DISTANCES IN THE CURVE COMPLEX AND THE HEEGAARD GENUS

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Abstract. We show, extending a result of Lackenby, that every minimal genus Heegaard splitting of a manifold which is obtained by gluing two simple manifolds along their boundaries is obtained as the amalgamation of Heegaard splittings of both pieces whenever the gluing map is complicated enough.

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Let M_1 and M_2 be simple 3–manifolds whose boundaries ∂M_1 and ∂M_2 are homeomorphic, connected and have at least genus two. Recall that a manifold is simple if it is irreducible, atoroidal and has incompressible and acylindrical boundary. In this note we are interested in the Heegaard genus $g(N_\phi)$ of the manifold $N_\phi = M_1 \cup_\phi M_2$. Amalgamating two minimal genus Heegaard splittings of M_1 and M_2 we obtain a Heegaard splitting of N_ϕ of genus $g(M_1) + g(M_2) - g(\partial M_1)$. If $\psi : \partial M \to \partial M$ is a pseudo–Anosov map then Lackenby [3] proved that there is n_0 such that every minimal genus Heegaard splitting of $N_{\psi^n \circ \phi}$ is constructed amalgamating splittings of M_1 and M_2 for all $n \geq n_0$ and hence

$$g(N_{\phi \circ \psi^n}) = g(M_1) + g(M_2) - g(\partial M_1)$$

Masur and Minsky [5] studied the curve complex $\mathcal{C}(S)$ of a surface S and proved that $d_{\mathcal{C}(S)}(\alpha, \psi^n(\alpha))$ tends to infinity for every curve α and for every pseudo–Anosov mapping class ψ where $d_{\mathcal{C}(S)}$ is the distance in $\mathcal{C}(S)$. Extending thus Lackenby's result we show

Theorem 1. *Let M_1 and M_2 be simple 3–manifolds with connected homeomorphic boundaries of genus at least two and fix an essential simple closed curve $\alpha_i \subset \partial M_i$ for $i = 1, 2$. Then there is a constant n_0 such that every minimal genus Heegaard splitting of N_ϕ is constructed amalgamating splittings of M_1 and M_2 and hence*

$$g(N_\phi) = g(M_1) + g(M_2) - g(\partial M_1)$$

for every diffeomorphism $\phi : \partial M_1 \to \partial M_2$ with $d_{\mathcal{C}(\partial M_2)}(\phi(\alpha_1), \alpha_2) \geq n_0$.

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Theorem 1 was suggested, among other possible generalizations, in Lackenby's paper [3]. See [7] for a second generalization of Lackenby's theorem.

After completing this work, the author has learned that Bachmann and Schleimer have announced an independent and different proof of Theorem 1.

Let us briefly describe the strategy of the proof of Lackenby's theorem and then outline the additional difficulties that we face. Considering generalized Heegaard splittings Lackenby remarked that it was sufficient to show that for large n there is in $N_{\psi^n\circ\phi}$ no minimal surface of genus less or equal $g(M_1)+g(M_2)-g(\partial M_1)$ which intersects the surface ∂M_1 essentially. After suitable choice of base points, the hyperbolic manifolds $N_{\phi\circ\psi^n}$ converge geometrically to the infinite cyclic cover of the mapping torus of the pseudo–Anosov ψ. In particular the injectivity radius of this geometric limit is uniformly bounded from below. Lackenby [4] proved that in the presence of a lower bound on the injectivity radius the diameter of a minimal surface in a hyperbolic 3–manifold can be bounded only in dependence of its genus. The Gromov–Hausdorff convergence and the diameter bound imply the desired result.
In our setting we cannot argue with geometric limits and we do not have uniform bounds on the injectivity radius. In section 2 we extend, in some sense, Lackenby's bound of the diameter of a minimal surface in a hyperbolic manifold to the case that there is no lower bound on the injectivity radius. In section 3 we obtain uniform control on the geometry of the manifolds N_ϕ. This is based on results of Minsky on the geometry of quasi–fuchsian manifolds. Finally, in the last section we prove Theorem 1 following Lackenby's strategy.

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Let S be a minimal surface in a hyperbolic 3–manifold N. The induced metric has curvature less or equal to -1 and hence we have that $\mathrm{vol}(S)\leq 2\pi|\chi(S)|$. For ϵ less than the Margulis constant let $S^{<\epsilon}$ (resp. $N^{<\epsilon}$) be the set of points in S (resp. in N) with injectivity radius less than ϵ. Following Thurston define the *length relative to* $S^{<\epsilon}$ of a segment γ in S to be the length of the intersection of γ with $S\setminus S^{<\epsilon}$. We define the pseudo–distance $d_{\mathrm{rel}\,S^{<\epsilon}}(x,y)$ of two points $x,y\in S$ as the minimal length relative to $S^{<\epsilon}$ of a segment in S joining x and y. The pseudo–distance $d_{\mathrm{rel}\,N^{<\epsilon}}$ on N is similarly defined. The area estimate and a simple comparison argument show that the diameter of S with respect to $d_{\mathrm{rel}\,S^{<\epsilon}}$ is at most $\frac{2}{\epsilon^2}|\chi(S)|$.

If the surface S is incompressible in N then the components of $S^{<\epsilon}$ are contained in components of $N^{<\epsilon}$ which allows to bound $d_{\mathrm{rel}\, N^{<\epsilon}}(x,y)$ in terms of $d_{\mathrm{rel}\, S^{<\epsilon}}(x,y)$ for all $x,y \in S \subset N$. In the case of a compressible minimal surface, then there may be some components of $S^{<\epsilon}$ which do not meet $N^{<\epsilon}$. We consider therefore the *essential* thin part $S^{<\epsilon}_{\mathrm{ess}}$ of S, i.e. the set of $x \in S$ for which there is a loop $\gamma_x \subset S$ based at x which is essential in N and whose length is at most 2ϵ. We define the pseudo distance $d_{\mathrm{rel}\, S^{<\epsilon}_{\mathrm{ess}}}$ on S as above. Now, every component of the essential thin part is contained in $N^{<\epsilon}$ and we have thus

Lemma 2. *Let S be a minimal surface in a hyperbolic manifold N. For every ϵ smaller than the Margulis constant holds:*

$$d_{\mathrm{rel}\, N^{<\epsilon}}(x,y) \leq d_{\mathrm{rel}\, S^{<\epsilon}_{\mathrm{ess}}}(x,y)$$

for all $x,y \in S \subset N$. □

This lemma is not of much use if we do not bound the diameter of S with respect $d_{\mathrm{rel}\, S^{<\epsilon}_{\mathrm{ess}}}$.

Proposition 3. *Let S be a minimal surface in a hyperbolic manifold N. The diameter of S with respect to the pseudo distance $d_{\mathrm{rel}\, S^{<\epsilon}_{\mathrm{ess}}}$ of S is bounded only in terms of ϵ and of $\chi(S)$.*

Proof. Following Lackenby [4] let Γ be a maximal set of disjoint, not necesarily simple, primitive closed geodesics in S which have length at most 2ϵ and which are compressible in N. The cardinality of Γ is at most $4|\chi(S)|$. Let $\mathcal{N}_1(\Gamma)$ be the set of all points in S which are at distance at most 1 of Γ. We claim that $S^{<\epsilon} \subset S^{<\epsilon}_{\mathrm{ess}} \cup \mathcal{N}_1(\Gamma)$. Seeking a contradiction assume that there is $x \notin \mathcal{N}_1(\Gamma)$ and an essential loop η in S, compressible in N, based at x and with length less than 2ϵ. The loop η is disjoint of Γ and thus maximality of Γ ensures that η is homotopic to some power of curve γ in Γ. In particular, η lifts to an essential closed curve η in the cover S_γ of S which corresponds to γ. The following proposition, due to Lackenby, implies that the length of η is at least π and, in particular, yields the desired contradiction:

Proposition 4. [4, pp.37–40] *Let S be a minimal surface in a hyperbolic manifold N. Given a geodesic γ in S which is compressible in N let S_γ be the covering of S corresponding to γ. Denote the core geodesic of S_γ again by γ and let η be an essential curve in S_γ which is contained in $\{x \in S_\gamma | d(x,\gamma) \geq u\}$ for some $u > 0$. Then*

$$l(\eta) \geq \pi(2u-1)$$

where $l(\eta)$ is the length of the curve η in S_γ.

We have proved that $S^{<\epsilon}$ is contained in $\mathcal{N}_1(\Gamma) \cup S^{<\epsilon}_{\mathrm{ess}}$. In particular, we obtain

$$d_{\mathrm{rel}\, S^{<\epsilon}_{\mathrm{ess}}}(x,y) \leq d_{\mathrm{rel}\, S^{<\epsilon}}(x,y) + 4|\chi(S)|(2+2\epsilon) \leq \left(8 + 4\epsilon + \frac{2}{\epsilon^2}\right)|\chi(S)|$$

for all $x,y \in S \subset M$. This concludes the proof of Proposition 3. □

We remark that a qualitative version of Proposition 4 follows from the compactness of minimal surfaces with bounded total curvature.

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Given manifolds M_1 and M_2 as in Theorem 1 and $\phi : \partial M_1 \to \partial M_2$ we can identify canonically, up to isotopy, M_1 and M_2 with submanifolds of N_ϕ. We also identify the curves α_1 and α_2 with their images in N_ϕ.
Let $(\phi_i : \partial M_1 \to \partial M_2)$ be a sequence of diffeomorphisms and consider the manifolds $N_{\phi_i} = M_1 \cup_{\phi_i} M_2$. The manifold N_{ϕ_i} is irreducible, atoroidal and Haken, and hence hyperbolic for all i. In particular, we have an isomorphism of $\pi_1(N_{\phi_i})$ with a subgroup of $\mathrm{PSL}_2\,\mathbb{C}$ for all i. Restricting this isomorphism to $\pi_1(M_1) \subset \pi_1(N_{\phi_i})$ we obtain a faithful and discrete representation $\rho_i : \pi_1(M_1) \to \mathrm{PSL}_2\,\mathbb{C}$. Since the manifold M_1 is simple, a theorem of Thurston [12] ensures that, up to conjugacy and the choice of a subsequence, the sequence (ρ_i) converges algebraically to a discrete and faithful representation ρ_∞. Let $M_1^i = \mathbb{H}^3/\rho_i$ be the associated quotient manifold for $i = 1, 2, \ldots, \infty$. By construction, the manifolds M_1^i cover the manifolds N_{ϕ_i}. After choice of suitable base points and of a subsequence, we may assume that the sequences $(N_{\phi_i})_i$ and $(M_1^i)_i)$ converge geometrically to manifolds N_G and M_G such that there is the following tower of coverings

$$M_1^\infty \to M_G \to N_G$$

A theorem of Scott [11] ensures that the manifold M_1^∞ contains a compact core, i.e. a compact submanifold C whose inclusion into M_1^∞ is a homotopy equivalence. Since C is simple we may assume without loss of generality that the core C_1^∞ embedds under the covering $M_1^\infty \to N_G$ [1]. Geometric convergence ensures that for all i large enough the core C_1^∞ can be pushed back to a core C_1^i in the manifold M_1^i which embedds under the covering $M_1^i \to N_{\phi_i}$ and such that the pairs (N_{ϕ_i}, C_1^i) converge geometrically to (N_G, C). Moreover, by construction, the subgroups $\pi_1(C_1^i)$ and $\pi_1(M_1)$ are conjugated in $\pi_1(N_{\phi_i})$ for all i. We deduce that C_1^i and M_1 are isotopic in N_{ϕ_i} for all i since they are both simple. The same applies obviously to M_2. The sequence (ϕ_i) being arbitrarily chosen we have proved:

Lemma 5. *There is a constant L such that for every $\phi : \partial M_1 \to \partial M_2$ holds:*

(1) *The geodesics homotopic to α_1 and α_2 in N_ϕ are shorter than L.*
(2) *For $j = 1, 2$ there is a compact submanifold C_j of N_ϕ isotopic to M_j with the property that for every $x \in C_j$ there is a loop η_x based at x, shorter than L and which represents an element in $\pi_1(C_j)$ which cannot be homotoped into $\pi_1(\partial C_j)$.* □

In the sequel we will say that the submanifolds C_1 and C_2 are the *uniform cores* of N_ϕ. If $C_1 \cap C_2 = \emptyset$, then $N_\phi \setminus (C_1 \cup C_2)$ is homeomorphic to $\partial M_1 \times \mathbb{R}$. The main result of this section is the following proposition which shows that a large distance in the curve complex ensures that a very thick collar separates both uniform cores of N_ϕ.

Proposition 6. *There is $\epsilon > 0$ smaller than the Margulis constant such that for every D there is d with*

$$d_{\mathrm{rel}\, N_\phi^{<\epsilon}}(x_1, x_2) \geq D$$

for every $x_1 \in C_1$ and $x_2 \in C_2$ and for every $\phi : \partial M_1 \to \partial M_2$ with $d_{\mathcal{C}(\partial M_2)}(\phi(\alpha_1), \alpha_2) \geq d$. Here C_1 and C_2 are the uniform cores of N_ϕ.

Proof. To begin with we may assume that the constant L provided by Lemma 5 is larger than $\operatorname{arccosh}(2|\chi(\partial M_1)| + 1)$. Choose ϵ smaller than the Margulis constant, such that each two components of the ϵ–thin part of N_ϕ are at least at distance 1 from each other and such that whenever σ_1, σ_2 are elements in $\mathrm{PSL}_2\,\mathbb{C}$ which generate a discrete subgroup and p is a point in $\mathbb{H}^3$ with $d_{\mathbb{H}^3}(p, \sigma_1 p) < \epsilon$ and $d_{\mathbb{H}^3}(p, \sigma_2 p) < L$, then σ_1 and σ_2 commute.
Let I be a minimal segment with respect to the pseudo distance $d_{\mathrm{rel}\, N^{<\epsilon}}$ joining x_1 and x_2 and η_{x_1} and η_{x_2} are the loops guaranteed by Lemma 5, then, by juxtaposition, we obtain a closed curve η in N_ϕ which cannot be homotoped into one of the cores and hence intersects essentially every surface homotopic to S. Moreover, we can use the length relative to $N_\phi^{<\epsilon}$ of η to bound $d_{\mathrm{rel}\, N_\phi^{<\epsilon}}(x_1, x_2)$ since

$$d_{\mathrm{rel}\, N_\phi^{<\epsilon}}(x_1, x_2) \geq \frac{1}{2} l_{\mathrm{rel}\, N_\phi^{<\epsilon}}(\eta) - L.$$

We are going to bound $l_{\mathrm{rel}\, N_\phi^{<\epsilon}}(\eta)$ in terms of $d_{\mathcal{C}(\partial M_2)}(\phi(\alpha_1), \alpha_2)$.
Choose essential closed curves $\beta_0, \dots, \beta_n$ on ∂M_2 with $\beta_0 = \alpha_1$, $\beta_n = \alpha_2$ and $d_{\mathcal{C}(\partial M_2)}(\beta_i, \beta_{i+1}) = 1$; remark that $n \geq d_{\mathcal{C}(\partial M_2)}(\phi(\alpha_1), \alpha_2)$. Let Q be the quasi–fuchsian covering of N_ϕ corresponding to the surface ∂M_1. The curves β_i lift to curves in Q which we denote again by β_i. There are pleated surfaces $S_1, \dots, S_n$ in Q such that S_i realizes β_{i-1} and β_i for all i. Let γ_i be a shortest essential curve on S_i for all i. It is a theorem of Minsky [6] that there is a constant c_1 which depends only on $\chi(\partial M_2)$ with $d_{\mathcal{C}(\partial M_2)}(\gamma_i, \gamma_{i+1}) \leq c_1$ for all i.
For all i let y_i be a point in S_i which projects to a point in η and choose τ_i a shortest essential loop in S_i based at y_i. The length of the geodesic representatives of β_0, β_n and γ_1, and of the loop τ_i are all bounded from above by the uniform constant L. In particular, there is a uniform constant c_2 with

$$d_{\mathcal{C}}(\gamma_1, \beta_1), d_{\mathcal{C}(S)}(\gamma_n, \beta_n), d_{\mathcal{C}}(\gamma_i, \tau_i) \leq c_2$$

We deduce that the cardinality of the set $\{[\tau_i]\}$ of simple closed curves in ∂M_2 homotopic to τ_i for some i is at least

$$|\{[\tau_i]\}| \geq \frac{1}{c_1 + c_2} n.$$

Since $\pi_1(\partial M_2)$ is malnormal in $\pi_1(N_\phi)$ we obtain that two loops τ_i and τ_j are homotopic in Q if and only if they are in N_ϕ and that τ_i is primitive in $\pi_1(N_\phi)$ for all i.

We claim that $[\tau_i] = [\tau_j]$ for all i, j such that the projections of y_i and y_j are contained in the same component of $N_\phi^{<\epsilon}$. Indeed, let σ be the element in $\pi_1(N_\phi)$ represented by the core of the Margulis tube containing x_i and x_j. The choice of ϵ and the fact that τ_i and τ_j have length less than L implies that σ and τ_i commute. Since the manifold N_ϕ is compact and hyperbolic we obtain that any maximal abelian subgroup of $\pi_1(N_\phi)$ is cyclic. In particular the claim follows from the fact that τ_i and τ_j represent primitive elements in $\pi_1(N_\phi)$.

Let J be a unit length subsegment of $\eta \cap (N_\phi \setminus N_\phi^{<\epsilon})$. The Margulis Lemma implies that there are at most c_3 distinct homotopy classes of loops of length less than L which are based at J and which represent primitive elements in $\pi_1(N_\phi)$. All this implies that $\eta \cap (N_\phi \setminus N_\phi^{<\epsilon})$ contains at least $\frac{1}{c_3(c_1+c_2)}n$ disjoint segments of length $\frac{1}{2}$. We deduce that

$$d_{\mathrm{rel}\, N_\phi^{<\epsilon}}(x_1, x_2) \geq \frac{1}{2} l_{\mathrm{rel}\, N_\phi^{<\epsilon}}(\eta) - L \geq \frac{1}{4(c_1+c_2)c_3} n - L$$

which yields the claim of Proposition 6. □

4

In this section we prove Theorem 1. As announced in the introduction we follow closely the arguments given in [3].

Let $N_\phi = U \cup V$ be a minimal genus Heegaard splitting of N_ϕ; as remarked in the introduction we have $g = g(N_\phi) \leq g(M_1) + g(M_2) - g(\partial M_1)$. We consider the associated generalized Heegaard splitting, i.e. a decomposition $N_\phi = W_1 \cup \cdots \cup W_n$ of N_ϕ as union of manifolds with incompressible boundary and for all $j = 1, \ldots, n$ a strongly irreducible Heegaard splitting, represented by a surface S_j, of W_j whose amalgamation yields back the original splitting $N_\phi = U \cup V$. Moreover, none of the manifolds W_i is an interval bundle. Let $F_1, \ldots, F_{2n+1}$ be the involved surfaces, i.e. the boundary components and the Heegaard surfaces of W_i for $i = 1, \ldots, n$ and $F_+ = \cup_i F_i$. Scharlemann and Thompson established equalities relating the genus g of the Heegaard splitting $U \cup V$ and the genus of the surfaces F_+ and in particular we have $|\chi(F_+)| \leq (2g-2)^2$ [3]. We are going to show that the surface ∂M_2 coincides, up to isotopy, with one of the boundary components of W_i for some i. Once this is proved Theorem 1 follows since amalgamating the associated Heegaard splittings of the set of all W_i contained in M_1 (resp. M_2) we get Heegaard splittings of M_1 (resp. M_2) such that their amalgamation yields the original splitting.

It is a theorem of Freedman, Hass and Scott [2] that up to isotopy we may assume that the boundary of W_i consists of minimal surfaces for all i. Using methods of geometric measure theory, Pitts and Rubinstein [9] established that it is possible to associated to every strongly irreducible Heegaard surface F in a Riemannian 3–manifold N with minimal boundary a minimal surface F'. More precisely they proved

Existence theorem of minimal surfaces. *Let F be an oriented strongly irreducible Heegaard surface in a Riemannian 3–manifold N with ∂N minimal. Then either F is isotopic to an embedded minimal surface, or is isotopic to to the boundary of a regular neighborhood $N(F')$ of a one–sided embedded minimal surface F' with a single handle added by taking the boundary of a regular neighborhood of a fiber of the bundle $N(F') \to F'$. In particular every component of $N \setminus F'$ is a compression body.*

Considering all the minimal surfaces corresponding to the boundary surfaces of $W_1, \ldots, W_n$ and to their associated strongly irreducible Heegaard splittings we obtain a minimal surface F'_+ with $|\chi(F'_+)| \leq (2g-2)^2$ such that every component of $M \setminus F'_+$ is a compression body. Moreover, the incompressible boundary components of this compression bodies correspond to the boundary components of the manifolds $W_1, \ldots, W_n$.
It follows from Lemma 2 and Proposition 3 that there is a constant D such that every component X of F'_+ has diameter with respect to $d_{\text{rel}\, N^{<\epsilon}}$ less or equal to D. In particular, Proposition 6 ensures that there is d such that for all $\phi : \partial M_1 \to \partial M_2$ with $d_{\mathcal{C}(\partial M_2)}(\phi(\alpha_1), \alpha_2) \geq d$ the surface X does not intersect at least one of the uniform cores C_1, C_2. In particular, the surface ∂M_2 can be isotoped to be disjoint of F'_+ and hence is contained in one of the components U of $N_\phi \setminus F'_+$. Since U is a compression body and S is incompressible we obtain that S is isotopic to one of the incompressible boundary components of U, i.e. to one of the boundary components of W_i for some i. This concludes the proof of Theorem 1.

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