537: Differential topology **Juan Souto**

VECTOR BUNDLES AND LINEAR ALGEBRA

No guarantee that things are correct. There are namely many more typos than normally!!!

A say smooth vector bundle $\pi : E \to M$ is given by a (surjective) projection π between two manifolds E (the total space) and B (the base) such that:

- For each $x \in M$, the *fiber* $E_x = \pi^{-1}(x)$ of x is a vector space of say dimension k.
- If M has say dimension n, then there are open sets $U \subset M$ and charts $\phi : U \to V \subset \mathbb{R}^n$ and $\Phi : \pi^{-1}(U) \to V \times \mathbb{R}^k$ such that the projection of $\Phi(e)$ to the first factor coincides with $\phi(\pi(e))$ for all $e \in \pi^{-1}(U)$. Moreover, the obvious map
$$\Phi|_{E_x} : E_x \to \mathbb{R}^k$$
is a vector space isomorphism for all $x \in U$. Such charts Φ are said to be trivializations.
- Given two trivializations $\Phi\pi^{-1}(U) \to V \times \mathbb{R}^k$ and $\Phi' : \pi^{-1}(U) \to V' \times \mathbb{R}^k$ then the map
$$\Phi' \circ \Phi^{-1} : V \times \mathbb{R}^k \to V' \times \mathbb{R}^k$$
is given by
$$(\Phi \circ \Phi^{-1})(x, v) = (f(x), A_x(v))$$
where $f : V \to V'$ and $x \mapsto A_x$ are smooth maps. The later with values in $\mathrm{GL}_k(\mathbb{R})$. This linearity is the key point.

Examples of vector bundles:

(1) The tangent bundle to a smooth manifold.
(2) The trivial bundle $M \times \mathbb{R}^k$.
(3) The cylinder and the Möbious bans are the total spaces of line (i.e. 1-dimensional) bundles overt $\mathbb{S}^1$.
(4) The canonical line bundle over $\mathbb{R}P^n$ and $\mathbb{C}P^n$ and their cousins over Grassmannians.
(5) Flat bundles (this is going into the next problem sheet).

Another important construction of vector bundles: **pull-back**. The setting is as follows. Assume that $\pi : E \to N$ is a say smooth vector bundle and that $f : M \to N$ is smooth. Then the pull-back is bundle $\sigma : f^*E \to M$ such that for each $x \in M$ we have $(f^*E)_x = E_{f(x)}$.

Moreover, this identification varies smoothly with x, meaning that we have a map $F : f^*E \to E$ such that the following diagram conmutes:

$$\begin{array}{ccc} f^*E & \xrightarrow{F} & E \\ \Big\downarrow{\scriptstyle \sigma} & & \Big\downarrow{\scriptstyle \pi} \\ M & \longrightarrow & N \end{array}$$

The precise construction you can find somewhere else; the idea is to construct the trivializations of f^*E using those of E and the charts of M.

A *section* of a bundle $\pi : E \to M$ is smooth map $\sigma : M \to E$ with $\pi \circ \sigma = \mathrm{Id}$. Intuitively, it means that one chooses smoothly an element of E_x for each $x \in M$. Some remarks:

- The set of all sections of $\pi : E \to M$ is a vector space: just add and multiply in each fiber.
- The set of all sections of $\pi : E \to M$ is a sheaf.
- The set of all sections of the trivial bundle $M \times \mathbb{R}^k$ is isomorphic to $C^\infty(M, \mathbb{R}^k)$. If $k = 1$, this is a homomorphism of algebras. For $k \geq 2$ this last sentence does not make sense since $\mathbb{R}^k$ is not an algebra.
- A section of the tangent bundle is said to be a *vector field*.

Vector bundles are in some way just ways to work with nice families of vector spaces at the same time. A basic idea to construct examples of vector bundles is to apply sufficiently canonical constructions in linear algebra. Let's say first what is a "not sufficiently canonical" construction. You know that every vector space has a **basis**. However, there are many choices for basis. In other words a vector space does not have a canonical basis. This is reflected by the fact that there are vector bundles where it is **impossible** to choose smoothly (continuously) a basis for each fiber (think of the **Möbious band**). Similarly, each vector space has an orientation... but there are two of them. This lack of uniqueness is again reflected in the fact that there are vector bundles where it is impossible to choose coherently an orientation of each fiber (think again of the Möbious band).

Constructions in linear algebra and constructions of bundles: I just make a list. Some notation first. Below V, W, U are (finite dimensional real) vector spaces, f, g, h are linear maps, $E \to M$, $F \to M$ and $L \to M$ and vector bundles over a common base and $\phi, \psi, ...$ are bundle homomorphisms with map for each x the fiber over x in say E to the fiber over x in F (they cover the identity).

(1) **Direct sum:**
- Given two vector spaces U, V there is a vector space $U \oplus V$ and homomorphisms $i_U : U \to U \oplus V$ and $i_V : V \to U \oplus V$ such that for every other vector space W and pair of homomorphism $f : U \to W$ and $g : V \to W$ there is $h : U \oplus V \to W$ with $h \circ i_U = f$ and $h \circ i_V = g$.
- Given two vector bundles $E \to M, F \to M$ there is a vector bundle $U \oplus V \to M$ and bundle homomorphisms $i_E : E \to E \oplus F$ and $i_F : F \to E \oplus F$ (covering the identity, i.e. $E_x \mapsto (E \oplus F)_x$) such that for every other vector bundle $L \to M$ and pair of bundle homomorphism $\phi_E : E \to F$ and $\phi_F : F \to L$ there is $\psi : E \oplus F \to L$ with $\psi \circ i_E = \phi_E$ and $\psi \circ i_F = \phi_F$.

(2) **Cartesian product:**
- Given two vector spaces U, V there is a vector space $U \times V$ and homomorphisms $\pi_U : U \times V \to U$ and $\pi_V : U \times V \to V$ such that for every other vector space W and pair of homomorphism $f : W \to U$ and $g : W \to V$ there is $h : W \to U \times V$ with $\pi_U \circ h = f$ and $\pi_V \circ h = g$.
- Given two vector bundles $E \to M, F \to M$ there is a vector bundle $E \times F \to M$ and bundle homomorphisms $\pi_E : E \times F \to E$ and $\pi_F : E \times F \to F$ (covering the identity) such that for every other vector bundle $L \to M$ and pair of bundle homomorphism $\phi_E : L \to E$ and $\phi_F : L \to F$ there is $\psi : L \to E \times F$ with $\pi_E \circ \psi = \phi_E$ and $\pi_F \circ \psi = \phi_F$.

(3) **Quotient:**
- Given a vector space V and a subspace $U \subset V$ there is a vector space V/U and a homomorphism $\pi : V \times V/U$ such that for every other vector space W and homomorphism $f : V \to W$ with $U \subset \operatorname{Ker}(f)$, there is $h : V/U \to W$ with $h \circ \pi = f$.
- Given a vector bundle $E \to M$ and a subbundle $F \subset V$ there is a vector bundle $E/F \to M$ and a homomorphism $\pi : E \times E/F$ such that for every other vector bundle $L \to M$ and homomorphism $\phi : E \to F$ with $F_x \subset \operatorname{Ker}(\phi : E_x \to L_x)$ for each $x \in M$, there is $\psi : E/F \to L$ with $\psi \circ \pi = \phi$.

(4) **Tensor product:**
- Given two vector spaces U, V there is a vector space $U \otimes V$ and a bilinear map $\phi : U \times V \times U \otimes V$ such that for every

other vector space W and bilinear map $f : U \times V \to W$ there is a unique linear map $F : U \otimes V \to W$ with $F \circ \phi = f$.

- Given two vector bundles $E \to M, F \to M$ there is a vector bundle $E \otimes V \to M$ and fiber-preserving smooth map $\pi : E \times F \to E \otimes F$, bilinear in each fiber, such that for every other vector bundle $L \to M$ and fiber-preserving smooth map $f : E \times F \to L$ there is a unique bundle homomorphism $F : E \otimes F \to L$ with $F \circ \pi = f$.

There are many other constructions, which are perhaps even more important than the one's listed above. For instance there are dual space V^* which produces **dual bundle** E^* and the exterior product which produces the **exterior product** $\Lambda^k E$. The later is the most important one and I would **very much** suggest that you read in say Guillemin-Pollack what they have to say on the exterior product.