537: Differential topology **Juan Souto**

9-9-2008

Hola,

You can leave your solutions until Wednesday in Johanna Mangaha's office (EH 3860). She will grade at least one of the problems. In any case, if you want to discuss any of them you are always well-come to talk to me.

Juan

Question 1. *By construction,* $(\mathbb{R}, |\cdot|)$ *is complete.*

(1) *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces and consider the* $C^0(X, Y)$ *be the set of all continuous maps* $f : X \to Y$ *and set*

$$d_{C^0}(f, g) = \max_{x \in X} d_Y(f(x), f(y))$$

Show that $(C^0(X, Y), d_{C^0})$ *is a complete metric space.*

(2) *Assume that* $U \subset \mathbb{R}^n$ *is open and let* $C^1(U)$ *be the set of continuously differentiable maps* $f : U \to \mathbb{R}^m$ *and set*

$$d_{C^1}(f, g) = d_{C_0}(f, g) + d_{C^0}(Df, Dg)$$

Show that $(C^1(U), d_{C^1})$ *is a complete metric space.*

(3) *Let* l^2 *be the set of all sequence* (a_i) *with index set* $\mathbb{N}$ *and* $a_i \in \mathbb{R}$ *with such that the series* $\sum_i |a_i|^2$ *converges and set*

$$d_{l^2}((a_i), (b_i)) = \sqrt{\sum_i |a_i - b_i|^2}$$

Prove that (l_2, d_{l_2}) *is a complete metric space.*

Question 2.

(1) *Assume that* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is continuously differentiable and that for all* $x \in \mathbb{R}^m$

$$\|Df|_x\| = \max_{v \in \mathbb{R}^n, v \neq 0} \frac{\|Df_x v\|}{\|v\|} \leq L$$

Show that f *is L-Lipschitz.*

(2) *Construct a continuous function* $f : \mathbb{R} \to \mathbb{R}$ *which is differentiable almost everywhere with* $f'(x) = 0$ *but which is not Lipschitz.*

Question 3.

(1) *Construct a 1-Lipschitz map* $\mathbb{R} \to \mathbb{R}$ *without fix-points.*

(2) *Prove that if* $f : (X, d) \to (X, d)$ *is continuous, surjective and* $d(f(x), f(y)) \geq 2d(x, y)$ *for all* x, y *then* f *has a fix-point.*
(3) *Construct an example showing that the condition that* f *is surjective in (2) is necessary.*

Question 4.

(1) *Construct a non-complete vector field on* $\mathbb{R}^2$*.*
(2) *Prove that any bounded vector field is complete.*
(3) *Assume that* F *is a complete vector field on* $\mathbb{R}^n$ *and let for* $x \in \mathbb{R}^n$ *be* γ_x *be the solution to the ODE* $\gamma_x'(t) = F(\gamma_x(t))$ *and* $\gamma_x(0) = 0$*. Prove that the map* $(x, t) \mapsto \gamma_x(t)$ *is continuous.*

Question 5. *Prove that* SO_3 *is a submanifold of* $\mathbb{R}^{3\times 3}$ *homeomorphic to the real projective space* $\mathbb{R}P^3$*. Similarly, prove that* SU_2 *is a (real) submanifold of* $\mathbb{C}^{2\times 2}$ *homeomorphic to* $\mathbb{S}^3$*.*

Question 6. *Realize the real projective plane* $\mathbb{R}P^2$ *as a submanifold of* $\mathbb{R}^n$ *for some* n*.*

Question 7. *Realize the co-tangent bundle* T^*M *and the endomorphism bundle* $\mathrm{End}(TM)$ *as submanifolds of* $\mathbb{R}^m$ *for some* m*.*