of the theorem. We assume that $S_i \cap S_j = \emptyset$ for all $i, j$. The end $[E]$ is tame if the corresponding component $E \subset M - C$ is a neighborhood of a rank–two cusp of $M$, so we are going to assume that this is not the case. The component $E$ of $M - C$ is a neighborhood of $[E]$; thus we may assume that $S_i \subset E$ for all $i$.

Proposition 6 asserts that there is a collection $\Gamma$ of disjoint simple closed curves on the boundary $\partial C$ of the core with the following properties:

(1) $\Gamma$ intersects at least three times every essential simple closed compressible curve on $\partial C$,
(2) $\Gamma$ intersects the boundary of every essential and properly embedded annulus $(A, \partial A) \subset (C, \partial C)$,
(3) $0 = [\Gamma] \in H_1(M; \mathbb{Z})$ and
(4) the collection $\Gamma$ is freely homotopic in $M$ to a collection $\Gamma_*$ of primitive geodesics.

The collection $\Gamma_*$ is perhaps not the disjoint union of simple geodesics in $M$ but this is a minor problem since Canary (Can89; Can93) proved:

**Lemma 14** *Let $U$ be a small neighbourhood of $\Gamma_*$ in $M$. There is a complete metric $g$ of pinched negative curvature which coincides with the hyperbolic metric of $M$ outside $U$ and such that $\Gamma$ is homotopic in $(M, g)$ to a disjoint union $\Gamma_*^g$ of simple geodesics in $(M, g)$ which are contained in $U$.* □

The collection $\Gamma_*^g$ is homologically trivial and, therefore, there is an embedded surface $\Sigma \subset M$ with $\partial\Sigma = \Gamma^g_*$. The surface $\Sigma$ induces a 3–fold cyclic branched cover $\pi : M^3 \to M$. In particular we have an action of the cyclic group $\mathbb{Z}_3$ on $M^3$ with $M^3/\mathbb{Z}_3 = M$. A priori the manifold $M^3$ might have infinitely generated fundamental group.

Let $K$ be a compact set which contains the surface $\Sigma$, the core $C$ and the track of a homotopy of $\Gamma^g_*$ to $\Gamma$, and let $K^3$ be the preimage of $K$ under $\pi$. We may assume that $K \cap S_i = \emptyset$ for all $i$; in particular, $S_i$ lifts homeomorphically to a surface $S^3_i$ in $M^3 - K^3$ for all $i$.

**Lemma 15** *(Can89; Can93) The surface $S^3_i$ is incompressible in $M^3$ for all $i$ and, if a simple curve $\gamma$ on $S^3_i$ can be homotoped into a cusp of $M^3$, then the curve $\pi(\gamma)$ can be homotoped in $M - C$ into a cusp of $M$.*

**Proof** Assume that $S^3_i$ is compressible in $M^3$. Then, by the equivariant Dehn lemma (MY81) there is an embedded disk $D^3$ in $M^3$ with $\partial D^3 \subset S^3_i$ which is either $\mathbb{Z}_3$–invariant or such that $\mathbb{Z}_3 D^3$ is the disjoint union of 3–disks. In any case $D = \pi(D^3)$ is an embedded disk in $M$ with $\partial D \subset S_i$ and $\pi|_{D^3} : D^3 \to D$ is a possibly trivial branched cover. The surface $S_i$ is incompressible in $E$ and thus, $D$ intersects $\Gamma$ essentially at least three times; $\Gamma$ is homotopic to $\Gamma^*_g$ outside of $\partial D \subset M - K$ and therefore, $D$ intersects $\Gamma^*_g$ essentially at least three times. In particular we obtain that $\pi|_{D^3} : D^3 \to D$ is a 3–fold branched cover with at least 3 branching points. The Riemann–Hurewitz formula yields a contradiction. We have proved that $S^3_i$ is incompressible in $M^3$.

The proof of the second claim follows in a similar way using the equivariant annulus theorem (MS86). □

In order to prove Theorem 1 our next step is:

**Proposition 16** *The manifold $M^3$ is irreducible and the surfaces $(S^3_i)_i$ represent only finitely many homotopy classes in $M^3$.*

Let us first finish the proof of Theorem 1, assuming Proposition 16. This proposition implies that, up to the choice of a subsequence, we may assume that the surfaces $S^3_i$ and $S^3_j$ are homotopic for all $i, j$. It is due to Waldhausen (Wal68) that two embedded, homotopic and incompressible surfaces in an irreducible 3–manifold bound a trivial interval bundle. This ensures that, for all $i, j$, the surfaces $S^3_i$ and $S^3_j$ bound a submanifold of $M^3$ homeomorphic to $S^3_i \times [0, 1]$. The covering $\pi$ is one–to–one on this interval bundle; thus, the surfaces $S_i$ and $S_j$ bound an interval bundle in $M$. This yields that the end $[E]$ is tame, and concludes the proof of Theorem 1.

It remains to show Proposition 16.

**Proof of Propositon 16** We continue with the same notation as above.

Gromov and Thurston (GT87) (see also Canary (Can89; Can93)) proved that that the manifold $M^3$ admits a metric $g^3$ of pinched negative curvature such that $\pi$ is a local isometry outside a small neighborhood of the branching locus; for simplicity we are going to write $M^3$ instead of $(M^3, g^3)$. In particular, since $M^3$ is negatively curved we deduce that it is irreducible. It remains to show that, up to choice of a subsequence, the surfaces $S_i^3$ and $S_j^3$ are homotopic in $M^3$ for all $i, j$.

For all $i$, let $P_i$ be a maximal collection of disjoint non–parallel simple closed curves on $S_i^3$ such that each curve in $P_i$ can be homotoped into a cusp of $M^3$. By Lemma 15 we know that, for all $\gamma$ in $P_i$, the curve $\pi(\gamma)$ can be homotoped in $M - C$ into a cusp of $M$. In particular $\gamma$ cannot be homotoped into a rank–two cusp of $M^3$. We have proved

**Lemma 17** *The maximal abelian subgroup of $\pi_1(M^3)$ which contains $\gamma \in P_i$ is cyclic* $\square$

For all $i$ let $\mathcal{N}_i$ be a regular neighborhood of $P_i$ in $S_i$ and let $Y_i^3$ be the surface $S_i - \mathcal{N}_i$. The surface $Y_i^3$ is $\pi_1$–injective, has no accidental parabolics and every boundary curve of $Y_i^3$ can be homotoped into a cusp of $M^3$. Lemma 7 shows that every component of $Y_i^3$ is homotopic to a simplicial ruled surface.

**Lemma 18** *There is a compact set $K_0^3 \subset M^3$ such that every component of the surface $Y_i^3$ is homotopic to a simplicial ruled surface which intersects $K_0^3$.*

**Proof** First recall the definition of the set $K$ above and that $K^3 = \pi_1(K^3)$. For a fixed $i$ let $Z^3$ be a simplicial ruled surface which is homotopic to a component of $Y_i^3$. If the surface $Z^3$ intersects $K^3$ we are done; so we may assume that $Z^3$ does not intersect the branching locus of $\pi$. Denote by $Z$ the projection $\pi(Z^3)$ of $Z^3$ to $M$; $Z$ is a simplicial hyperbolic surface in $M$. Since $Z^3$ is $\pi_1$-injective in $M^3$ we deduce as in Lemma 15 that $Z$ is $\pi_1$-injective in $M - K$. Further, notice that there is $A > 0$, independent of $Z$ and of $i$ such that $|\chi(Z)| \leq A$.

If $Z$ is not $\pi_1$–injective in $M$ then Proposition 13 yields an interpolation $(S_t)_{t\in[0,1]}$ with $S_0 = Z$ and $S_1 \cap \mathcal{N}_1(K) \neq \emptyset$; set $t_0 = \inf\{s | S_t \cap K \neq \emptyset\}$. The homotopy $(S_t)_{t\in[0,t_0]}$ does not intersect $K$ and thus, lifts to a homotopy $(S_t^3)_{t\in[0,t_0]}$ in $M^3$ such that $S_0^3 = Z^3$ and $S_{t_0}^3$ is a simplicial ruled surface with $S_{t_0}^3 \cap \mathcal{N}_1(K^3) \neq \emptyset$.

Now assume that the surface $Z$ is $\pi_1$–injective in $M$. If $Z$ has an accidental parabolic, then we deduce as above that every homotopy of a curve on $Z$ which represents an accidental parabolic must intersect $K$. Proposition 13 yields again an interpolation which connects $Z$ to a simplicial ruled surface $Z'$ which intersects $\mathcal{N}_1(K)$. We proceed as in the previous case.

It remains to consider the case that $Z$ is $\pi_1$–injective and type–preserving. By Corollary 9 there are only finitely many proper homotopy classes of type preserving $\pi_1$–injective simplicial ruled surfaces $S$ in $M$ with $|\chi(S)| \leq A$. Let $K_0$ be a compact set in $M$ containing $\mathcal{N}_1(K)$ and intersecting a simplicial ruled surface in each one of these homotopy classes and set $K_0^3 = \pi^{-1}(K_0)$. By Proposition 13 there is an interpolation $(S_t)_{t\in[0,1]}$ with $S_0 = Z$, and $S_1 \cap K_0 \neq \emptyset$. The same argument as above yields a simplicial ruled surface in $M^3$ which intersects $K_0^3$ and which is homotopic to $Z^3$. □

We can now complete the proof of Proposition 16. The surfaces $Y_i^3$ are $\pi_1$–injective, type preserving and have bounded Euler–characteristic; it follows from Lemma 18 and from Proposition 8 that, up to the choice of a subsequence, the surfaces $Y_i^3$ and $Y_j^3$ are homotopic for all $i, j$. This implies that the surface $S_i^3$ is homotopic to the union of $Y_1^3$ and a collection $\tilde{\mathcal{N}}_i$ of annuli which connect boundary components of $Y_1^3$ for all $i$. The surfaces $S_i^3$ and $S_j^3$ are homotopic if $\tilde{\mathcal{N}}_i$ and $\tilde{\mathcal{N}}_j$ are homotopic by a homotopy fixing $\partial Y_1^3$.

Taking if necessary a subsequence we can assume that for all $i, j$ and for each annulus $A_i$ in $\tilde{\mathcal{N}}_i$ there is an annulus $A_j$ in $\tilde{\mathcal{N}}_j$ such that $A_i$ and $A_j$ connect the same boundary components of $Y_1^3$. The union of both annuli is a torus which is compressible by Lemma 17. This implies that the annulus $A_i$ can be homotoped to $A_j$ by a homotopy fixing $\partial Y_1^3$. We have proved that, passing if necessary to a subsequence, the surfaces $S_i^3$ and $S_j^3$ are homotopic for all $i, j$. □

## 5 Proof of Corollary 2 and Theorem 3

**Corollary 2** *If a complete orientable hyperbolic 3–manifold $M$ with finitely generated fundamental group is a nested union of cores, then $M$ is tame.*

**Proof** Let $(C_i)_i$ be a family of cores of $M$ with $C_i \subset C_{i+1}$ for all $i$ and such that $M = \cup C_i$; remark that $C_0$ is a core of $C_i$ for all $i$. Recall that the cores $C_i$ and $C_j$ are homeomorphic for all $i, j$ (MMS85). Corollary 2 follows from Theorem 1 if we show that, for all $i$, the surface $\partial C_i$ is incompressible in $M - C_0$. Let $D \subset M - C_0$ be an embedded disk with $\partial D = D \cap \partial C_i$; the disk $D$ is either contained in $C_i$ or in $M - C_i$.

If $D \subset C_i$, then $D$ must separate $C_i$: otherwise $\pi_1(C_i) = \pi_1(C_i - D) * _1$, contradicting the fact that $C_0 \subset C_i - D$ is a core of $C_i$. Let $X$ be the component of $C_i - D$ which contains $C_0$ and set $Y = C_i - X$. It follows from the Seifert–van Kampen theorem that $\pi_1(Y) = 1$; but this implies that $D$ can be homotoped fixing the boundary, to $\partial C_i$.

Consider now the case $D \subset M - C_i$. The curve $\partial D \subset \partial C_i$ is compressible in $M$; therefore, since $C_i$ is a core of $M$, there is an embedded disk $D' \subset C_i$ with $\partial D' = \partial D$. The sphere $D \cup D'$ bounds a ball $B$; thus there is a disk $\bar{D} \subset \partial C_i$ such that $D' \cup \bar{D}$ is a boundary component of $B \cap C_i$; hence $\partial \bar{D} = \partial D' = \partial D$.

We have proved that $\partial C_i$ is incompressible in $M - C_0$ for all $i$. As remarked above, the claim follows from Theorem 1. □

**Theorem 3** *Let $[E]$ be an end of a complete oriented hyperbolic 3–manifold $M$ such that $\pi_1(M)$ is finitely generated but not free. If there is a sequence $(X_i)_i$ of surfaces homotopic in $M$ to $\partial E$ which converge to $[E]$ when $i$ goes to $\infty$, then the end $[E]$ is tame.*

**Proof** As above we assume that $X_i \subset E$ for all $i$. We are going to show that there is a sequence $(S_i)_i$ of embedded incompressible surfaces in $E$ with $|\chi(S_i)| \le |\chi(\partial E)|$ and such that $S_i$ is near to $X_i$ for all $i$. Once this is achieved, Theorem 1 yields Theorem 3.

To begin with, we claim that for all $i$, the surface $X_i$ is not homologically trivial in $E$. Indeed, let $B_E$ be the relative compression body associated to the surface $\partial E$ and let $M_E$ be the cover of $M$ induced by $\pi_1(B_E)$; the compression body $B_E$, as well as the component $E$, the surface $X_i$ and the homotopy between $X_i$ and $\partial E$ lift to $M_E$. Since $\pi_1(M)$ is not free we obtain that $B_E$ is not a handlebody and this implies that $0 \neq [\partial E] \in H_2(M_E; \mathbb{Z})$; thus, $0 \neq [X_i] \in H_2(E; \mathbb{Z})$, as claimed.

It follows from a theorem of Gabai (see Person (Per93)) that in every neighborhood of $X_i$, there is a, possibly disconnected, embedded surface $Z_i$ with $[Z_i] = [X_i] \in H_2(E; \mathbb{Z})$ and $\chi(Z_i) \ge \chi(X_i)$ and such that no component of $Z_i$ is a sphere or a projective plane. Let $S_i$ be a component of $Z_i$ which is not homologically trivial in $E$. We contend that the surface $S_i$ is incompressible in $E$.

Assume, on the contrary, that $S_i$ is compressible in $E$. We may perform surgery on $S_i$ and find a possibly disconnected embedded surface $S_i' \subset E$ with $[S_i'] = [S_i] \in H_2(E; \mathbb{Z})$ and $\chi(S_i') > \chi(S_i)$ and such that no component of $S_i'$ is a sphere or a projective plane. Let $S_i''$ be a component of $S_i'$ which is homologically not trivial in $E$. We have

$$\chi(S_i'') \geq \chi(S_i') > \chi(S_i) \geq \chi(Z_i) \geq \chi(X_i) = \chi(\partial E) \tag{4}$$

The surface $S_i''$ separates $M$ because it is not homologically trivial in $E$. Let $A_1$ and $A_2$ be the two components of $M - S_i''$ and assume that $A_1$ contains the core. The component $A_2$ contains one of the surfaces $X_j$ for some $j$. In particular, the images of $\pi_1(A_2)$ and $\pi_1(\partial E)$ in $\pi_1(M)$ are conjugated in $\pi_1(M)$. This implies that the images of $\pi_1(S_i'')$ and $\pi_1(\partial E)$ in $\pi_1(M)$ are also conjugated in $\pi_1(M)$; by Lemma 5 we have $\chi(S_i'') \leq \chi(\partial E)$, which is a contradiction to (4).

We have proved that the embedded surface $S_i$ is incompressible in $E$, and by construction, contained in a small neighborhood of the surface $X_i$, hence Theorem 3 follows from Theorem 1. □

As remarked in the introduction the same proof yields

**Theorem 3 (second version)** *Let $[E]$ be an end of a complete oriented hyperbolic 3–manifold $M$ with $\pi_1(M)$ finitely generated. If there is a sequence $(X_i)_i$ of surfaces homotopic in $M$ to $\partial E$ which converge to $[E]$ when $i$ goes to $\infty$ and such that $0 \neq [X_i] \in H_2(M - C; \mathbb{Z})$ then the end $[E]$ is tame.* □


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