537: Differential topology **Juan Souto**

MANIFOLDS 1

Definition of manifold: countable basis, locally euclidean, Hausdorff, paracompact (Every open cover admits a locally finite refinement: more or less equivalent to the existence of a partition of unity).

C^1-manifold, smooth manifold, analytic manifold, complex manifold, euclidean manifold, foliated manifold. Summing up: $(\mathcal{G}, \mathcal{X})$-structure on a manifold.

Definition. *Let $\mathcal{X}$ be a manifold. A* pseudo-group $\mathcal{G}$ *of a homeomorphisms of $\mathcal{X}$ is a collection of maps $f : U \to V$ where $U, V \subset \mathcal{X}$ are open sets and f is a homeomorphism such that the following holds:*

- *For every open set U, the identity map on U is in $\mathcal{G}$.*
- *If $f : U \to V$ is in $\mathcal{G}$, then so is $f^{-1} : V \to U$.*
- *If $f : U \to V$ is in $\mathcal{G}$, then the restriction of f to an arbitrary open subset of U is in $\mathcal{G}$.*
- *If U is open, then U is the union of the open sets U_i, f is a homeomorphism from U to an open subset of $\mathcal{G}$, and the restriction of f to U_i is in $\mathcal{G}$ for all i, then f is in $\mathcal{G}$.*
- *If $f : U \to V$ and $f' : V \to W$ are in $\mathcal{G}$, then $f' \circ f \in \mathcal{G}$.*

A manifold has a $(\mathcal{X}, \mathcal{G})$ structure if all the charts take values in $\mathcal{X}$ and the changes of charts are elements of $\mathcal{G}$.

Examples of possible $(\mathcal{X}, \mathcal{G})$ structures on manifolds are say C^1, C^∞, analytic, holomorphic...

From now on assume that all manifolds are C^∞. A map $f : M \to N$ is smooth if the composition of "coming up from a chart", f, and "going down with a chart" is smooth. The most important thing is to check that this is well-defined; this is so because changing say the chart one goes up with differers from the original by the obvious change of charts, which is smooth by assumption. **It is very important to understand this well.**

If M is a C^∞-manifold and $U \subset M$ is open, let $C^\infty(U)$ be the set of all smooth functions $f : M \to \mathbb{R}$. It follows directly from the definition that for each such U the set $C^\infty(U)$ is a $\mathbb{R}$-algebra, meaning that one can add functions, can multiply functions with reals and with other functions and that these operations satisfy the obvious conditions of associativity, commutativity and distributivity.
The map $U \to C^\infty(U)$ forms a sheaf (check the precise definition somewhere such as wikipedia) meaning that

- if $W \subset V \subset U$ is are further open subsets then one can restricts functions on U to V and then to W and one gets the same things as directly restricting to W.
- if $U = \cup U_i$ and one has functions f_i defined on U_i with $f_i = f_j$ on $U_i \cap U_j$, then all the f_i are the restriction to U_i of some globally defined $f \in C^\infty(U)$; just glue the f_i together.

Given $p \in M$ we say that a smooth function f is defined around p if the domain of f contains p. Two functions f and g defined around p are equivalent $f \sim g$ if there is a small neighborhood U of p with $f|_U = g|_U$. An equivalence class is said to be a "germ at p". Obviously, one can multiply and add germs. In other words, the set C^∞_p of all germs at p forms an algebra. Wanting to sound fancy, C^∞_p is the *stalk* of the sheaf C^∞ at p. Wanting to be even more fancy, one can say that C^∞_p is the projective limit of the $C^\infty(U)$ over all open neighborhoods of p. However fancy one sounds, it is just the set of equivalence classes defined above.

Important note: All these words like sheaf, stalk, pseduo-group and so on could sound sexy or scary. However, it is important to remember that, as one says in Spanish, "aunque la mona se vista de seda, mona se queda" which translates as "even if the monkey dresses with silk, it is still a monkey".

Definition of tangent space: Two (germs of) curves $\gamma, \eta : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = \eta(0) = x$ are equivalent $\gamma \simeq \eta$ if for every smooth function defined around x we have that

$$(f \circ \gamma)'(0) = (g \circ \eta)'(0)$$

The tangent space $T_x M$ is the set of equivalence classes of curves.

It follows directly from the definition that if $U \subset M$ and $V \subset N$ are open subsets, $x \in U$ and $\phi : U \to V$ is a differentiable then there is map

$$df_x : T_M \to T_{f(x)}N, \quad df_x[\gamma] = [f \circ \gamma]$$

this is the *differential* of f at x. It is also clear that if f has a differentiable inverse g then $dg_{f(x)}$ is the inverse of df_x. In particular, df_x is bijective.

So far it makes no sense to say that df_x is linear or anything like this. The tangent space $T_x M$ is so far just a set. In order to solve this problem, one wants to identify $T_x M$ with some vector space and defined the operations in $T_x M$ to be the induced operations. So, let U be a neighborhood of x and $\phi : U \to \mathbb{R}^n$ a chart with $\phi(x) = 0$. By the above, the map $d\phi_x$ yields a bijection between $T_x M$ and $T_0\mathbb{R}^n$.

Lemma 0.1. *We have a canonical isomorphism between $T_0\mathbb{R}^n$ and $\mathbb{R}^n$. Moreover, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable with a differentiable inverse and $f(0) = 0$ then the map $\mathbb{R}^n \to \mathbb{R}^n$ given by the identifcation of $T_0\mathbb{R}^n$ and $\mathbb{R}^n$ and df_0 is the linear map Df_0.*

This lemma asserts that first, any chart yields a vector space structure on T_xM and that any two charts yields compatible structures.

I also discussed the identification between T_xM and the space of derivations of the algebra C^∞_x. For this last equivalence, as well as for the former definition, see the book of Bröcker-Jänich. I also identified T_xM with the dual space to the vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ where $\mathfrak{m}_x$ is the maximal idea of C^∞_x consisting of germs of functions which vanish at x. (This last is the definition of the Zariski tangent space in algebraic geometry.)

It is very important to understand the different definitions of T_xM and how they relate to each other!!!

Finally, I constructed the tangent bundle $TM = \cup_{x\in M} T_xM$ as a manifold. Assume that M is an n-dimensional smooth manifold. The idea was that

(1) if $V \subset \mathbb{R}^n$ is open then TV is canonically bijective to $V \times \mathbb{R}^n$, and
(2) if $U \subset M$ is open and $\phi : U \to V \subset \mathbb{R}^n$ is a chart then the map
$$d\phi : TU \to TV, \quad d\phi(x, v) \mapsto d\phi_x v$$
is a bijection.

By (1), TV is naturally identified with an open subset of $\mathbb{R}^n \times \mathbb{R}^n$. Pulling back the topology of TV by the bijection $d\phi$ we obtain a topology on TU. If $\psi : U \to V' \subset \mathbb{R}^n$ is another chart, then both topologies agree (in fact, both induced smooth structures agree). A set W in TM is open if $W \cap TU$ is open for each U as above. This topology is Hausdorff, second countable, paracompact and, by definitions, TM is locally euclidean. In other words, TM is a manifold with charts $d\phi$. Since, as remarked above, the chart changes are smooth, we obtain that TM is a smooth manifold.

Remarks.
(1) If M is only a topological manifold, TM makes no sense!!!
(2) If M has more structure (say M is complex) then TM tends to have more structure (TM is a complex manifold and each of the vector spaces T_xM is complex as well).