537: Differential topology **Juan Souto**

BASICS

I assume that everyone knows the theorem of implicit functions and the inverse mapping theorem. However, these theorems are so central that I will prove them. I also prove the theorem of existence and uniqueness of solutions of ODEs. The proofs here are not absolutely complete; for complete proofs look at any good calculus book.

A metric space (X, d_X) is complete if every Cauchy sequence in X converges, i.e. every sequence (x_i) such that for all $\epsilon > 0$ there is i_ϵ with $d_X(x_i, x_j) < \epsilon$ for all $i, j > i_\epsilon$.

Question 1. *By construction,* $(\mathbb{R}, |\cdot|)$ *is complete.*

(1) *Let* $(X, d_X), (Y, d_Y)$ *be metric spaces and consider the* $C^0(X, Y)$ *be the set of all continuous maps* $f : X \to Y$ *and set*

$$d_{C^0}(f, g) = \max_{x \in X} d_Y(f(x), f(y))$$

Show that $(C^0(X, Y), d_{C^0})$ *is a complete metric space.*

(2) *Assume that* $U \subset \mathbb{R}^n$ *is open and let* $C^1(U)$ *be the set of continuously differentiable maps* $f : U \to \mathbb{R}^m$ *and set*

$$d_{C^1}(f, g) = d_{C_0}(f, g) + d_{C^0}(Df, Dg)$$

Show that $(C^1(U), d_{C^1})$ *is a complete metric space.*

(3) *Let* l^2 *be the set of all sequence* (a_i) *with index set* $\mathbb{N}$ *and* $a_i \in \mathbb{R}$ *with such that the series* $\sum_i |a_i|^2$ *converges and set*

$$d_{l^2}((a_i), (b_i)) = \sqrt{\sum_i |a_i - b_i|^2}$$

Prove that (l_2, d_{l_2}) *is a complete metric space.*

A map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is L-Lipschitz if for all $x_1, x_2 \in X$ we have

$$d_Y(f(x_1), f(x_2)) \leq L d_X(x_1, x_2)$$

Question 2.

(1) *Assume that* $f : \mathbb{R}^n \to \mathbb{R}^m$ *is continuously differentiable and that for all* $x \in \mathbb{R}^m$

$$\|Df|_x\| = \max_{v \in \mathbb{R}^n, v \neq 0} \frac{\|Df_x v\|}{\|v\|} \leq L$$

Show that f *is* L*-Lipschitz.*

(2) *Construct a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable almost everywhere with $f'(x) = 0$ but which is not Lipschitz.*

A map $f : X \to X$ has a fix-point if there is $x \in X$ with $f(x) = x$. The point x is then said to be a fix-point.

Banach fix-point theorem. *Let (X, d_X) be a complete metric space and $f : X \to X$ a L-Lipschitz map with $L < 1$. Then f has a fix-point.*

Proof. Pick $x \in X$, set $x_0 = x$ and define inductively $x_{n+1} = f(x_n)$. For all $n \geq 1$ we have then

$$d_X(x_{n+1}, x_n) = d_X(f(x_n), f(x_{n-1})) \leq L d_X(x_n, x_{n-1})$$

Inductively one has $d_X(f(x_{n+1}, x_n) \leq L^n d(x_1, x_0)$. This implies then that for all $n, k \geq 1$ one has

$$d_X(x_{n+k}, x_n) \leq \sum_{j=0}^{k-1} d_X(x_{n+j+1}, x_{n+j}) \leq \sum_{j=0}^{k-1} L^{n+j} d_X(x_1, x_0)$$

$$= \frac{1 - L^k}{1 - L} L^n d_X(x_1, x_0)$$

Since $L < 1$, this implies that the sequence (x_i) is a Cauchy-sequence. In particular, it converges to some x_∞. We have then

$$x_\infty = \lim_{i \to \infty} x_i = \lim_{i \to \infty} x_{i+1} = \lim_{i \to \infty} f(x_i) = f(x_\infty)$$

Hence x_∞ is the desired fix-point. □

Question 3.

(1) *Prove that the fix-point is unique.*
(2) *Construct a 1-Lipschitz map $\mathbb{R} \to \mathbb{R}$ without fix-points.*
(3) *Prove that if $f : (X, d) \to (X, d)$ is continuous, surjective and $d(f(x), f(y)) \geq 2d(x, y)$ for all x, y then f has a fix-point.*
(4) *Construct an example showing that the condition that f is surjective in (2) is necessary.*

Implicit function theorem. *Let $U_1 \subset \mathbb{R}^k$ and $U_2 \subset \mathbb{R}^m$ be open sets and*

$$F : U_1 \times U_2 \to \mathbb{R}^m, \ (x, y) \mapsto F(x, y)$$

be a continuously differentiable map. Given $(a, b) \in U_1 \times U_2$ with $F(a, b) = 0$ assume that the linear map $\frac{\partial F}{\partial y}|_{(a,b)}$ is invertible. Then there are $V_1 \subset U_1$ and $V_2 \subset U_2$ open neighborhoods of a and b respectively and a continuously differentiable map

$$f : V_1 \to V_2$$

with $f(a) = b$ such that $F(x, f(x)) = 0$ for all $x \in V_1$. Moreover, the map $x \mapsto (x, f(x))$ maps onto the set $(V_1 \times V_2) \cap F^{-1}(0)$.

Proof. Assume without loss of generality that $(a, b) = (0, 0)$ and let $B = \frac{DF}{\partial Y}|_{(0,0)} \in \mathrm{GL}_m \mathbb{R}$. Set

$$G : U_1 \times U_2 \to \mathbb{R}^m, \quad G(x, y) = y - B^{-1}F(x, y)$$

and observe that

$$G(0, 0) = 0, \quad \frac{\partial G}{\partial y}|_{(0,0)} = 0$$

Observe also that $F(x, y) = 0$ if and only if $G(x, y) = y$.

By continuity we have small closed balls $V_1 \subset U_1$ and $V_2 \subset U_2$ of 0 with

$$\|\partial G_{(x,y)}\| \leq \frac{1}{2}$$

for all $x, y \in V_1 \times V_2$.

> **For simplicity** we assume that $G(V_1 \times V_2) \subset V_2$; otherwise one has to choose further smaller balls.

For all $x \in V_1$, the map

$$V_2 \to V_2, \quad y \mapsto G(x, y)$$

is $\frac{1}{2}$-Lipschitz and hence has a unique fixed point y. Set $f(x) = y$.

So far, we have that $f : V_1 \to V_2$ parametrizes the set $(V_1 \times V_2) \cap F^{-1}(0)$. The continuity of the map follows directly. For the smoothness one has to work a bit more. □

Question 4. *Prove that the map f in the implicit function theorem is smooth and compute its differential.*

Inverse mapping theorem. *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ a continuously differentiable function. Assume that for some $x \in U$ the differential $Df|_x$ has maximal rank. Then there are small neighborhoods V of x and W of $f(x)$ with $f(V) = W$ and a smooth map $g : W \to V$ with $f \circ g = \mathrm{Id}$ and $g \circ f = \mathrm{Id}$.*

Proof. Consider the map $F : \mathbb{R}^n \times U \to \mathbb{R}^n$ given by $F(y, x) = y - f(x)$. The assumptions on the theorem of implicit functions are satisfied. The map g provided by the implicit function theorem fulfills

$$0 = F(y, g(y)) = y - f(g(y))$$

and hence it does the job. □

Let $U \subset \mathbb{R}^n$ be open and $F : U \to \mathbb{R}^n$ be a map. Under a solution of the first order ODE $\gamma' = F(\gamma)$ we understand a map

$$\gamma : I \to U$$

with $\gamma'(t) = F(\gamma(t))$ for all $t \in I$. Here I is a subinterval of $\mathbb{R}$. Fixing $x \in U$ we say that a solution solves the initial value problem determined by x if $\gamma(0) = x$.

The definition of a second, third, etc... order ordinary differential equation is similar. One can reduce the discussion of the higher order ODEs as follows. In order to understand $y'' = y$ it suffices to solve $(y_1', y_2') = (y_2, y_1)$ and take the y_1-component. One can use a similar trick to solve time-dependent equations.

The main theorem here is the following:

Existence and uniqueness theorem (Picard-Lindelöf)**.** *Assume that $U \subset \mathbb{R}^n$ is open and $F : U \to \mathbb{R}^n$ satisfies a local Lipschitz condition. Then for every $x \in U$ there is $\epsilon > 0$ and a curve $\gamma : (-\epsilon, \epsilon) \to U$ with*

$$\gamma(0) = x \quad \text{and} \quad \gamma'(t) = F(\gamma(t)) \quad \text{for all } t$$

Moreover, any two solutions γ, η coincide on a small neighborhood of $0 \in \mathbb{R}$.

Proof. Up to reducing U we may assume that it is the ball of radius $1 > r > 0$ around $x = 0$ and that F is L-Lipschitz on U for some L. Observe also that for $\lambda > 0$ solutions of the original problem are related by solutions of

$$\gamma' = \lambda F(\gamma(x))$$

by a time change; replace namely $\gamma(t)$ by $\gamma(\lambda t)$. Choosing then λ very small we may assume that F is actually $\frac{1}{2}$-Lipschitz and that $\max_{y \in U} \|F(y)\| \leq \frac{1}{2}$.

We consider now the set X of C^1-curves $\gamma : [0, r) \to U$ with $\gamma(0) = 0$ and such that $\max_t |\gamma'(t)| \leq 1$. With respect to the d_{C^1}-distance, X is a complete metric space.

Given $\gamma \in X$ let Φ_γ be the curve

$$\Phi_\gamma(t) = \int_0^t F(\gamma(t))dt$$

Observe that $\Phi_\gamma(0) = 0$ and that for all t we have

$$\Phi_\gamma'(t) = \frac{d}{dt} \int_0^t F(\gamma(s))ds = F(t)$$

In particular, $\|\Phi_\gamma'(t)\| \leq \frac{1}{2}$ for all t and hence $\Phi_\gamma \in X$.

Let's estimate the Lipschitz constant of $\gamma \mapsto \Phi_\gamma$.

$$\begin{aligned}\|\Phi_\gamma(t) - \Phi_\eta(t)\| &\le \int_0^t \|F(\gamma(s)) - F(\eta(s))\| ds \\ &\le t\frac{1}{2}\max_t \|\gamma(t) - \eta(t)\| \\ \|\Phi'_\gamma(t) - \Phi'_\eta(t)\| &\le \|F(\gamma(t)) - F(\eta(t))\| \\ &\le \frac{1}{2}\|\gamma(t) - \eta(t)\|\end{aligned}$$

Taking both computations into account we obtain that

$$d_{C^1}(\Phi_\gamma, \Phi_\eta) \le \frac{1}{2} d_{C^0}(\gamma, \eta) \le \frac{1}{2} d_{C^1}(\gamma, \eta)$$

This means that $\Phi : X \to X$ is $\frac{1}{2}$-Lipschitz and hence has a unique fix-point γ. The curve γ is our desired solution. $\square$

Things to say:

(1) Maximal solutions
(2) Complete vector fields.
(3) Flow.

Question 5.

(1) *Construct a non-complete vector field on* $\mathbb{R}^2$*.*
(2) *Prove that any bounded vector field is complete.*
(3) *Assume that* F *is a complete vector field on* $\mathbb{R}^n$ *and let for* $x \in \mathbb{R}^n$ *be* γ_x *be the solution to the ODE* $\gamma'_x(t) = F(\gamma_x(t))$ *and* $\gamma_x(0) = 0$*. Prove that the map* $(x,t) \mapsto \gamma_x(t)$ *is continuous.*

Dependence on initial values. *Assume that* $F : \mathbb{R} \to \mathbb{R}$ *is continuously differentiable and let for all* $x \in \mathbb{R}$ *be* γ_x *be the solution to the initial value problem* $\gamma'_x = F \circ \gamma_x$ *and* $\gamma_x(0) = x$*. Then the map* $(x,t) \mapsto \gamma_x(t)$ *is continuously differentiable.*