2.8 Divergence As a Function of Wavenumber 41
2.9 Sensitivity to Perturbations 44
2.10 Padé Approximants and Other Summability Methods 44
2.11 Summary 45

3 HYPERASYMPTOTIC PERTURBATION THEORY 48
3.1 Introduction: Superasymptotics, Optimal Truncation and All That 48
3.2 Optimal Truncation and Superasymptotics for the Stieltjes Function 51
3.3 Hyperasymptotics for the Stieltjes Function 54
3.4 Soliton Hyperasymptotics: Outline of Procedure 57
3.5 Hyperasymptotics for Solitons: An Overview 58
3.6 Superasymptotic Assertion 60
3.7 Newton’s Iteration 62
3.8 Solving a Forced, Constant Coefficient ODE 64
3.9 Simplifying the Residual: Rearrangement 66
3.10 Simplifying the Residual: Approximations 69
3.11 Extrapolation: $\nu_0$ 73
3.12 Optimal Truncation 75
3.13 Extrapolation: $\nu_1$ 77
3.14 Summary 78

4 MATCHED ASYMPTOTIC EXPANSIONS IN THE COMPLEX PLANE 80
4.1 Perturbation Theory “Beyond All Orders” 80
4.2 A Primer of Matched Asymptotic Expansions 81
4.3 Exponentially Small Reflections 82
4.4 Forced Linear BVP 89
4.5 Borel-Logarithm Function 93
4.6 General Singularities 94
4.7 Nonlinearity 96
4.8 Critique: Complex Plane-Matched Asymptotics 100
5 Stokes’ Expansion, Resonance & Polycnoidal Waves

5.1 Introduction: Solitons from Sine Waves 106
5.2 Stokes’ Expansions 108
5.3 Stokes’ Expansion: Radius of Convergence 112
5.4 Stokes’ Expansion: Accuracy 114
5.5 Resonant Stokes’ Expansion 116
5.6 Regimes of the FKdV Cnoidal Wave 122
5.7 Polycnoidal waves and resonance 127

6 Existence, Non-Existence & Symmetry

6.1 Introduction 132
6.2 Formal Methods: Perturbation Theory and Numerical Methods 133
6.3 Nonexistence Theorems and Other Negative Results 133
6.4 Existence of Nanopterons 135
6.5 Symmetry 136

Part III Numerical Methods

7 Pseudospectral and Galerkin Methods

7.1 Introduction 141
7.2 Choice of Basis Set 145
7.3 Multidimensional Basis Sets 149
7.4 Accuracy: Geometric and Subgeometric 150
7.5 Iteration 155
7.6 Partial Summation and the Fast Fourier Transform 157
7.7 Embedding Pseudospectral Algorithms in Finite Difference or Finite Element Codes 159
7.8 Parity and Other Discrete Symmetries 159
7.9 Continuous Symmetries: Translation and Dilation 161
7.10 A Worked Example 164
7.11 Travelling Kinks and Solitary Waves That Asymptote to Constants 167
7.12 Modons and Peakons 169
7.13 Summary 170

8 NONLINEAR ALGEBRAIC EQUATIONS 172
8.1 Introduction: Initialize, Continue, Iterate 172
8.2 A Tale of Two Flows: the Davidenko Predictor and the Newton Corrector 175
8.3 Initialization 177
8.4 Continuation 182
8.5 Branch-Crossing and Fold Points 189
8.6 Newton’s Iteration: Basic Properties 195
8.7 Minimizing Residual Norm 197
8.8 The Geometry of Phase Space 197
8.9 Newton’s Iteration: Underrelaxation 198
8.10 Tunneling Away from a Minimum of the Residual Norm 203
8.11 Quasi-Newton Methods 208
8.12 Non-Newtonian Iterations 212
8.13 Direct Methods for Special Low Order Systems 219
8.14 Summary 222

9 SPECIAL ALGORITHMS FOR EXPONENTIALLY SMALL PHENOMENA 224
9.1 Introduction 224
9.2 Fourier methods 225
9.3 Infinite interval, I: The triumph and folly of pure rational Chebyshev basis 229
9.4 Infinite interval, II: The radiation basis function 232
9.5 Infinite interval, III: Cnoidal matching 235
9.6 Sensitivity 238
9.7 Summary 239

Part IV APPLICATIONS 241

10 WATER WAVES: FIFTH-ORDER KORTEWEG-DEVRIES EQUATION 243
10.1 Introduction 243
## Contents

10.2 A generic derivation and scaling of the FKdV equation 246  
10.3 Multiple Scales Perturbation Theory 250  
10.4 Parameters: $\epsilon$, Phase Speed and $\Phi$ 255  
10.5 Eigenfunctions and resonant phase shifts 257  
10.6 Symmetry 262  
10.7 Complex Plane Matched Asymptotics 264  
10.8 Numerical Solutions: FKdV 265  
10.9 Radiatively Decaying FKdV Solitons 268  
10.10 Bound States of Solitons 269  
10.11 Water Waves 271

### 11 ROSSBY & INTERNAL GRAVITY WAVES: NONLOCAL HIGHER MODES 279  
11.1 Introduction 279  
11.2 Model Equations 283  
11.3 Eigenfunction Expansions, the Method of Multiple Scales, and the KdV Equation 285  
11.4 Two-Mode (Coupled-KdV) Model 289  
11.5 Improved Single-Mode Model: The Korteweg-deVries Equation with Mixed Cubic and Quadratic Nonlinearity 292  
11.6 Variations on a Theme: One-Mode & Two-Mode Models for Equatorial Rossby Waves 294  
11.7 Modified Korteweg-deVries Equation 297  
11.8 Magnetic Modons in a Shear Flow 301  
11.9 Observations of Nonlocal Gravity and Rossby Waves 302  
11.10 Theoretical and Numerical Studies 304

### 12 THE $\phi^4$ BREATHER 306  
12.1 Introduction 306  
12.2 The Multiple Scales Series 310  
12.3 Far Field Analysis 314  
12.4 Eigenfunctions 316  
12.5 Numerical calculations of the nanopteron and nanopteroidal wave 318  
12.6 Radiative Decay of a Localized Initial Condition 321  
12.7 Summary 323
# 13 **ENVELOPE SOLITARY WAVES**

13.1 Introduction 325  
13.2 Optical Solitons 329  
13.3 TNLS Equation: An Overview 331  
13.4 TNLS Equation: Scaling and Parameters 333  
13.5 The TNLS Equation: Perturbation Theory 337  
13.6 Periodic Solutions 339  
13.7 Eigenfunctions 341  
13.8 Numerical calculations of nanopterons 344  
13.9 Bions 348  
13.10 Radiative Decay in the TNLS Equation 349  
13.11 Direct Approach to Envelope Solitons: KG Eq. 352  
13.13 The Fourier Method in Group and Carrier Coordinates 355  
13.14 KG Equ.: Far Field & Phase Condition 357  
13.15 Klein-Gordon numerical solutions 360  
13.16 Summary 363

# 14 **SEPARATRIX SPLITTING & SLOW MANIFOLD**

14.1 Introduction 366  
14.2 Separatrix Splitting for the Perturbed Pendulum 368  
14.3 The Slow Manifold: Background 373  
14.4 Simplest Model of the Slow Manifold: The LK Quintet 375  
14.5 The Slow Manifold: The Method of Multiple Scales & Ambiguity 379  
14.6 Dynamical Systems Theory 383  
14.7 The Slow Manifold: Summary 384

# 15 **MICROPTERONS**

15.1 Introduction 387  
15.2 The Rotation-Modified Korteweg-deVries (RMKdV) Equation: a One-Dimensional Micropterion 389  
15.3 Non-Hydrostatic Gravity Waves with Rotation in Two Horizontal Dimensions: RMKP Equation 396  
15.4 The Morning Glory: Internal Gravity Solitons 399
# Contents

15.5 Modons: Basic Theory for Dipolar Vortices 407  
15.6 Tribbia-Verkley-Boyd Nonlocal Modons 412  
15.7 Nonlocal Modons: One-Sided, East Wing-Only Vortex Pairs of Swaters 417  
15.8 Monopole Vortices 421  
15.9 Gulf Stream Rings 422  
15.10 Equatorial Rossby: Leakage-to-Barotropic 427  
15.11 Summary 429

## Part V RADIATIVE DECAY & OTHER EXPONENTIALLY SMALL PHENOMENA 431

16 RADIATIVE DECAY 433  
16.1 Introduction 433  
16.2 Perturbation Theory 435  
16.3 Near-Soliton Initial Conditions: Analysis of Perturbative Wave Packets 438  
16.4 Extensions: Linearity, Forcing, Transience and Breathers 440  
16.5 Limitations: Dispersion versus Resonance 443  
16.6 Subtleties 446  
16.7 Multiple Scales in Time: Derivation of Model Equations for Radiative Damping 447  
16.8 An ODE for the Decay of the Amplitude Due to Radiation 449  
16.9 Multiple Scales in Time: Micropterons and the Morning Glory 451  
16.10 Summary 454

17 NON-SOLITON EXPONENTIALLY SMALL PHENOMENA 455  
17.1 Introduction 455  
17.2 Dendritic Crystal Growth 455  
17.3 Viscous Fingering in a Hele-Shaw Cell (Saffman-Taylor Problem) 459  
17.4 Quantum Mechanics 461  
17.5 Diffusion on Exponentially Long Time Scales 462  
17.6 Steepest Descent: Asymptotic Expansions for Integrals 464  
17.7 Numerical Analysis 468
17.8 Isolation of Exponential Smallness, I: Equatorial Kelvin Wave with Critical Latitude and Other Eigenproblems 471
17.9 Isolation of Exponential Smallness II: Laminar Flow in a Porous Pipe or Channel with Suction (Berman-Robinson and Terrill Problems) 476
17.10 Summary 478

18 THE FUTURE 479

A TRIGONOMETRIC AND SECH IDENTITIES 482
A.1 Differentiation Identities 482
A.2 Hyperbolic Identities 484
A.3 Trigonometric Identities 485

B SECH/TANH SERIES 486
B.1 Polynomialization 486

C ELLIPTIC FUNCTIONS 489
C.1 Basic Properties 489
C.2 Elliptic Nome and Modulus 490
C.3 Series: Complete Elliptic Integral and Modulus 491
C.4 Elliptic Functions: Fourier & Imbricate Series 492
C.5 Elliptic Functions: Differentiation Identities 493
C.6 Elliptic Functions: Integration 493

D SOLITONS AND CNOIDAL WAVES 494
D.1 Solitons 494
D.2 Spatially Periodic Solutions 497

E TIME INTEGRATION & FOURIER PSEUDOSPECTRAL ALGORITHM 501
E.1 Spectral Methods and Wave Equations 501
E.2 Time Integration Schemes 501
E.4 Complication I: Library Fast Fourier Transform Software 506