537: Differential topology **Juan Souto**

SUBMANIFOLDS

A subset $M \subset \mathbb{R}^n$ is a *k-dimensional submanifold* if every point $x \in M$ has an open neighborhood $U \subset \mathbb{R}^n$ such that there is a homeomorphism onto its image

$$\phi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$$

with $\phi(M \cap U) = \phi(U) \cap (\mathbb{R}^k \times \{0\})$. If the map ϕ can be chosen to be C^1, C^2, smooth (meaning C^∞), analytic (meaning the Taylor seires converges) or holomorphic (meaning that $\mathbb{R}^n = \mathbb{C}^{\frac{n}{2}}$ and ϕ is complex differentiable) then M is a C^1, C^2,... submanifold.

How to construct submanifolds? Different versions of the implicit function theorem.

Lemma 0.1. *Assume that $F : \mathbb{R}^k \to \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ is a smooth function and that $(a, b) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$ is a point with $F(a, b) = 0$ and $\frac{\partial F}{\partial y}|_{(a,b)}$ non-singular. Then there is an open neighborhood U of (a, b) in $\mathbb{R}^n$ such that $U \cap F^{-1}(0)$ is a submanifold.*

Proof. Implicit function theorem. □

If $U \subset \mathbb{R}^n$ is an open set then a map $F : U \to \mathbb{R}^k$ is a *submersion* if $DF : \mathbb{R}^n \to \mathbb{R}^k$ has maximal rank.

Theorem 0.2. *Assume that $F : U \to \mathbb{R}^k$ is a submersion for every $x \in F^{-1}(0) \cap U$. Then $F^{-1}(0)$ is a submanifold.*

Proof. Implicit function theorem composed with rotations. □

Examples: $\mathrm{SL}_n \mathbb{R}, \mathbb{S}^2, \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^4, \mathrm{SO}_n, \mathrm{SU}_n$.

Question 1. *Realize the real projective plane $\mathbb{R}P^2$ as a submanifold of $\mathbb{R}^n$ for some n.*

Let $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$ be submanifolds, then a function $f : M \to N$ is smooth if there are U and V neighborhoods of M and N and a smooth function $F : U \to V$ extending f. Equivalence on smooth functions.

Question 2. *Assume that M and N are submanifolds and $f : M \to N$ is a function such that for each $x \in M$ there is a neighborhood U_x in $\mathbb{R}^m$ such that $f|_{U_x \cap M}$ is smooth. Prove that f is smooth.*

Two equivalent definitions of tangent space:

- Velocities of curves: T_xM is the set of those $v \in \mathbb{R}^n$ such that there is a differentiable curve with $\gamma(0) = x$ and $\gamma'(0) = v$.

- Image under the differential of the inverse of one of the maps in the definition. Namely, if ϕ is a map as above, $T_xM = D(\phi^{-1})|_{\phi(x)}\mathbb{R}^k$.

The first definition is much more intuitive and it is on what one is supposed to think. On the other hand, with the first definition it is not easy to see that T_xM is a linear subspace of $\mathbb{R}^n$; this is trivial with the second one.

Lemma 0.3. *Assume that M is a k-dimensional submanifold of $\mathbb{R}^n$. Then the tangent bundle $TM = \cup_{x\in M}T_xM$ is a $2k$-dimensional submanifold of $\mathbb{R}^{2n}$.*

Proof. For simplicity assume that $M = F^{-1}(0)$ is the 0-set of a map $F : \mathbb{R}^n \to \mathbb{R}^k$ such that DF_x has maximal rank for all $x \in M$. Consider then

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k \times \mathbb{R}^k, \quad \Phi(x, v) = (F(x), DF_xv)$$

The map Φ has maximal rank and $\Phi^{-1}(0, 0) = TM$. □

Question 3. *Realize the normal bundle $\mathcal{N}(M)$, the co-tangent bundle T^*M and the endomorphism bundle $\mathrm{End}(TM)$ as submanifolds of $\mathbb{R}^n \times (\mathbb{R}^*)^n$ and $\mathbb{R}^n \times \mathbb{R}^{n\times n}$ respectively.*

Assume that $f : M \to N$ is a differentiable map and recall that this means that f extends to a smooth map between neighborhoods of M and N. Let

$$df_x : T_xM \to T_{f(x)}N$$

be the restriction of DF_x to T_xM. Taking the point of view, that T_xM is a set of velocities of curves it is clear that the image of df_x is contained in $T_{f(x)}N$.