537: Differential topology **Juan Souto**

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Question 1. Check the definition of pseudogroup and convince yourself that the set $\mathcal{G}$ consisting of all biholomorphisms $\phi : U \to V$ where U and V are open subsets of $\mathbb{C}$ is a pseudo-group. Assume that a 2-dimensional manifold M has a $(\mathbb{C}, \mathcal{G})$-structure (in other words, M is a Riemann surface).

(1) Convince yourself that each tangent space has a natural complex structure (i.e., T_xM is in a natural way a $\mathbb{C}$-vectorspace, where "natural" means that you find the same structure as your neighbor).
(2) Given $U \subset M$ open let $\mathcal{O}(U)$ and $C^\infty(U)$ be the sets of holomorphic and smooth functions on U. Check that $\mathcal{O}(U)$ and $C^\infty(U)$ are algebras. Check the definition of *sheaf* and prove that both $\mathcal{O}$ and C^∞ are sheaves of algebras.
(3) Prove that if M is compact then there are no non-constant holomorphic functions defined on the whole of M (Lioville's theorem).
(4) Prove that for all $x, y \in M$ with $x \neq y$ there is a smooth function, meaning C^∞, with $f(x) \neq f(y)$.
(5) Check the definitions of *germ of a function* and *stalk of a sheaf.* Prove that the stalks $\mathcal{O}_x$ and C^∞_x of $\mathcal{O}$ and C^∞_x at x are algebras. Prove that the set $\mathfrak{m}_x$ (resp. $\mathfrak{a}_x$) consisting of the germs of those holomorphic (resp. smooth) functions which vanish at x is a maximal ideal of $\mathcal{O}_x$ (resp. C^∞_x). (In fact, $\mathcal{O}_x$ and C^∞_x are local rings with maximal ideal $\mathfrak{m}_x$ and $\mathfrak{a}_x$ respectively).
(6) Prove that $\cap_{l \in \mathbb{N}} \mathfrak{a}_x^l$ is a proper ideal, meaning that it is not 0. Here $\mathfrak{a}_x^l$ is the ideal generated by products of l elements in $\mathfrak{a}_x$.

Note: To a large extent, this problem amounts essentially to check the meaning of words (wikipedia does it). The point of the problem is to see that the perhaps scary words used above are actually just hot air.

Question 2. Assume that X is a manifold and that Γ is a group acting freely and properly discontinuously on X. Endow $M = X/\Gamma$ with the quotient topology and let $\pi : X \to M$ be the associated projection map. Endow M with the structure of a manifold with a (X, Γ)-structure.
Note: I will do this briefly in class.

Question 3. Construct a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that for no $x \in \mathbb{R}$ the level set $f^{-1}(x)$ is a manifold.

Question 4. Assume that M is a connected 1-dimensional manifold. Prove that M is either homeomorphic to the real line $\mathbb{R}$ or to the circle $\mathbb{S}^1$.

Question 5. Construct an atlas for $\mathbb{S}^n$, $\mathbb{R}P^n$, $\mathbb{C}P^n$, and $Gr_k(\mathbb{R}^n)$ (the Grassmannian of k-planes in $\mathbb{R}^n$). If you don't know what these things are, check the definition in say wikipedia.