VOLUME RIGIDITY FOR FINITE VOLUME MANIFOLDS

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Abstract. We extend to finite volume manifolds some volume rigidity results of Besson-Courtois-Gallot.

1. Introduction

In this paper we prove the following result.

Main Theorem (Volume Theorem). *Let (M, g) and (M_o, g_o) be two oriented complete finite volume riemannian manifolds of the same dimension $n \geq 3$ and suppose that*

$$Ric_g \geq -(n-1)g, \quad and \quad -a \leq K_{g_o} \leq -1.$$

Then for all proper continuous maps $f : M \to M_o$,

$$\mathrm{vol}(M, g) \geq |\deg(f)| \,\mathrm{vol}(M_o, g_o),$$

and equality holds if and only if f is proper homotopic to a Riemannian covering.

When M and M_o are compact, the Volume Theorem follows from a real Schwarz lemma proved by Besson, Courtois, and Gallot in [BCG98], and the reader may consult [BCG95], [BCG96], and [BCG98] for related results. To apply the method of [BCG96] and [BCG98], we face the fundamental difficulty of proving the properness of the natural map. Though immediate in the compact case, we use the Bishop-Gromov comparison theorem to prove it under our finite volume assumption.

Recall that the minimal volume $\mathrm{Minvol}(M)$ of a manifold M is the infimum of the volumes of all metrics on M with sectional curvature in $[-1, 1]$. From the Main Theorem we get the

Corollary 1.1. *Under the hypotheses of the Main Theorem,*

$$\mathrm{Minvol}(M) \geq \deg(f) \,\mathrm{vol}(M_o).$$

Now restrict f to be degree 1, and consider what happens when $\mathrm{Minvol}(M) = \mathrm{vol}(M_o)$. Bessieres proved in [Bessieres98] that if there is a degree 1 map $f : M \to M_o$ from the compact n-dimensional M to the compact n-dimensional real hyperbolic M_o such that $\mathrm{Minvol}(M) = \mathrm{vol}(M_o)$, then M and M_o are diffeomorphic. He also gave examples in [Bessieres99] of a finite volume manifold M and a hyperbolic manifold M_o such that there is a degree 1 map and $\mathrm{Minvol}(M) = \mathrm{vol}(M_o)$ but M and M_o are not even homeomorphic. In spite of this however, our next result shows that asymptotically, they are actually isomteric.

Theorem 1.2 (Minimal Volume). *Let M and M_o be finite volume manifolds of the same dimension $n \geq 3$, M_o real hyperbolic, and $f : M \to M_o$ a continuous, proper, degree 1 map. If* $\mathrm{Minvol}(M) = \mathrm{vol}(M_o)$*, then for any sequence of metrics*

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g_i realizing the minimal volume of M, there are $p_i \in M$ and a subsequence g_{i_j} such that (M, p_{i_j}, g_{i_j}) converges in the pointed Lipschitz topology to Riemannian manifold isometric to M_o.

We conclude this introduction by stating for reference without proof a volume growth entropy rigidity result for finite volume manifolds. The reader may consult [BCG95] for the defintion of volume growth entropy.

Theorem 1.3 (Entropy Rigidity)**.** *Let (M, g) be an n-dimensional finite volume manifold of nonpositive sectional curvature, $n \geq 3$, and $h(g)$ its volume growth entropy. Let (M_o, g_o) be an n-dimensional finite volume rank one locally symmetric manifold and $h(g_o)$ its volume growth entropy. If $f : M \to M_o$ is a continuous, proper map of degree $deg(f) > 0$, then $h(g)^n \operatorname{vol}(M, g) \geq \deg(f) h(g_o)^n \operatorname{vol}(M_o, g_o)$ and equality holds if and only if f is proper homotopic to a Riemannian covering.*

As in [BCG95], this gives a quick proof of the Mostow rigidity theorem for finite volume negatively curved locally symmetric manifolds.

2. Preliminaries

We begin by recalling the construction of the natural map. For each $s > 0$ and $x \in \widetilde{M}$ consider the measure μ_x^s on $\widetilde{M}$ in the Lebesgue class with density

$$\frac{d\mu_x^s}{d\operatorname{vol}_g}(z) = e^{-sd(x,z)}.$$

It follows from the Bishop-Gromov comparison theorem (stated here since we will use it again later) that for $s > n-1$ and $x \in \widetilde{M}$, the integral $\int_{\widetilde{M}} e^{-sd(x,z)} d\operatorname{vol}_g(z)$ is finite, so the measure μ_x^s is finite.

Theorem 2.1 (Bishop-Gromov comparison theorem)**.** *Let (N, g) be an n-dimensional complete Riemannian manifold and suppose $Ric_g \geq -(n-1)g$. Denoting by $V(R)$ the volume of a ball of radius R in $\mathbb{H}^n$, for all $0 < r \leq R$*

$$\frac{\operatorname{vol}(B_p(R))}{\operatorname{vol}(B_p(r))} \leq \frac{V(R)}{V(r)}.$$

Consider the push-forward measure $\widetilde{f}_*\mu_x^s$ on $\widetilde{M}_o$, and define a measure σ_x^s on $\partial\widetilde{M}_o$ in the following way. For $z \in \widetilde{M}_o$, let ν_z be the visual probability measure on $\partial\widetilde{M}_o$, and for $U \subset \partial\widetilde{M}_o$ measurable define

$$\sigma_x^s(U) = \int_{\widetilde{M}_o} \nu_z(U) d(\widetilde{f}_*\mu_x^s)(z).$$

That is, we take σ_x^s to be convolution of the push-forward measure $\widetilde{f}_*\mu_x^s$ with the visual measures ν_z. (This is a difference between our approach and that of [BCG96], [BCG98], where the push-forward measure is convolved with the harmonic measures.) Notice that for all s, x, $\|\mu_x^s\| = \|\sigma_x^s\|$, so the measure σ_x^s is finite for $s > n-1$.

Denoting by $B_o(y, \theta)$ the Busemann function of (M_o, g_o) (normalized so that $B_o(O, \cdot) = 0$ for some fixed origin $O \in \widetilde{M}_o$), consider the function on $\widetilde{M}_o$ defined by

$$\mathcal{B}_{s,x}(y) = \int_{\partial\widetilde{M}_o} B_o(y, \theta) d\sigma_x^s(\theta).$$

This is a proper strictly convex function, hence it has a unique minimum [BCG95], which we call the *barycenter* of the measure σ_x^s and denote by $\mathrm{Bar}(\sigma_x^s)$. For all $s > n-1$, this gives us a map $\widetilde{F}_s : \widetilde{M} \to \widetilde{M}_o$ defined by $x \mapsto \mathrm{Bar}(\sigma_x^s)$. It is equivariant under the action of $\pi_1(M)$ and $\pi_1(M_o)$ and so descends to the *natural map* $F_s : M \to M_o$. The following is a collection and restatement of some of the important properties of the natural map.

Theorem 2.2 (Besson-Courtois-Gallot)**.** [BCG95], [BCG98] *For all $s > n-1$ and all $x \in M$,*

- *The natural map is C^1.*
- *The map $\Psi_s : [0,1] \times M \to M_o$ defined by $\Psi_s(t,x) = F_{s+\frac{t}{1-t}}(x)$ gives a continuous homotopy between $\Psi_s(0,x) = F_s(x)$ and $\Psi_s(1,x) = f(x)$.*
- $|\operatorname{Jac}(F_s)(x)| \leq \left(\frac{s}{n-1}\right)^n$.

¿From the definition of the barycenter as a minimum, $\widetilde{F}_s(x) = y$ if and only if the gradient of $\mathcal{B}_{s,x}$ vanishes at y, $\nabla_y \mathcal{B}_{s,x}(y) = 0$. Later we will need the following expression for this gradient:

$$\nabla_y \mathcal{B}_{s,x}(y) = \int_{\partial \widetilde{M}_o} n_\theta(y) d\sigma_x^s(\theta) = \int_{\widetilde{M}_o} \int_{\partial \widetilde{M}_o} n_\theta(y) d\nu_z(\theta) d(\widetilde{f}_* \mu_x^s)(z),$$

where $n_\theta(y)$ is the unit vector in $S_y \widetilde{M}_o$ pointing to $\theta \in \partial \widetilde{M}_o$.

3. Properness of the Natural Map

In this section we prove the following result, which is essential for proving the Main Theorem.

Theorem 3.1. *For all $s > n-1$, the natural map F_s is proper, and* $\deg(F_s) = \deg(f)$.

First we collect some facts we need. The first tells us that if a convex set contains most of the push-forward measure, then the barycenter is close.

Lemma 3.2. *There is $D > 0$ such that for any convex set $K \subset \widetilde{M}_o$ and $x \in \widetilde{M}$, if $\widetilde{f}_* \mu_x^s(K) \geq \frac{9}{10} |\widetilde{f}_* \mu_x^s|$, then $d_{\widetilde{M}_o}(\widetilde{F}_s(x), K) \leq D$.*

Proof. For $y \in \widetilde{M}_o - K$ let $v_K(y)$ be the unit vector in $T_y \widetilde{M}_o$ pointing to the point in K closest to y [BGS85] and

$$\Sigma_y(K) = \{\theta \in \partial \tilde{M}_o \mid g_o(n_\theta(y), v_K(y)) \leq \frac{9}{10}\}.$$

By Toponogov's comparison theorem [Sakai96] and the upper curvature bound on $\widetilde{M}_o$, there is D such that for all convex sets K, all y with $d_{\widetilde{M}_o}(y, K) > D$, and all $z \in K$, the visual measure is small: $\nu_z(\Sigma_y(K)) < \frac{1}{10}$. It follows that

$$\begin{aligned}\int_{\partial \widetilde{M}_o} g_o(n_\theta(y), v_K(y)) d\nu_z(\theta) &\geq -\nu_z(\Sigma_y(K)) + \int_{\partial \widetilde{M}_o - \Sigma_y(K)} g_o(n_\theta(y), v_K(y)) d\nu_z(\theta) \\ &\geq -\nu_z(\Sigma_y(K)) + \frac{9}{10}(1 - \nu_z(\Sigma_y(K))) \\ &\geq -\frac{1}{10} + \frac{9}{10}\frac{9}{10} \geq \frac{1}{3}.\end{aligned}$$

Therefore

$$\begin{aligned}
\|\nabla_y \mathcal{B}_{s,x}(y)\| &= \left\| \int_{\widetilde{M}_o} \int_{\partial \widetilde{M}_o} n_\theta(y) d\nu_z(\theta) d(\widetilde{f}_* \mu_x^s)(z) \right\| \\
&\geq \int_K \int_{\partial \widetilde{M}_o} \langle n_\theta(y), v_y \rangle \, d\nu_z(\theta) d(\widetilde{f}_* \mu_x^s)(z) - \widetilde{f}_* \mu_x^s(\widetilde{M}_o - K) \\
&\geq \|\mu_x^s\| \left(\frac{\widetilde{f}_* \mu_x^s(K)}{\|\mu_x^s\|} \frac{1}{3} - \frac{\|\mu_x^s\| - \widetilde{f}_* \mu_x^s(K)}{\|\mu_x^s\|} \right) \\
&\geq \|\mu_x^s\| \left(\frac{9}{10} \frac{1}{3} - \frac{1}{10} \right) = \frac{1}{5} \|\mu_x^s\| \neq 0.
\end{aligned}$$

This shows that $y \neq \widetilde{F}_s(x)$ and so the barycenter $\widetilde{F}_s(x)$ must be within D of K, as claimed. □

Recall that $V(R)$ denotes the volume of a ball of radius R in $\mathbb{H}^n$. We define a function $f_s : \mathbb{R}^+ \cup \{\infty\} \to \mathbb{R}^+ \cup \{\infty\}$ by

$$f_s(R) = \int_{\{d_{\mathbb{H}^n}(0,z) \leq R\}} e^{-s d_{\mathbb{H}^n}(0,z)} d \operatorname{vol}_{\mathbb{H}^n}(z) = \int_0^R e^{-st} V'(t) dt.$$

It is well known that $V(R) = \operatorname{vol}(\mathbb{S}^{n-1}) \int_0^R \sinh^{n-1} r dr$, and so if $s > n-1$, then $f_s(\infty) < \infty$. We will also need later that for $t > s > n-1$, $\frac{f_t(R)}{f_t(\infty)} \geq \frac{f_s(R)}{f_s(\infty)}$.

Lemma 3.3. *For $s > n-1$, $x \in \widetilde{M}$, and $R > 0$ we have*

$$\begin{aligned}
\mu_x^s(B_x(R)) &\geq f_s(R) \frac{\operatorname{vol}(B_x(R))}{V(R)} \quad \textit{and} \\
\mu_x^s(\tilde{M} - B_x(R)) &\leq (f_s(\infty) - f_s(R)) \frac{\operatorname{vol}(B_x(R))}{V(R)}.
\end{aligned}$$

Proof. These both follow from the Bishop-Gromov comparison theorem and integration by parts:

$$\begin{aligned}
\mu_x^s(B_x(R)) &= \int_{B_x(R)} e^{-s d(x,z)} d \operatorname{vol}(z) = \int_0^R e^{-st} \operatorname{vol}'(B_x(t)) dt \\
&= e^{-sR} \operatorname{vol}(B_x(R)) + s \int_0^R e^{-st} \operatorname{vol}(B_x(t)) dt \\
&\geq \frac{\operatorname{vol}(B_x(R))}{V(R)} \left(e^{-sR} V(R) + s \int_0^R e^{-st} V(t) dt \right) \\
&= \frac{\operatorname{vol}(B_x(R))}{V(R)} f_s(R),
\end{aligned}$$

and

$$\begin{aligned}\int_{\tilde{M}-B_x(R)} e^{-sd(x,z)} d\,\mathrm{vol}(z) &= \int_R^\infty e^{-st}\,\mathrm{vol}'(B_x(t))dt\\ &= -e^{-sR}\,\mathrm{vol}(B_x(R)) + s\int_R^\infty e^{-st}\,\mathrm{vol}(B_x(t))dt\\ &\leq \frac{\mathrm{vol}(B_x(R))}{V(R)}\left(-e^{-sR}V(R) + s\int_0^R e^{-st}V(t)dt\right)\\ &= \frac{\mathrm{vol}(B_x(R))}{V(R)}(f_s(\infty) - f_s(R)).\end{aligned}$$

□

Proof of Theorem 3.1. Choose $R > 0$ such that for $s > n-1$,

$$\frac{f_s(R)}{f_s(\infty)} \geq \frac{10}{11}.$$

Applying Lemma 3.3, we get that for all $x \in \widetilde{M}$,

$$\begin{aligned}\frac{\mu_x^s(B_x(R))}{\mu_x^s(\tilde{M})} &= 1 - \frac{\mu_x^s(\tilde{M} - B_x(R))}{\mu_x^s(\tilde{M})} \geq 1 - \frac{\mu_x^s(\tilde{M} - B_x(R))}{\mu_x^s(B_x(R))}\\ &\geq 1 - \frac{f_s(\infty) - f_s(R)}{f_s(R)} = 2 - \frac{f_s(\infty)}{f_s(R)} \geq \frac{9}{10}\end{aligned}$$

Ley x_i be a sequence of points in M tending to infinity. By passing to a subsequence, we may assume that the x_i travel down a single end in M. By properness of f, the sets $f(B_{x_i}(R))$ march to infinity down a single cusp. Let $y_i \in \tilde{M}$ be a lift of the x_i in a single fundamental domain for M. Then the $\tilde{f}(y_i)$ tend to a single cusp point $\theta \in \partial\tilde{M}_o$.

Therefore, in the universal cover there is a nested sequence of open convex horoballs $\tilde{M}_o \supset U_1 \supset U_2 \supset \ldots$ based at θ with $d_{\tilde{M}_o}(O, U_i)$ tending to infinity such that the projection, $V_i \subset M_o$, of $U_i \subset \tilde{M}_o$ to the base M_o contains $f(B_{x_i}(R))$. Since $\tilde{f}(B_{y_i}(R))$ is connected, it follows from the Margulis Lemma that U_i contains $\tilde{f}(B_{y_i}(R))$ for all sufficiently large i.

Let K_i be the convex hull of $\tilde{f}(B_{y_i}(R))$. Then by convexity of the U_i, we have $K_i \subset U_i$. Hence the convex sets K_i go to infinity in $\widetilde{M_o}$.

However, Lemma 3.2 implies that for all i, $d_{\widetilde{M_o}}\left(\widetilde{F_s}(y_i), K_i\right) < D$. Consequently, $d_{M_o}(F_s(x_i), V_i) < D$, and therefore $F_s(x_i)$ is unbounded. It follows that F_s is proper, and hence so is the homotopy Ψ_s. Finally, by degree theory for proper homotopies [Spanier66], $\deg(F_s) = \deg(f)$ as claimed.

□

4. Main Theorem and Minimal Volume

Proof of Main Theorem. By Theorem 3.1, $\deg(F_s) = \deg(f) \neq 0$ for $s > n-1$, and from Theorem 2.2, $|\operatorname{Jac}(F_s)(x)| \leq \left(\frac{s}{n-1}\right)^n$. Therefore,

$$\begin{aligned}|\deg(f)| \operatorname{vol}(M_o, g_o) = |\deg(F_s)| \int_{M_o} d\operatorname{vol}_{(M_o,g_o)} = \left| \int_M F_s^*(d\operatorname{vol}_{(M_o,g_o)}) \right| \\ \leq \int_M |\operatorname{Jac}(F_s)| d\operatorname{vol}_{(M,g)} \leq \left(\frac{s}{n-1}\right)^n \operatorname{vol}(M,g),\end{aligned}$$

which in the limit as $s \to n-1$ yields the desired inequality.

For the equality case, suppose $\operatorname{vol}(M,g) = |\deg(f)| \operatorname{vol}(M_o, g_o)$. Then in the calculation above the functions $x \mapsto \operatorname{Jac} F_s(x)$ must converge to the constant function $x \mapsto 1$ in $\mathcal{L}^1(M)$ as $s \to n-1$.

At this point we follow Section 7 and 8 of [BCG95] making changes where necessary. We note that the proofs of the lemmas in Section 7 of [?] (done for the case $f = \mathrm{Id}$) are identical so long as we restrict the uniformity of Lemmas 7.5 and 7.6 to be uniform only on compact subsets. These proofs go through with only minor modification in the case that f has (local) degree $\deg(f) \neq 1$; this is explained in Section 8.2 of [?]. In this case we obtain the general versions of Lemma 7.6 and 7.7 of [?],

Lemma 4.1. *There is a subsequence s_i such that the maps F_{s_i} converges uniformly on compact sets to a continuous map $F : N \to M$ such that $||d_y F_{s_i}||$ is uniformly bounded on compact subsets of N and converges to 1 almost everywhere.*

The proof of Lemma 7.8 of [?] then goes through without modification to obtain

Lemma 4.2. *The map F is Lipschitz with Lipschitz constant less than or equal to one.*

Before we proceed we must show,

Lemma 4.3. *The map F is proper.*

Proof. By the previous lemma, F is a contracting Lipschitz map.

We note that the local notion of *deg*(F) given by

$$\deg F(x) = \sum_{y \in F^{-1}(x)} \operatorname{sign}(\operatorname{Jac} F(y))$$

is well defined for a.e. $x \in N$. Let $P \subset M$ be the set of points which have unbounded preimage under F. The set P is clearly closed and of measure 0, and F acts properly on $M \setminus P$. Since F is homotopic to f, $\deg F(x) = \deg f(x)$ for a.e. x in one connected component U of $M \setminus P$, and $\deg F(x) = 0$ a.e. on the other components. We note that Lemma C.2 and C.4 in [?] only require that the injectivity radius be bounded for any finite set of points. The proof of these two lemmas show that for every $x \in U$, $\operatorname{card}(F^{-1}(x) \leq \deg(f)$.

Now we will show that F is proper on the closure of U. This implies that $U = M$.

If the map F were not proper then there would be a sequence y_i tending down an end of N such that $F(y_i) \in U$ limits to a point $x_0 \in P$. After pasing to a subsequence (also denoted y_i) we may find compact rectifiable curves of finite length which pass through all of the $F(y_i)$ and x_0. For any such c, by continuity, the pre-image $F^{-1}(c)$ is therefore contained in at most $\deg(f)$ curves $\alpha_1, \dots, \alpha_{\deg(f)}$ one of which (say $\alpha = \alpha_1$) can be chosen to pass through the y_i.

By possibly slightly perturbing the points y_i, Fubini's theorem guarantees that we can choose a curve c such that the derivatives of $F|_\alpha$ on the pre-image curves α are a.e. equal to one. On the other hand, curves α are Lipschitz since F is and therefore by the fundamental theorem of calculus it must have the same length as c which is finite. This contradicts that the y_i are unbounded. $\square$

To complete the inequality case we must prove the following,

Proposition 4.4. *Consider two n-dimensional complete oriented Riemannian manifolds of finite volume, M and M_o. Suppose $F : M \to M_o$ is a proper Lipschitz map satisfying $d_o(F(x), F(y)) \leq d(x, y)$ for all $x, y \in M$. Then if* $\mathrm{vol}(M) = |\deg f| \mathrm{vol}(M_o)$*, the map F is a Riemannian covering homotopic to f.*

We establish this following Appendix C of [BCG95] through a series of lemmas. If we set $N(y) = \mathrm{card}\left\{F^{-1}y\right\}$ The proof of Lemma C.2 in [BCG95] establishes,

Lemma 4.5. *For almost every $x \in M$, and a.e. $y \in M_o$, we have $N(y) = \deg(f)$ and D_xF is an isometry between T_xM and $T_{F(x)}M_o$.*

Since F is proper, the preimage set $\left\{F^{-1}(x)\right\}$ is compact and hence lies in a region with injectivity radius bounded from below. The rest of the proofs of Appendix C can then be followed verbatim to show that for every $x \in M$, $N(x) = \deg(f)$ and hence F is a local isometry.

$\square$

Proof of Theorem 1.2. Suppose that $\mathrm{Minvol}(M) = \mathrm{vol}(M_o)$, and let $f : M \to M_o$ be a degree 1 map, M_o a real hyperbolic manifold, and $(g_i)_{i \in N}$ a sequence of metrics on M with curvatures in $[-1, 1]$ such that $\lim_i \mathrm{vol}(M, g_i) = \mathrm{Minvol}(M)$.

By a theorem of Thurston [Gromov82], the simplicial volume $\|M_o\|$ is not 0. Hence the simplicial volume of M is also not 0. The *injectivity radius estimate* in [Gromov82] shows that for all i there are points $p_i \in M$ with $\mathrm{inj}_{(M,g_i)}(p_i) > \epsilon_n$, where ϵ_n depends only on the dimension.

It follows from the $C^{1,1}$*-compactness theorem* [Gromov99], that there is a subsequence of (M, p_i, g_i), say the whole sequence, converging to (X, p_X, g_X) in the pointed Lipschitz topology, where (X, g_X) is a $C^{1,1}$ complete Riemannian manifold. This means that for all $R > 0$ there exists $i_R > 0$ such that for $i \geq i_R$ there is a bi-Lipschitz map $\phi_{R,i} : X \supset B_{p_X}(R) \to M$ such that $\phi_{R,i}(p_X) = p_i$ and $\|d\phi_{R,i}\|, \|d\phi_{R,i}^{-1}\| \leq 1 + \frac{1}{R}$.

For all i consider $F_i = F_{s_i} : (M, g_i) \to M_o$, where $s_i = \frac{1}{i} + n - 1$. Bessieres considered the functions $F_i \circ \phi_{R,i_R} : X \supset B_{p_X}(R) \to M_o$ and in sections 3 to 5 of [Bessieres98] proved

Theorem 4.6. [Bessieres98] *There is a subsequence of the $F_i \circ \phi_{R,i_R}$, say the whole sequence, which for all $r > 0$ and $R > r$ converges uniformly to a local (smooth) isometry $\psi_r : X \supset B_{p_X}(r) \to M_o$. If $r' > r$, then $\psi_{r'}|_{B_{p_X}(r)} = \psi_r$. So we have a well defined local (smooth) isometry $\phi : X \to M_o$.*

Since (X, g_X) is complete and ϕ is a local isometry, it is a covering, hence $\mathrm{vol}(X, g_X) \geq \mathrm{vol}(M_o, g_o)$. On the other hand,

$$\mathrm{vol}(X) \leq \lim_i \mathrm{vol}(M, g_i) = \mathrm{Minvol}(M) = \mathrm{vol}(M_o),$$

so ϕ is an isometry, concluding the proof of Theorem 1.2. $\square$

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