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ALGEBRAIC LIMITS OF GEOMETRICALLY FINITE MANIFOLDS ARE TAME

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1 Introduction

An irreducible 3-manifold M is *tame* if it is homeomorphic to the interior of a compact 3-manifold. In this paper we address the following conjecture of A. Marden.

Marden's Tameness Conjecture. *Let M be a complete hyperbolic 3-manifold with finitely generated fundamental group. Then M is tame.*

In his 1986 article [T2], W. Thurston proposed that one might approach Marden's conjecture from a dynamical point of view via a study of limits in the natural deformation space: its interior consists of geometrically finite manifolds, and promoting their tameness to *algebraic limits* on the boundary has proven to be a successful strategy to address Marden's conjecture in special cases. In this paper we complete this part of his approach.

Theorem 1.1. *Each algebraic limit of geometrically finite hyperbolic 3-manifolds is tame.*

Theorem 1.1 reduces Marden's tameness conjecture to the density conjecture of Bers, Sullivan, and Thurston, which predicts that every hyperbolic 3-manifold with finitely generated fundamental group is an algebraic limit of geometrically finite hyperbolic 3-manifolds.

Our result finishes the cases left unaddressed by our previous work with K. Bromberg and R. Evans (see [BrBES]). We first outline some further consequences of our results and then review the history of our approach.

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Strong convergence. A central difficulty arising in the consideration of algebraic convergence is the lack of continuity of many important geometric and topological properties. A better topology for understanding geometric changes under deformations is the Gromov–Hausdorff or *geometric* topology on the set $\{(M, \omega)\}$ of complete hyperbolic 3-manifolds M equipped with *base-frames* ω, specified by a choice of orthonormal frame at a basepoint.

After passing to a subsequence, manifolds in an algebraically convergent sequence $M_n \to M$ may be equipped with base-frames ω_n so that the sequence (M_n, ω_n) converges geometrically to a limit (N_G, ω_G) with a locally isometric cover

$$\pi : (M, \omega) \to (N_G, \omega_G)$$

by the natural framed algebraic limit (M, ω). When π is an isometry and the algebraic and geometric limits agree, we say the convergence to the limit is *strong*.

If cusps of the limit M of a sequence M_n correspond algebraically to cusps of the approximates, we say the convergence $M_n \to M$ is *type-preserving*. This condition is motivated by a conjecture of Jørgensen.

JØRGENSEN'S CONJECTURE. *Let (M_n) be an algebraically convergent sequence of geometrically finite hyperbolic 3-manifolds with limit M. If the convergence $M_n \to M$ is type-preserving then it is strong.*

In [BrBES], our strategy was to show that each algebraic limit of geometrically finite manifolds is approximated by a type-preserving sequence $M_n \to M$. Applying cases of Jørgensen's conjecture due to J. Anderson and R. Canary [AC1,2], tameness of the limit M follows provided M has non-empty conformal boundary by results of Canary–Minsky and Evans [CM], [Ev]. Here, while we employ a similar philosophy of improving our approximating sequences, we engage in a more specific topological investigation of the geometric limit to show tameness of the limit M directly. As a corollary we confirm Jørgensen's conjecture.

Theorem 1.2. *Let (M_n) be a sequence of geometrically finite manifolds converging in a type-preserving manner to a limit M. Then the convergence of (M_n) to M is strong.*

Proof. When M has non-empty conformal boundary, the theorem follows from the main results of [AC1,2]. Otherwise, applying Theorem 1.1, the limit M is tame, and the strong convergence follows from [C3, Theorem 9.2]. □

Conformal dynamical systems. Theorem 1.1 has important dynamical consequences for the action of the associated Kleinian group on $\widehat{\mathbb{C}}$. In [BrBES] we applied the work of Thurston, Bonahon, and Canary (see [T1], [Bo], [C2]) to show that for each algebraic limit $M = \mathbb{H}^3/\Gamma$ of geometrically finite hyperbolic 3-manifolds the Ahlfors measure conjecture holds, namely that either the limit set $\Lambda(\Gamma)$ is all of $\widehat{\mathbb{C}}$ or $\Lambda(\Gamma)$ has Lebesgue measure zero on $\widehat{\mathbb{C}}$.

The tameness of general algebraic limits gives stronger consequences.

Corollary 1.3. *Let $M = \mathbb{H}^3/\Gamma$ be an algebraic limit of geometrically finite manifolds. Then either $\Lambda(\Gamma)$ has measure zero, or $\Lambda(\Gamma) = \widehat{\mathbb{C}}$ and Γ acts ergodically on $\widehat{\mathbb{C}} \times \widehat{\mathbb{C}}$.* □

The latter conclusion guarantees the geodesic flow on the unit tangent bundle $T_1(M)$ is ergodic, as well as the non-existence of measurable Γ-invariant line fields on $\Lambda(\Gamma)$ established in more general contexts by Sullivan early on (cf. [Su], [T1], [Mc]).

Deformation spaces. In the course of the proof of Theorem 1.1 we verify certain conjectural features of the deformation space of hyperbolic 3-manifolds with a given homotopy type. In particular, Thurston conjectured that each algebraic limit of geometrically finite manifolds is a strong limit of a perhaps different sequence of such manifolds (see [T3]). After [BrBES, Theorem 1.9], each limit of geometrically finite manifolds is a type-preserving limit of geometrically finite manifolds, so Thurston's conjecture then follows from Theorem 1.2.

Theorem 1.4. *Let M be an algebraic limit of geometrically finite hyperbolic 3-manifolds. Then there is a strongly convergent sequence $M_n \to M$ with M_n geometrically finite.* □

Also of interest in the proof of the main theorem are certain uniform estimates we obtain for limits of surfaces in 3-manifolds. We plan in a future paper to employ these estimates to study the relationship between the geometry of closed hyperbolic 3-manifolds and the combinatorics of their Heegaard splittings.

History. Our main theorem is a culmination of a series of similar results. In his original lecture notes [T1], Thurston introduced the notion of *geometric* tameness for an incompressible end of a hyperbolic 3-manifold M. This condition posits the existence of closed geodesics exiting the end that are homotopic to simple curves on the boundary of a compact core for M.

Thurston showed that this notion of geometric tameness persists in limits of geometrically finite hyperbolic 3-manifolds with freely-indecomposable fundamental group and with certain assumptions on cusps. He showed moreover that the *topological* tameness predicted by Marden's conjecture follows in these cases, as well as the dynamical conclusions of Corollary 1.3. F. Bonahon later proved geometric tameness holds generally for hyperbolic 3-manifolds with incompressible ends [Bo], but the compressible case has remained open.

In [C2], Canary demonstrated that topological tameness always guarantees geometric tameness for hyperbolic 3-manifolds, once this notion is appropriately generalized to the setting of compressible ends, and also that the conclusions of Corollary 1.3 hold as a consequence. The condition of topological tameness has been a central focus since this work. Renewing the limiting approach, Canary and Y. Minsky [CM] established that tameness persists in cusp-free limits of cusp-free hyperbolic manifolds, under the extra assumption that the convergence is strong. Work of Evans [Ev] generalized these results to the type-preserving (and *weakly* type-preserving) setting.

Recent developments in the deformation theory of hyperbolic cone-manifolds have improved our ability to choose a desirable sequence of approximating manifolds for a given limit of geometrically finite manifolds. Indeed, this is the central technique of our recent work with Bromberg and Evans in [BrBES], which applies a *drilling theorem* of [BrB] to establish that each algebraic limit of geometrically finite manifolds has a sequence of type-preserving approximates.

Combining this fact with theorems of Anderson and Canary (see [AC1,2]) giving criteria for algebraic and geometric limits to agree (cf. Theorem 1.2), the main theorem of [Ev] guarantees that the limit M is tame whenever either

1. M has non-empty conformal boundary; or
2. $\pi_1(M)$ is not a compression body group

(recall G is a *compression body group* if it is isomorphic to a non-trivial free product of orientable surface groups and infinite cyclic groups). It is the remaining recalcitrant case that the limit M has empty conformal boundary and $\pi_1(M)$ is a compression body group that we address in the present treatment.

Plan of the paper. We describe the plan of the paper and suggest the structure of the argument. In section 2 we give a condition on a sequence that guarantees that all cusps of the geometric limit of a sequence M_n correspond *geometrically* to cusps in M_n; like the type-preserving condition for algebraic convergence, this condition gives us substantially more control over degenerations that can occur. Section 3 proves a combination theorem for tame hyperbolic manifolds along essential incompressible surfaces.

Applying these techniques, in section 4 we give the proof of the main theorem assuming the existence of a tame degenerate (relative) end E in the geometric limit bounded by a surface S that is either compressible or incompressible with an essential curve that is homotopic to a cusp by a homotopy that intersects the core essentially. Cutting along essential disks or annuli we decompose $\pi_1(M)$ into subgroups with non-empty domain of discontinuity, reducing the theorem to the main theorem of [BrBES].

The remainder of the paper is devoted to finding the surface S and the tame end E of the geometric limit bounded by S. The techniques here are generalizations of the interpolation arguments of Canary–Minsky, together with a crucial application of the geometrically type-preserving condition defined in section 2 to make the arguments work in the presence of cusps.

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Remark. During this manuscript's final stages of preparation, Ian Agol announced a proof of the tameness conjecture, and Danny Calegari and David Gabai announced an independent proof shortly thereafter.

2 Deformations, Drilling and Strong Convergence

In this section we establish the central tool that will allow us to control the behavior of parabolics in an algebraically convergent sequence. We begin by introducing the necessary background on algebraic and geometric convergence, and then discuss how to find a sequence approximating a given algebraic limit M of geometrically finite manifolds in such a way that the cusps of the *geometric* limit correspond to cusps in the approximates M_n.

Background. Recall, a *Kleinian group* is a discrete torsion-free subgroup of $\mathrm{Isom}^+ \mathbb{H}^3$, the orientation-preserving isometries of hyperbolic 3-space. To fix notation, we will refer to the quotient $M = \mathbb{H}^3/\Gamma$ of hyperbolic

3-space by a Kleinian group as a *hyperbolic 3-manifold*, assuming implicitly that M is complete and oriented by the standard orientation on $\mathbb{H}^3$. Unless otherwise stated, all Kleinian groups will be assumed non-elementary. The extension of the action of Γ to the Riemann sphere partitions $\widehat{\mathbb{C}}$ into its *domain of discontinuity* Ω where Γ acts properly discontinuously and its complementary limit set $\Lambda = \widehat{\mathbb{C}} \setminus \Omega$. We denote by ∂M the *conformal boundary* Ω/Γ obtained as the quotient of Ω.

Deformation spaces and the geometric topology. Let N be a compact 3-manifold whose interior $\mathrm{int}(N)$ is homeomorphic to $\mathbb{H}^3/\Gamma$ for some Kleinian group Γ. Then the representation space

$$AH(N) = \big\{\rho : \pi_1(N) \to \mathrm{Isom}^+ \mathbb{H}^3 \bigm| \rho \text{ is discrete and 1-1}\big\}/\mathrm{conj.}$$

parameterizes complete hyperbolic 3-manifolds homotopy equivalent to N. Convergence $\rho_n \to \rho$ of such representations is called *algebraic convergence*, and $AH(N)$ inherits the *algebraic topology* as the quotient of the topology of algebraic convergence. The convergence $\rho_n \to \rho$ is called *type-preserving* when $\rho_n(g)$ is parabolic if and only if $\rho(g)$ is as well.

As each conjugacy class in $AH(N)$ determines a complete hyperbolic 3-manifold up to isometry, we will often refer to the hyperbolic manifold itself as an element $M \in AH(N)$ assuming an implicit isomorphism $\rho : \pi_1(N) \to \pi_1(M)$.

Unfortunately, fine geometric information can be lost in the passage to limits. Each algebraically convergent sequence $\rho_n \to \rho$ admits a subsequence that converges *geometrically* as well: if $\rho_n(\pi_1(N)) = \Gamma_n$, then (Γ_n) converges geometrically to its Kleinian *geometric limit* Γ_G if

1. for each $\gamma \in \Gamma_G$ we have $\gamma_n \to \gamma$ for some $\gamma_n \in \Gamma_n$; and
2. if a subsequence (γ_{n_j}) converges then its limit lies in Γ_G.

It is evident that the limit representation $\rho(\pi_1(N))$ is a subgroup of Γ_G when Γ_n converges geometrically to Γ_G.

A complete hyperbolic 3-manifold M determines a Kleinian group only up to conjugacy; the additional data of a *base-frame* ω, namely an orthonormal frame at a basepoint, determines a unique Kleinian group via the condition that the standard base-frame $\widetilde{\omega} \in \mathbb{H}^3$ descends to ω under the locally isometric covering projection $(\mathbb{H}^3, \widetilde{\omega}) \to (M, \omega)$. Then a sequence of such *framed* hyperbolic 3-manifolds (M_n, ω_n) converges geometrically to its *geometric limit* $(N_G, \omega) = (\mathbb{H}^3, \widetilde{\omega})/\Gamma_G$ if the associated Kleinian groups converge geometrically to Γ_G.

A more geometric formulation of geometric convergence of a sequence (M_n, ω_n) of framed hyperbolic manifolds to the limit (N_G, ω_G) is the existence of a sequence $\phi_n : K_n \to M_n$ of smooth embeddings defined on an exhaustion of N_G by compact subsets K_n with $\omega_G \in K_n$ so that for each i and $n \geq i$ the mappings ϕ_n have 1-jet sending ω_G to ω_n and ϕ_n converge C_∞ on K_i to an isometry. We call these associated mappings *virtually defined almost isometries* and use the notation

$$\phi_n : N_G \dashrightarrow M_n$$

to refer to these mappings and their implicitly defined compact domains $K(\phi_n) = K_n$.

The sequence $(M_n) \subset AH(N)$ converges *strongly* to a limit $M \in AH(N)$ if M_n converges to M in $AH(N)$ and there are base-frames $\omega_n \in M_n$ and $\omega \in M$ so that (M_n, ω_n) converge geometrically to (M, ω). As a point of terminology, we will say that an algebraically convergent sequence $M_n \to M$ *converges geometrically to a limit N_G covered by M* to refer to the existence of base-frames for which (M_n, ω_n) converges geometrically to (N_G, ω_G), and N_G is locally isometrically covered by M.

The thick-thin decomposition. By the Margulis lemma (see [BP, Thm. D.3.3]), there is a uniform constant $\mu > 0$, so that for any $\epsilon < \mu$ and any complete hyperbolic 3-manifold M, each component T of the ϵ-thin part $M^{\leq \epsilon}$ of M where the injectivity radius is at most ϵ has a standard form: either

1. T is a *Margulis tube*: a solid torus neighborhood $\mathbb{T}^\epsilon(\gamma)$ of a short geodesic γ in M with $\ell_M(\gamma) < 2\epsilon$ (T is the short geodesic itself if $\ell_M(\gamma) = 2\epsilon$); or
2. T is a *cusp*: the quotient of a horoball $B \subset \mathbb{H}^3$ by the action of a $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$ parabolic subgroup of $\mathrm{Isom}^+ \mathbb{H}^3$ with fixed point at $\overline{B} \cap \widehat{\mathbb{C}}$.

When $T = B/\mathbb{Z} \oplus \mathbb{Z}$, the component T is called a *rank-2 cusp*, and when $T = B/\mathbb{Z}$, T is called a *rank-1 cusp*. The constant μ is called the *3-dimensional Margulis constant*.

Given a complete hyperbolic manifold M and $\epsilon < \mu$, we will typically denote by P^ϵ the *cuspidal ϵ-thin part of M*, namely, the union of components of $M^{\leq \epsilon}$ corresponding to cusps of M, and we will frequently denote single cusp components of $M^{\leq \epsilon}$ by $\mathbb{P}^\epsilon$. When reference to the hyperbolic manifold is required we will use the notation $P^\epsilon(M)$, though we will often omit the reference to M when there is no danger of confusion.

Given a complete hyperbolic 3-manifold M with cusps P, we will refer to the complement $M \setminus \mathrm{int}(P)$ as the *pared submanifold* of M. Each end of the pared submanifold of M will be called a *relative end* of M. A relative end of M is *degenerate* if it has a neighborhood that embeds in the convex core of M, and it is *tame* if it has a neighborhood homeomorphic to $S\times\mathbb{R}_+$ that is a component of the complement of a compact surface $(S,\partial S)\hookrightarrow(M,\partial P)$. When such a surface determines a neighborhood of a relative end of M, we will frequently make the usual notational abuse that refers to such a neighborhood of an end as the "end" itself.

The hyperbolic 3-manifold M is *geometrically finite* if its convex core $CC(M)$, namely, the minimal geodesically convex subset of M whose inclusion is a homotopy equivalence, has finite volume. The *convex core boundary* $\partial CC(M)$ is a collection of finite-area hyperbolic Riemann surfaces with the intrinsic path metric induced from M. By a theorem of Marden [M], geometrically finite hyperbolic 3-manifolds are tame; for geometrically finite M the interior $\mathrm{int}(CC(M))$ is homeomorphic to M (see [T1], [EM]).

When the geometrically finite manifold M has cusps, the convex core $CC(M)$ is naturally compactified by adjoining compact annuli and tori at infinity corresponding to each cusp of M (otherwise M is *convex cocompact*). We denote by $\mathcal{M}$ this compactification, and by the pair $(\mathcal{M},\mathcal{P})$ the *associated pared manifold* for M, where $\mathcal{P}\subset\partial\mathcal{M}$ denotes the union of such parabolic annuli and tori in $\partial\mathcal{M}$, the *parabolic locus*. The pared manifold $(\mathcal{M},\mathcal{P})$ plays primarily the role of recording topological and algebraic information concerning cusps of the manifold M.

Embedding cores in the geometric limit. A *compact core* of a 3-manifold N is a compact submanifold C such that the inclusion $C\hookrightarrow N$ is a homotopy equivalence. If the manifold N has boundary ∂N then a *relative compact core* is a compact core C such the inclusion of $C\cap\partial N$ in ∂N is also a homotopy equivalence. By a theorem of Peter Scott [S], each irreducible 3-manifold with finitely generated fundamental group admits a compact core.

After [BrBES], we may constrain our investigation to algebraic limits M homotopy-equivalent to a *compression body*, namely, a compact, irreducible, orientable 3-manifold N that has a privileged boundary component $\partial_{\mathrm{ext}}N$ called the *exterior boundary* such that $\pi_1(\partial_{\mathrm{ext}}N)$ surjects onto $\pi_1(N)$. The remaining components $\partial N\setminus\partial_{\mathrm{ext}}N$ are called the *interior boundary* of N and denoted $\partial_{\mathrm{int}}N$. Each component of $\partial_{\mathrm{int}}N$ is incompressible.

REMARK. In fact, we may further constrain our working assumptions, but we must first pause to address an omitted case of [BrBES]. Though the hypotheses of [BrBES] focus on algebraic limits M that are not homotopy equivalent to a compression body, its techniques are sufficient to cover all cases in which the algebraic limit M has a compact core that is not *homeomorphic* to a compression body. To see this, note that in this context the remark following Corollary 3.3 of [AC2] applies to show that any type-preserving sequence $M_n \to M$ converges strongly, and thus M is tame provided each M_n is geometrically finite by the main theorem of [Ev]. Since Theorem 1.9 of [BrBES] guarantees the existence of a type-preserving sequence converging to M, it follows that M is tame provided its compact core is not homeomorphic to a compression body.

Following the above remark, many of our preparatory discussions will be constrained to treat the case when the limit M has a compact core homeomorphic to a compression body. In particular, we now prove the following lemma allowing us to choose in certain cases a core for such an M that embeds in the geometric limit (cf. [AC1,2]). We remark that the proof of this lemma was suggested to us by Richard Evans.

LEMMA 2.1. *Let $M_n \to M$ be an algebraically convergent sequence with geometric limit N_G covered by M, and assume M has empty conformal boundary. If M has a compact core C_0 homeomorphic to a compression body, then there is a compact core C for M that embeds in N_G under the covering projection $\pi : M \to N_G$.*

Proof. Let

$$\partial_{\text{int}} C_0 = \partial_1 C_0 \sqcup \ldots \sqcup \partial_k C_0$$

be the interior boundary of C_0. The topology of the ends cut off by $\partial_1 C_0, \ldots, \partial_k C_0$ is understood after work of Bonahon (see [Bo]), which we record for future reference.

Bonahon's Tameness Theorem. *Let M be a complete hyperbolic 3-manifold with cuspidal thin part P. If each end of the pared submanifold $M \setminus \text{int}(P)$ is incompressible then M is tame.*

It follows that the boundary components $\partial_1 C_0, \ldots, \partial_k C_0$ of $\partial_{\text{int}} C_0$ bound ends $E_1, \ldots, E_k$ of M with $E_i \simeq \partial_i C_0 \times \mathbb{R}_+$. Further, all of these ends are degenerate by our assumption that the conformal boundary of M is empty. We apply the covering theorem of Thurston and Canary (see [T1], [C3]).

The Covering Theorem. *Let M be a complete hyperbolic 3-manifold with parabolic locus P and let N be a hyperbolic 3-manifold covered by M by a local isometry $\pi : M \to N$. Then if E is a tame degenerate end of $M \setminus P$ then either N has finite volume and fibers over the circle, the restriction $\pi|_E$ is finite-to-one.*

The covering theorem together with an application of [JM, Lemma 3.6] (see, e.g. [KT], [AC1]) implies that the locally isometric cover from an algebraic limit to a corresponding geometric limit is an embedding on each tame degenerate relative end. Hence, there is an $a \in \mathbb{R}_+$ such that the subset $\partial_i C_0 \times [a,\infty)$ of E_i embeds under π for $i = 1,\dots,k$. After increasing a we may assume that $\pi(\partial_i C_0 \times [a,\infty)) \cap \pi(\partial_j C_0 \times [a,\infty)) = \emptyset$ for all $i \neq j$. Up to changing C_0 by an isotopy we may assume that $a = 0$, so E_i itself embeds.

Set $K = \cup_{i=1}^k \partial_i C_0 \times [0,1]$. Now we can choose a graph $G \subset M \setminus \cup_{i=1}^k E_i$ which intersects K only in its endpoints, with $K \cup G$ connected and such that the induced homomorphism $\pi_1(K \cup G) \to \pi_1(M)$ is an isomorphism. Further, a general position argument shows that we can isotope G to guarantee that π is an embedding when restricted to G.

Assume that $\pi(G)$ intersects $\pi(E_i)$ for some $i = 1,\dots,k$. Since $\pi(E_i)$ is homeomorphic to $\partial_i C \times \mathbb{R}_+$ we can isotope $\pi(G)$ relative to its endpoints in N_G to a graph which only intersects $\cup_i \pi(E_i)$ at its endpoints. This isotopy lifts to an isotopy from G to a graph G' such that $K \cup G'$ is connected and embeds under π and such that $\pi_1(K \cup G') \to \pi_1(M)$ is an isomorphism. Any regular neighborhood C of $K \cup G'$ is a compact core of M and if it is small enough then it embeds under π. □

Recall that we have the virtually defined maps

$$\phi_n : N_G \dashrightarrow M_n .$$

When M is as in the above lemma, the submanifold $C_n = \phi_n(C) \subset M_n$ is a compact core for all sufficiently large n, say for all n.

Uniform length decay. We now define a condition on a geometrically convergent sequence that will give us a substantially greater degree of control on the degenerations that can occur in the geometric limit.

DEFINITION 2.2. *A sequence (M_n, ω_n) of framed hyperbolic 3-manifolds has* uniform length decay *if for every n and each $R > 0$ there is an $\epsilon > 0$, so that if the R-ball $B_R(\omega_n) \subset M_n$ intersects a Margulis tube $\mathbb{T}_\alpha^\mu$ about a closed geodesic α, then we have*

$$\ell(\alpha) > \epsilon .$$

In a similar spirit to the argument of [BrBES], we employ the drilling theorem of [BrB] to prove that each limit of geometrically finite manifolds is approximated by a sequence with uniform length decay.

Theorem 2.3. *Let $M_n \in AH(N)$ be an algebraically convergent sequence of geometrically finite manifolds with algebraic limit M. Then there is a*