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Question 1. Check all the definitions of tangent space and prove the equivalence between them.

Let $M \subset \mathbb{R}^n$ be a submanifold and consider it as a manifold. Prove that the tangent space (bundle) as defined for M as a submanifold and as a manifold coincide.

Question 2. A *Lie group* is a smooth manifold G together with a group structure (G, \cdot) such that the map

$$G \times G \rightarrow G, \quad (g, h) \mapsto g \cdot h^{-1}$$

is smooth. Denote by e the identity element.

- (1) Prove that the map

$$G \rightarrow G \quad g \mapsto g^{-1} \quad \text{and} \quad G \times G \rightarrow G \quad (g, h) \mapsto g \cdot h$$

are smooth as well.

- (2) Prove that $\mathrm{SO}_n, \mathrm{SU}_n, \mathrm{GL}_n \mathbb{R}$ and $\mathrm{SL}_n \mathbb{R}$ are Lie groups.
 (3) Given $g \in G$ consider the links translation $L_g : G \rightarrow G$ given by $L_g(h) = g \cdot h$. A vector field X (a section of the tangent bundle) is *lefts-invariant* if for all g we have

$$X(g \cdot h) = dL_g|_h X(h)$$

Let \mathfrak{g} be the set of all left-invariant vector fields. Prove that \mathfrak{g} is a vector space and that the map

$$\mathfrak{g} \rightarrow T_e G, \quad X \mapsto X(e)$$

is an isomorphism. (\mathfrak{g} is the *Lie-algebra* of the group G .)

- (4) Deduce from (3) that the tangent bundle TG is trivial. More precisely, construct an isomorphism $TG \simeq G \times \mathfrak{g}$.
 (5) Compute \mathfrak{so}_n and \mathfrak{sl}_n as subspaces of $\mathbb{R}^{n \times n}$ (in other words, interpret the tangent space as the tangent space for a submanifold).
 (6) Given $g \in G$ let $\kappa_g : G \rightarrow G$ be the map $\kappa_g(h) = g \cdot h \cdot g^{-1}$. Prove that for $X \in \mathfrak{g}$,

$$h \mapsto d\kappa_g|_h X(h)$$

is also a left-invariant vector field. Prove that, via the isomorphism given in (3), the corresponding action on $T_e G$ is given by

$$\mathrm{Ad}_g : v \mapsto d\kappa_g|_e(v)$$

Check that $\text{Ad}_g \circ \text{Ad}_h = \text{Ad}_{g \cdot h}$. The so obtained homomorphism $\text{Ad} : G \rightarrow \text{End}(\mathfrak{g})$ is the *adjoint representation* of G .

- (7) Given $v \in T_e G$ let X_v be the associated left-invariant vector-field. Deduce from the theorem on existence of solutions of differential equations that there is

$$\exp(tv) : \mathbb{R} \rightarrow G$$

with $\frac{d}{dt} \exp(tv)|_t = X_v(\exp(tv))$ for all t . (You need to show that X_v is a complete vector field.)

- (8) Consider the map

$$\text{Exp} : T_e G \rightarrow G, \quad v \mapsto \exp(v)$$

Compute the differential at $0 \in T_e G$ and show that it is a local diffeomorphism, meaning that there is a small neighborhood $U \subset T_e G$ of 0 such that $\text{Exp}|_U$ is a diffeomorphism onto its image. The locally defined inverse $\log : \text{Exp}(U) \rightarrow U$ is a *logarithm*.

- (9) Assume now that $G \subset \text{GL}_n \mathbb{R}$ is a matrix group, say $G = \text{SO}_n$, and identify $T_e G$ with a subspace of $T_e \text{GL}_n \mathbb{R} = \mathbb{R}^{n \times n}$. Prove that for each $A \in T_e G$ we have

$$\exp(A) = e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

and that for all $\text{Id} + B \in \text{Exp}(U)$ we have

$$\log(\text{Id} + B) = \text{Id} + B - \frac{B^2}{2} + \frac{B^3}{3} - \frac{B^4}{4} + \dots$$

- (10) Still with notation as in (9) let $A, B \in T_e G$. Use the formula in (9) to compute (for a matrix group) the Taylor expansion at $t = 0$ of the map

$$(-\epsilon, \epsilon) \rightarrow G, \quad t \mapsto \log(e^{tA} \cdot e^{tB})$$

You should at least compute the first say 3 terms. The result should look like the first terms of the so-called *Baker-Campbell-Hausdorff formula*:

$$\begin{aligned} \log(e^A \cdot e^B) = & A + B + \frac{1}{2}[A, B] + \\ & + \frac{1}{12}[A, [A, B]] - \frac{1}{2}[B, [A, B]] + \dots \end{aligned}$$

Here $[X, Y] = XY - YX$; see wikipedia for the complete formula.

Question 3. Prove that every \mathbb{R} -line bundle over \mathbb{R} is trivial. Prove that every \mathbb{R} -line bundle over \mathbb{S}^1 is isomorphic to either the trivial bundle or to the Möbius band.

Question 4. Let $E = \{([x], v) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} \mid v \in [x]\}$ and $\pi : E \rightarrow \mathbb{C}P^n$ given by $\pi([x], v) = [x]$. Constructing a trivialization, prove that $\pi : E \rightarrow \mathbb{C}P^n$ is a \mathbb{C} -line bundle over complex projective space $\mathbb{C}P^n$. This is the so-called canonical line bundle over $\mathbb{C}P^n$.

Similar "canonical bundles" can also be constructed for every Grassmannian. In the case of the Grassmannian of k -planes in \mathbb{R}^n it is a bundle with (real) k -dimensional fibers.

Question 5. If you are into that, check that the sections of a bundle form a sheaf; observe that unless the fiber has some additional structure, this is just a sheaf of abelian groups and not of algebras. Prove also that the sheaf C^∞ of smooth functions on a manifold M is isomorphic to the sheaf of sections of the trivial bundle $M \times \mathbb{R}$.