TWO APPLICATIONS OF THE NATURAL MAP

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Abstract. Using a method which has been recently introduced by Besson, Courtois and Gallot we give new simple proofs of Ahlfors' measure theorem and of Canary's estimate of the Hausdorff-dimension of the limit set of a Kleinian group in terms of the volume of the convex-core.

Let Γ be a finitely generated discrete torsion free non-elementary subgroup of $\text{Isom}(\mathbb{H}^n)$, the isometry-group of n-dimensional hyperbolic space. The quotient $M_\Gamma = \mathbb{H}^n/\Gamma$ is an n-dimensional hyperbolic manifold with fundamental group Γ. The action of $\text{Isom}(\mathbb{H}^n)$ on $\mathbb{H}^n$ extends to an action by Möbious transformations of $\mathbb{S}^{n-1}$, the boundary at infinity of $\mathbb{H}^n$ and there is a unique minimal Γ-invariant subset Λ_Γ of $\mathbb{S}^{n-1}$ called *limit set* of Γ. The convex-hull $CH(\Lambda_\Gamma)$ of the limit set Λ_Γ in $\mathbb{H}^n$ is also Γ-invariant and the quotient $CC(M_\Gamma) = CH(\Lambda_\Gamma)/\Gamma$, the *convex-core* of M_Γ, is the smallest convex closed submanifold of M_Γ whose inclusion is a homotopy equivalence. In the case that the convex-core has finite volume then the group Γ is said to be *geometrically finite*. Ahlfors [2] proved

Theorem 1 (Ahlfors' measure theorem)**.** *Let Γ be a geometrically finite subgroup of $\text{Isom}(\mathbb{H}^n)$. Either Γ has finite co-volume or the Lebesgue measure of its limit set vanishes.*

The Ahlfors measure theorem is also valid for all finitely generated subgroups of $\text{Isom}(\mathbb{H}^3)$ by the work of Thurston [10], Canary [6], Agol [1] and Calegary-Gabai [5].

A more subtle invariant of the group Γ is the Hausdorff-dimension $\dim_{\mathcal{H}}(\Lambda_\Gamma)$ of the limit set. Again in the case $n = 3$, the following theorem due to Canary [7] links the geometry of the manifold M_Γ with $\dim_{\mathcal{H}}(\Lambda_\Gamma)$.

Theorem 2 (Canary)**.** *There is a constant $K > 0$ with*

$$\dim_{\mathcal{H}}(\Lambda_\Gamma) \geq 2 - K\frac{|\chi(M_\Gamma)|}{\text{vol}(CC(M_\Gamma))}$$

for every geometrically finite subgroup Γ of $\text{Isom}(\mathbb{H}^3)$.

Bishop and Jones [4] proved that the Hausdorff-dimension of a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ is equal to 2 if and only if it has either finite co-volume or it is not geometrically finite.
The purpose of this note is to give new and simple proofs of these two theorems using a method which has been recently introduced by Besson, Courtois and Gallot. Let us sketch this method and the proof of Theorem 2. Given such a group Γ we construct a map f from M_Γ to itself, homotopic to the identity and with

$$\mathrm{vol}(f(CC(M_\Gamma))) \leq \frac{\dim_{\mathcal{H}}(\Lambda_\Gamma)}{3}\,\mathrm{vol}(CC(M_\Gamma))$$

On the other hand, the map f is 3-Lipschitz. This implies that there is K independent of Γ with

$$\mathrm{vol}(CC(M_\Gamma)) - \mathrm{vol}(f(CC(M_\Gamma))) \leq K\,\mathrm{vol}(\partial CC(M_\Gamma))$$

A little bit of algebra yields Theorem 2.
The possibility of giving new proofs of these two results with this method arose in a conversation with Pete Storm. I would like to thank Gerard Besson and Gilles Courtois who pointed out a mistake in the proof of Lemma 6.

1. The natural map reloaded

In this section we present a construction due to Besson, Courtois and Gallot which is the main tool we are going to use.
Let Γ be a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$. A *conformal density* of dimension δ for Γ is a family $\{\mu_x | x \in \mathbb{H}^n\}$ of finite measures on $\mathbb{S}^{n-1} = \partial\mathbb{H}^n$ with the following properties:

(1) $\gamma_*\mu_x = \mu_{\gamma(x)}$ for all $\gamma \in \Gamma$.
(2) The measures μ_x and μ_y are absolutely continuous to each other and $\frac{d\mu_x}{d\mu_y}(\theta) = e^{-\delta B_y(x,\theta)}$ for every $\theta \in \mathbb{S}^{n-1}$.

Here $B_y(\cdot,\theta)$ is the Busemann function centered at θ and normalized at y. We consider the two conformal densities described in the following examples:

Example 3. *Let x be a point in $\mathbb{H}^3$. The exponential map determines a homeomorphism from the unit tangent space $T^1_x\mathbb{H}^n$ to the boundary at infinity $\mathbb{S}^{n-1}$. Denote by ν_x the push-forward of the Lebesgue measure on $T^1_x\mathbb{H}^3$. The measure ν_x is said to be the* visual measure *at x. The family $\{\nu_x\}_x$ is a conformal density for the whole isometry group* $\mathrm{Isom}(\mathbb{H}^n)$ *of dimension $n-1$.*
This example can be modified as follows: Assume that U is a Γ-invariant set of positive Lebesgue measure with characteristic function χ_U. Then

the family ν_x^U of measures which are absolutely continuous with respect to ν_x and with $\frac{d\nu_x^U}{d\nu_x}(\theta) = \chi_U(\theta)$ is a conformal density for Γ of dimension $n-1$.

Example 4. *Let Γ be a geometrically finite group. Sullivan [9] proved that there is a conformal density, called Patterson-Sullivan measure, for Γ of dimension $\dim_{\mathcal{H}}(\Lambda_\Gamma)$ and supported by the limit set Λ_Γ of Γ.*

Let from now on Γ be a non-elementary discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ and μ_x be a conformal density for Γ of dimension δ. Following Besson, Courtois and Gallot consider for $x \in \mathbb{H}^n$ the following function on $\mathbb{H}^3$ given by

$$\mathcal{B}_x(y) = \int_{\mathbb{S}^{n-1}} e^{B_x(y,\theta)} d\mu_x(\theta).$$

The function $\mathcal{B}_x$ is smooth and the convexity of the Busemann function implies that $\mathcal{B}_x$ is strictly convex and proper. In particular, $\mathcal{B}_x$ has a unique minimum and we may define a map

$$F : \mathbb{H}^n \to \mathbb{H}^n \quad F(x) = \text{minimum of } \mathcal{B}_x$$

The formula for the change of variables shows that $\mathcal{B}_{\gamma x}(\gamma y) = \mathcal{B}_x(y)$ and hence that the map F is Γ-equivariant. In particular, the map F induces a map $f : M_\Gamma \to M_\Gamma$. Both maps F and f are said to be the *natural maps*[1] induced by the conformal density μ_x.

Proposition 5 (Besson-Courtois-Gallot)**.** *The map F is smooth, homotopic to identity and, for all $p = 1, \ldots, n$, holds $|\mathrm{Jac}_p F| \leq \left(\frac{1+\delta}{p}\right)^p$.*

Before going further, recall that $\mathrm{Jac}_p F$ is the p-Jacobian of F which is formally defined on the Grassmannian of p-dimensional planes of $\mathbb{H}^n$. The proof of Proposition 5 has not yet appeared in the literature, that is why we give a proof for the case $p = n$ in section 3. The general case is proved similarly.

We conclude this section with the following simple lemma.

Lemma 6. *Let H be an open half-space with closure $\bar{H}$. Denote by $\partial_\infty \bar{H}$ the closed half sphere which bounds $\bar{H}$ at infinity and by $\partial \bar{H}$ the totally geodesic hyperplane which bounds $\bar{H}$ in $\mathbb{H}^3$.*

1. *Assume that $\mu_x(\partial_\infty \bar{H}) = 0$ for some, hence for all, $x \in \mathbb{H}^n$, then the image of F does not intersect $\bar{H}$.*
2. *Let U be a proper Γ-invariant subset of $\hat{\mathbb{C}}$ of positive Lebesgue measure and let ν_x^U be the conformal density constructed in Example 3. If $\partial_\infty \bar{H} \subset U$, then $F(\bar{H}) \subset H$.*

[1] Remark that this map is not the original *natural map* used by Besson, Courtois and Gallot in [3].

Proof. Assume that we are in the first case and let y be a point in $\bar{H}$ and $x \in \mathbb{H}^n$. We have to show that y is not the minimum of the function $\mathcal{B}_x$. Let N_y be the unit tangent vector at y pointing perpendicularly to the hyperplane $\partial\bar{H}$. For every θ in the support of μ_x we have $dB_x(\cdot,\theta)(N_y) < 0$; it follows that

$$d\mathcal{B}_x(N_y) = \int_{\mathrm{supp}(\mu_x)} e^{B_x(y,\theta)} dB_x(\cdot,\theta)(N_y) d\mu_x(\theta) < 0.$$

We have proved that $d\mathcal{B}_x$ does not vanish at y and, therefore, $F(x) \neq y$. We consider now the second case. Let x be a point in H and $y \notin \bar{H}$ and let $P \subset H$ be the unique hyperplane which contains x and which is perpendicular to the geodesic starting at y with direction N_y. As above we want to show that $d\mathcal{B}_x(N_y)$ does not vanish. We compute

$$\begin{aligned}(1) \quad d\mathcal{B}_x(N_y) &= \int_U e^{B_x(y,\theta)} dB_x(\cdot,\theta)(N_y) d\mu_x \\ &= \int_{U\cap\sigma U} e^{B_x(y,\theta)} dB_x(\cdot,\theta)(N_y) d\mu_x + \int_{U\setminus\sigma U} e^{B_x(y,\theta)} dB_x(\cdot,\theta)(N_y) d\mu_x\end{aligned}$$

where σ is the isometric reflection at the hyperplane P. The set $U \setminus \sigma U$ is contained in H, hence, by the same argument as in the first case we deduce that

$$\int_{U\setminus\sigma U} e^{B_x(y,\theta)} dB_x(\cdot,\theta)(N_y) d\mu_x < 0. \tag{2}$$

On the other hand, $\int_{U\cap\sigma U} e^{B_x(y,\theta)} dB_x(\cdot,\theta)(N_y) d\mu_x$ is the differential of the convex function $y \mapsto \int_{U\cap\sigma U} e^{B_x(y,\theta)} d\mu_x$. This function is σ-invariant since $\sigma(x) = x$ and therefore its value at y and $\sigma(y)$ coincide. We deduce from the convexity that

$$\int_{U\cap\sigma U} e^{B_x(y,\theta)} dB_x(\cdot,\theta)(N_y) d\mu_x \leq 0 \tag{3}$$

The equations (2) and (3) together yield the claim. □

2. The proofs

We show first the Ahlfors' measure theorem. Let Γ be a geometrically finite subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ and assume that $M_\Gamma = \mathbb{H}^n/\Gamma$ has infinite volume. In particular, the complement Ω of the limit set Λ_Γ of Γ in $\mathbb{S}^{n-1}$ is not empty. Seeking for a contradiction assume that the set Ω fails to have full measure. Consider the conformal density ν_x^Ω for Γ of dimension $n-1$ constructed in Example 3 and let $f : M_\Gamma \to M_\Gamma$ be the natural map induced by ν_x^Ω.

Lemma 7. *f is proper and has degree one.*

Proof. If M_Γ is convex-cocompact then we can compactify M_Γ by $\bar{M}_\Gamma = (\mathbb{H}^n \cup (\partial_\infty \mathbb{H}^n \setminus \Lambda_\Gamma))/\Gamma$ and Lemma 6 implies that the map f can be extended continuously to a map $\bar{f} : \bar{M}_\Gamma \to M_\Gamma$ which is the identity on $\bar{M}_\Gamma \setminus M_\Gamma$. This implies that f is proper and has degree one.
In general $\bar{M}_\Gamma$ is not a compactification of M_Γ for there may be cusps. It remains to show the for any sequence (x_i) in $\mathbb{H}^n$ which converges to a point $\theta \in \mathbb{S}^{n-1}$ whose stabilizer Γ_θ in Γ is not trivial and consists of parabolic elements we have $F(x_i) \to \theta$. To begin with, assume that the points x_i remain at a bounded distance of the convex-hull $CH(\Lambda_\Gamma)$. Then there is $\gamma \in \Gamma_\theta$ with $d(x_i, \gamma x_i) \to 0$; since F is Lipschitz and Γ-equivariant we obtain $d(F(x_i), \gamma F(x_i)) \to 0$ which implies $F(x_i) \to \theta$. It remains to consider the case that the points x_i are further and further away from the convex-hull. We may choose a nested sequence of half-spaces $H_1 \supset H_2 \supset \dots$ which are disjoint of the convex-hull, with $x_i \in H_i$ for all i and such that for any sequence (y_i) with $y_i \in H_i$ we have $y_i \to \theta$. We may apply Lemma 6 to the half-space H_i for i and obtain $F(x_i) \in H_i$. The result follows. □

Lemma 6 (2) and Lemma 7 imply that f maps $CC(M_\Gamma)$ onto a neighborhood of itself; in particular $\mathrm{vol}(CC(M_\Gamma)) < \mathrm{vol}(f(CC(M_\Gamma)))$. Proposition 5 yields the desired contradiction; this concludes the proof of the Ahlfors measure theorem.

We prove now Theorem 2. Let Γ be a geometrically finite subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ and let M_Γ be the associated hyperbolic manifold. Without loss of generality we may assume that the convex-core $CC(M_\Gamma)$ has not vanishing volume. It is due to Thurston [8] that the boundary of the convex-core is, with its induced metric, a complete hyperbolic surface of finite type with $\chi(\partial CC(M_\Gamma)) = 2\chi(M_\Gamma)$. In particular, $\partial CC(M_\Gamma)$ has volume equal to $4\pi|\chi(M_\Gamma)|$.
Let δ be the Hausdorff-dimension of the limit set Λ_Γ, μ_x the Patterson-Sullivan measures for the group Γ (Example 4) and let $f : M_\Gamma \to M_\Gamma$ and $F : \mathbb{H}^3 \to \mathbb{H}^3$ the natural maps corresponding to the conformal density μ_x. To begin with remark that the same argument as in Lemma 7 implies that the restriction $f|_{CC(M_\Gamma)}$ of f to the convex-core of M_Γ is a proper map. It follows from Proposition 5 that

$$\mathrm{vol}(f(CC(M_\Gamma))) \leq \left(\frac{1+\delta}{3}\right)^3 \mathrm{vol}(CC(M_\Gamma)) \leq \frac{1+\delta}{3}\,\mathrm{vol}(CC(M_\Gamma)).$$

On the other hand we have

$$\mathrm{vol}(CC(M_\Gamma)) \leq \mathrm{vol}(f(CC(M_\Gamma))) + \mathrm{vol}(g(\partial CC(M_\Gamma) \times [0,1])) \tag{4}$$

where $g(\{x\}\times[0,1])$ is the geodesic segment, in the correct homotopy class, which joins x and $f(x)$. For every unit tangent vector $v\in T^1_x\partial CC(M_\Gamma)$ let $J_v(t)\in T_{g(x,t)}M_\Gamma$ be the component of $dg|_{(x,t)}(v)$ which is orthogonal to $dg|_{(x,t)}\frac{\partial}{\partial t}$. Jacobi-fields in constants curvature spaces are given by explicit formulas; a computations yields that

$$\|J_v(t)\|\leq 2\frac{\max\{\|J_v(0)\|,\|J_v(1)\|\}}{\sinh(d_x)}\left(\sinh(d_x t)+\sinh(d_x(1-t))\right)$$

where d_x is the length of the geodesic segment $g(\{x\}\times[0,1])$. The map f is 3-Lipschitz by Proposition 5 hence $\max\{\|J_v(0)\|,\|J_v(1)\|\}\leq 3$. The transformation formula implies now that $\operatorname{vol}(g(\partial CC(M_\Gamma)\times[0,1]))$ is bounded from above by

$$\int_{\partial CC(M_\Gamma)}\int_0^{d_x}\left(2\cdot 3\frac{\sinh(t)+\sinh(d_x-t)}{\sinh(d_x)}\right)^2 dtdx$$

Setting

$$k=\max_{d\geq 0}\left\{\int_0^d\left(\frac{\sinh(t)+\sinh(d-t)}{\sinh(d)}\right)^2 dt\right\}$$

we obtain from equation (4)

$$\operatorname{vol}(CC(M_\Gamma))-144\pi k\operatorname{vol}(\chi(M_\Gamma))\leq\frac{1+\delta}{3}\operatorname{vol}(CC(M_\Gamma)).$$

A little bit of algebra yields the desired result. This concludes the proof of Canary's estimate.

3. Proof of Proposition 5

As promised above we show now

Proposition 5 (Besson-Courtois-Gallot)**.** *The map F is smooth and for all $p=1,\dots,n$ holds $|\mathrm{Jac}_pF|\leq\left(\frac{1+\delta}{p}\right)^p$.*

We restrict ourselves to give a proof for $p=n$ but the general case can be easily proved in the same way.

Proof. Recall that $F(x)=y$ iff $d(\mathcal{B}_x)|_y=0$. In particular we obtain the implicit defining equation $d(\mathcal{B}_x)|_{F(x)}=0$ for F. If x_t is a curve in $\mathbb{H}^n$ then we obtain for all t

$$0=d(\mathcal{B}_{x_t})|_{F(x_t)}=\int e^{B_{x_t}(F(x_t),\theta)}dB_{x_t}(\cdot,\theta)|_{F(x_t)}d\mu_{x_t}(\theta)$$

where we integrate over $\mathbb{S}^{n-1}$. We can differentiate this equality, taking care of the fact that $dB_o(\cdot,\theta)=dB_x(\cdot,\theta)$ for all $o,x\in\mathbb{H}^3$ and we obtain that

$$
(5)\quad 0 = \int [\left(\frac{d}{dt} B_{x_t}(F(x_t),\theta)\right) dB_o(\cdot,\theta)|_{F(x)} \\
+ \nabla_{dF\dot{x}} dB_o(\cdot,x) - \delta dB_o(\cdot,\theta)|_{F(x)} dB_o(\cdot,\theta)(\dot{x})] e^{B_x(F(x),\theta)} d\mu_x(\theta)
$$

For simplicity we consider the measure $\nu_x = e^{B_x(F(x),\theta)}\mu_x$. If we plug the formulas

$$
B_x(F(x),\theta) = B_o(F(x),\theta) - B_o(x,\theta)
$$

$$
\nabla dB_o(\cdot,\theta) = g - dB_o(\cdot,\theta) \otimes dB_o(\cdot,\theta)
$$

in equation (5) we obtain

$$
0 = \int g(dF\dot{x},\cdot) - (1+\delta) dB_o(\cdot,\theta)|_{F(x)} dB_o(\cdot,\theta)(\dot{x}) d\nu_x,
$$

and hence, normalizing ν_x to a probability measure, we have proved that

$$
dF = (1+\delta) \int \nabla B_o(\cdot,\theta)|_{F(x)} \otimes dB_o(\cdot,\theta)_x d\nu_x.
$$

Now the Cauchy-Schwarz inequality implies that for all $u \in T_x\mathbb{H}^n$ we have

$$
(6)\qquad |dFu| \leq (1+\delta)\langle H_x dFu, dFu\rangle^{\frac{1}{2}} \langle H'_x u, u\rangle^{\frac{1}{2}}
$$

where H_x and H'_x are the endomorphism of $T_{F(x)}\mathbb{H}^3$ and $T_x\mathbb{H}^3$ given by

$$
H_x = \int \nabla B_o(\cdot,\theta)|_{F(x)} \otimes dB_o(\cdot,\theta)|_{F(x)} d\nu_x,
$$

$$
H'_x = \int \nabla B_o(\cdot,\theta)|_x \otimes dB_o(\cdot,\theta)|_x d\nu_x.
$$

Inequality (6) and an elemental lemma of linear algebra [3, Lemma 2.4] imply that

$$
|\mathrm{Jac}_n F| \leq (1+\delta)^n (\det H_x)^{\frac{1}{2}} (\det H'_x)^{\frac{1}{2}}.
$$

Since both H_x and H'_x are symmetric positive definite endomorphisms with trace $\operatorname{Tr} H_x, \operatorname{Tr} H'_x \leq 1$ we obtain that $\det H_x, \det H_x \leq n^{-n}$. We obtain the desired inequality

$$
|\mathrm{Jac}_n F| \leq \left(\frac{1+\delta}{n}\right)^n.
$$

□

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